NONLINEAR DYNAMICAL SYSTEMS OF MATHEMATICAL PHYSICS

Spectral and Symplectic Integrability Analysis

Denis Blackmore
Anatoliy K Prykarpatsky
Valeriy Hr Samoylenko

World Scientific

NONLINEAR DYNAMICAL SYSTEMS OF MATHEMATICAL PHYSICS

Spectral and Symplectic Integrability Analysis

This page intentionally left blank

NONLINEAR DYNAMICAL SYSTEMS OF MATHEMATICAL PHYSICS

Spectral and Symplectic Integrability Analysis

Denis Blackmore

New Jersey Institute of Technology, USA

Anatoliy K Prykarpatsky

AGH University of Science and Technology, Poland

The Ivan Franko State Pedagogical University, Ukraine

Valeriy Hr Samoylenko

Kyiv National Taras Shevchenko University, Ukraine

World Scientific

NEW JERSEY • LONDON • SINGAPORE • BEIJING • SHANGHAI • HONG KONG • TAIPEI • CHENNAI

Published by

World Scientific Publishing Co. Pte. Ltd.

5 Toh Tuck Link, Singapore 596224

USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601

UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

Library of Congress Cataloging-in-Publication Data

Nonlinear dynamical systems of mathematical physics : spectral and symplectic integrability
analysis / by Denis Blackmore ... [et al.].
p. cm.
Includes bibliographical references and index.
ISBN-13: 978-981-4327-15-2 (hardcover : alk. paper)
ISBN-10: 981-4327-15-8 (hardcover : alk. paper)
1. Differentiable dynamical systems. 2. Nonlinear theories. 3. Symplectic geometry.
4. Spectral analysis--Mathematics. I. Blackmore, Denis L.
QA614.8.N656 2011
530.15'539--dc22
2010028336

British Library Cataloguing-in-Publication Data

A catalogue record for this book is available from the British Library.

Printed in Singapore.

We dedicate this book to our families, whose love, devotion and patience sustained us; our teachers and mentors, who enlightened and guided us; our colleagues, students and friends, who aided and encouraged us; and the pioneers in the field, whose magnificent contributions inspired us.

Preface

This book grew out of our lecture notes for advanced undergraduate and graduate students of mathematics and physics over the last several years at the New Jersey Institute of Technology (Newark, NJ, USA), the AGH-University of Science and Technology (Krakow, Poland), the Ivan Franko State Pedagogical University (Drohobych, Ukraine) and T. Shevchenko National University (Kyiv, Ukraine), as well as our recent research. We first introduce readers to the standard modern backgrounds of dynamical systems theory on finite-dimensional symplectic manifolds; in particular, the main differential-geometric and Lie-algebraic aspects of the abelian Hamilton–Jacobi and Liouville–Arnold integrability theories along with a modern grounding in the non-abelian Mishchenko–Fomenko integrability aspects of nonlinear Hamiltonian systems on Poisson manifolds with symmetries. We have also devoted a chapter to a rather extensive introduction to the differential-geometric properties of reduced canonically-symplectic manifolds with symmetry, and their relationship with structures on principal fiber bundles. As natural applications, we analyze the Hamiltonian properties of classical electromagnetic Maxwell and Yang–Mills equations. Keeping in mind the numerous applications of the results for studying quantum-mechanical problems, we also describe some of Lie-algebraic aspects of integrable dynamical systems related to Hopf and quantum algebras.

Next, we provide a detailed introduction to the theory of infinite-dimensional Hamiltonian dynamical systems on functional manifolds and describe the formulation of their Lax type integrability analysis in the framework of a new and very effective gradient-holonomic algorithm, and present some typical examples of testing the integrability of nonlinear infinite-dimensional dynamical systems. Of particular note is the integrability analysis of a Whitham type nonlocal dynamical system describing the nontrivial wave processes in a medium with spatial memory: its Riemann hydrodynamical generalized regularization and related infinite hierarchies of conservation laws are constructed, and the Lax integrability is proved. In an effort to generalize the class of integrable dynamical systems, we also develop a new parametric-spectral version of the gradient-holonomic integrability algorithm for nonuniform and non-autonomous nonlinear dynamical systems on functional manifolds.

Also presented are some interesting and useful results on the versal deformation structure of a one-dimensional Dirac type linear spectral problem on an axis. A whole section is devoted to an analysis of the classical integrability in quadratures of Riccati and Riccati–Abel ordinary differential equations. By making novel use of Bäcklund transformations in the context of a Lie-algebraic integrability scheme, we study integrable three-dimensional coupled nonlinear dynamical systems related to centrally extended operator Lie algebras.

The Lie-algebraic theory of Lax integrable nonlinear dynamical systems is used to thoroughly analyze the integrability aspects of differential-difference nonlinear dynamical systems. We show how this approach can be employed to obtain new results on tensor Poisson structures related to factorized operator dynamical systems, and in the process give a detailed Hamiltonian analysis of the systems and provide a complete description of their integrability properties.

A chapter is devoted to modern aspects of a generalized de Rham–Hodge theory related to the classical problem of describing Delsarte–Lions transmutation operators in multi-dimensions within a spectral reduction approach. This theory naturally gives rise to characteristic Chern classes, which we show can be used to develop a new differential-geometric integrability analysis of multi-dimensional differential systems of Gromov type on Riemannian manifolds.

Developments over past century have clearly demonstrated that the methods of quantum physics provide powerful tools for studying many mathematical problems, so we also included some introductory material that illustrates this by way of recent results obtained by means of quantum mathematical approaches. In particular, we present a regularized approach based on the classical Fock space embedding technique that allows a complete linearization of a wide class of nonlinear dynamical systems in Hilbert spaces. A brief description of how the geometric approach can be used to obtain some relatively new results on quantum holonomic computing is included as well. We also revisit in detail modern relativistic electrodynamic and hadronic string models using Lagrangian and Hamiltonian formalisms.

In order to make the overall treatment as self contained as possible, we have included at the end a short Supplement covering the essential differential-geometric preliminaries used throughout the text.

Many parts of the manuscript were discussed with our students and colleagues, to whom we express our heartfelt thanks. Especially, we express our cordial appreciation to Maciej Błaszak, Nikolai Bogolubov (Jr.), Vasyl Gafiychyk, Roy Glauber, Vladislav Goldberg, Roy Goodman, Lech Górniewicz, Ilona Gucwa, Petro Holod, Yuriy Kozicky, Vasyl Kuybida, Chjan Lim, Mirosław Luśtyk, Anatoliy Logunov, Robert Miura, Ryszard Mrugała, Ihor Mykytyuk, Jolanta Napora-Golenia, Maxim Pavlov, Zbigniew Peradzyński, Ziemowit Popowicz, Yarema Prykarpatsky, Mykola Prytula, Anthony Rosato, Anatoliy Samoilenko, Roman Samulyak, Andrey Shoom, Jan Sławianowski, Ufuk Taneri, John Tavantzis, Lu Ting, Xavier Tricoche and Kevin Urban. Special thanks are due to Natalia Prykarpatsky for her constant support and help in editing the bulk of the manuscript. And last, but not least, we express thanks to World Scientific Publishing for the opportunity of publish this book as part of their outstanding collection of titles in mathematics and mathematical physics.

The contribution to this book by Denis Blackmore was supported in part by the National Science Foundation (Grant CMMI-1029809) and Center for Applied Mathematics and Statistics at NJIT.

The research by Anatoliy K. Prykarpatsky was supported in part by the European Scientific Foundation through an ESF-grant-06 at the SISSA and the ICTP-scholarship-07 in Trieste, Italy.

Newark-Cracow:Drohobych-Kyiv

Denis Blackmore
Anatoliy K. Prykarpatsky
Valeriy H. Samoylenko

This page intentionally left blank

Contents

This page intentionally left blank

Chapter 1

General Properties of Nonlinear Dynamical Systems

We begin in this chapter with a brief treatment of some elements of dynamical systems theory, mainly in finite dimensions, that are indispensable for modern mathematical physics.

1.1 Finite-dimensional dynamical systems

1.1.1 *Invariant measure*

Let M be a finite-dimensional differentiable manifold of dimension $n = \dim M$, and $\mathfrak{A}$ be a σ-algebra of subsets of the manifold M on which the normalized and complete measure μ is defined, so that $\mu(M) = 1$ and all subsets of measure zero belong to $\mathfrak{A}$.

Suppose that there is a mapping $\varphi : M \to M$ with properties: 1) φ is bijective; 2) if $A \in \mathfrak{A}$, then sets $\varphi A, \varphi^{-1}A \in \mathfrak{A}$ and $\mu(\varphi A) = \mu(\varphi^{-1}A) = \mu(A)$.

In this case the mapping φ is called an automorphism of the manifold M with measure μ, and the measure μ is said to be *φ-invariant.*

Let $\{\varphi^t\}$, $t \in \mathbb{R}$, be a one-parameter group of automorphisms of M with respect to a measure μ, so that $\varphi^t \circ \varphi^s = \varphi^{t+s}$ for all $t, s \in \mathbb{R}$ and the measure μ is φ^t-invariant for every member of the group. Then $\{\varphi^t\}$, $t \in \mathbb{R}$, is called a *finite-dimensional dynamical system* on the manifold M, or a *flow* if for any measurable function $f : M \to \mathbb{R}$ the function $f \circ \varphi^t : M \to \mathbb{R}$ defined on the direct product $M \times \mathbb{R}$ is also measurable.

The manifold M with measure μ is called the *phase space* of the dynamical system.

1.1.2 *The Liouville condition*

There is a classical characterization of invariant measures for flows that can be described in simple terms. Define $\mathcal{F}(M)$ to be the module of smooth vector fields [3, 14] on a manifold M. Then a vector field $X \in \mathcal{F}(M)$ defines in local coordinates $(x_1, x_2, \ldots, x_n)$ on M a system of ordinary differential equations

$$\frac{dx_j}{dt} = X_j(x_1, x_2, \ldots, x_n), \quad j = 1, \ldots, n, \tag{1.1}$$

which generates automorphisms $\varphi^t : M \to M$, $t \in \mathbb{R}$, by means of the mapping $\varphi^t x = x(t)$, where $x(t) \in M$ is the (unique) solution of the equations (1.1) satisfying the initial condition or Cauchy data $x(0) = x \in M$.

The set $\{\varphi^t\}$, $t \in \mathbb{R}$, is obviously a one-parameter group if the manifold M is compact, oriented and closed.

An invariant measure μ on M can be (locally) determined in the following way by means of the formula: $\mu(dx) = p(x)\, dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n$, where the weight function $p : M \to \mathbb{R}_+$ satisfies *the Liouville condition* [3, 14]

$$\sum_{j=1}^{n} \frac{\partial}{\partial x_j}(pX_j) = 0. \tag{1.2}$$

Indeed, for the measure μ to be invariant it is necessary and sufficient that for all $f \in \mathcal{D}(M)$, where $\mathcal{D}(M) = L^1(M; \mu)$ is the space of integrable functions on M, the following equation holds:

$$\int_M f(x)\mu(d^n x) = \int_M f(\varphi^t x)\mu(d^n x) \tag{1.3}$$

for all $t \in \mathbb{R}$.

To prove this it suffices to work locally (owing to the additivity of the integral) and assume that f is C^1 (since such functions are dense in $\mathcal{D}(M)$). Accordingly we assume that f is C^1 and has the support $\operatorname{supp} f$ concentrated in some coordinate neighborhood U of a point $x \in M$ that is diffeomorphic to an open ball $V \subset \mathbb{R}^n$. Then by differentiating (1.3) with respect to $t \in \mathbb{R}$ for $0 < |\, t \,| \leq \delta$ with δ sufficiently small, we obtain:

$$\begin{aligned} 0 = \frac{d}{dt} \int_V f(\varphi^t x)\ p\ (x) d^n x \ |_{t=0} &= \int_V \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} X_j(x) p\ (x) d^n x \\ &= - \int_V \sum_{j=1}^{n} \frac{\partial}{\partial x_j}(p\ X_j) f(x) d^n x, \end{aligned} \tag{1.4}$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$and $d^n x = dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n$. Hence the Liouville condition (1.2) follows by virtue of the fact that $f \in \mathcal{D}(M)$ is arbitrary. □

Further details can be found in the fundamental papers by M.M. Krylov and N.N. Bogolubov [55], [214] devoted to general problems of the global theory of invariant measures for dynamical systems on metric spaces.

As an illustration, consider the case of system (1.1) where $M = \mathbb{T}^n$ is the n-dimensional torus and $X_j(x) = \omega_j \in \mathbb{R}$, $j = 1, \ldots, n$, are incommensurate (rationally independent) constants. Then $\mu(d^n x) = d^n x$ is clearly an invariant measure for the flow, and it follows directly that the motion of system (1.1) on $\mathbb{T}^n$ is *almost periodic* with the numbers ω_j, $1 \leq j \leq n$, playing the roles of the *frequencies* of motion.

1.1.3 *The Poincaré theorem*

One of the most important properties of a dynamical system is that of *recurrence* as formulated by Poincaré. A point $x \in A \subset M$ is *recurrent* (with respect to the set A) for a dynamical system $\{\varphi^t\}$ if for any $\tau \in \mathbb{R}_+$ there exists a $t > \tau$ such that $\varphi^t x \in A$.

Theorem 1.1 (Poincaré, [173, 205]**).** *For any dynamical system $\{\varphi^t\}$ on M and for any $A \in \mathfrak{A}$ almost all points of A are recurrent (with respect to the associated invariant measure μ).*

Proof. Denote by N the subset of A consisting of all non-recurrent points of A. Then $N := A \bigcap \left(\bigcap_{t>\tau_0} \varphi^{-t}(M \setminus A)\right)$. Obviously, for all $x \in N$ and for all $t > \tau_0$ we have $\varphi^{-t} x \notin N$.

Thus $N \cap \varphi^{-t}(N) = \varnothing$ for all $t > \tau_0$. It follows that $\varphi^{-t_1}(N) \cap \varphi^{-t_2}(N) = \varnothing$ for all $t_2 > t_1 + \tau_0 > 2\tau_0$ as $\varphi^{-t_1}(N) \cap \varphi^{-t_2}(N) = \varphi^{-t_1}(N \cap \varphi^{-(t_2 - t_1)}(N)) = \varphi^{-t_1}(\varnothing) = \varnothing$.

Therefore, choosing a sequence of numbers $\{t_j \in \mathbb{R}_j,\ j \in \mathbb{Z}_+ \backslash \{0\}\}$ such that $t_{j+1} > t_j + \tau_0$ we obtain,

$$1 = \mu(M) \geq \mu\left(\bigcup_{j=1}^{\infty} \varphi^{-t_j}(N)\right) = \sum_{j=1}^{\infty} \mu(\varphi^{-t_j}(N)) = \sum_{j=1}^{\infty} \mu(N),$$

which can hold only if $\mu(N) = 0$. □

1.1.4 *The Birkhoff–Khinchin theorem*

Another important property of certain dynamical systems is that of ergodicity. We shall now formulate a general theorem on ergodicity that is one of the most fundamental and useful in dynamical systems theory [205].

Theorem 1.2 (Birkhoff–Khinchin). *Let $f \in L^1(M;\mu)$ and $\{\varphi^t\}$, $t \in \mathbb{R}$, be a dynamical system on M. Then for almost all $x \in M$ the following limits exist and are equal, i.e.:*

$$\begin{aligned}\lim_{t\to\infty} \frac{1}{t}\int_0^t f(\varphi^\tau x)d\tau &= \lim_{t\to\infty}\frac{1}{t}\int_0^t f(\varphi^{-\tau}x)d\tau \\ &= \lim_{t\to\infty}\frac{1}{2t}\int_{-t}^t f(\varphi^\tau x)d\tau \overset{a.e.}{=} \bar{f}(x)\end{aligned} \tag{1.5}$$

with $\bar{f}(\varphi^t x) = \bar{f}(x)$ if the value $\bar{f}(x)$ exists, $\bar{f} \in L^1(M;\mu)$ and

$$\int_M \bar{f}(x)\mu(dx) = \int_M f(x)\mu(dx).$$

The limit function $\bar{f} \in L^1(M;\mu)$ is called the *time average along orbits* of a dynamical system.

A function $f : M \to \mathbb{R}$ (which is measurable with respect to the measure μ) satisfying $f(x) = f(\varphi^t x)$ for all $x \in M$ is called *invariant with respect to the dynamical system* $\{\varphi^t\}$, $t \in \mathbb{R}$. Thus, the limit function $\bar{f}(x) : M \to \mathbb{R}$ is invariant owing to the above theorem.

The ergodicity property can as a result of the above be neatly formulated in terms of invariant functions; namely, a dynamical system $\{\varphi^t\}$ is called *ergodic* if any invariant function is almost everywhere constant.

An equivalent definition can be formulated in terms of invariant sets. A set $A \in \mathfrak{A}$ is called *invariant* if its characteristic function χ_A is invariant. Then obviously a dynamical system $\{\varphi^t\}$ is ergodic if and only if the measure $\mu(A) = 0$ or $\mu(A) = 1$ for any invariant set. As an invariant function is constant on almost every orbit, it follows from the Birkhoff–Khinchin theorem that $\bar{f}(x) = \int_M f(x)\mu(dx)$ for almost all $x \in M$, i.e. the time average of a function $f \in L^1(M;\mu)$ is equal almost everywhere to its *spatial average*, and this fact has a wide variety of applications in various physical problems involving ergodicity.

There is another useful characterization of ergodicity that follows from the Birkhoff–Khinchin theorem when the function $f \in L^1(M;\mu)$ is taken as χ_A for a set $A \in \mathfrak{A}$. Then it is easy to see that a dynamical system $\{\varphi^t\}$ is ergodic if and only if the relative period of time the orbit points $\{\varphi^t x : t \in [0,T]\}$ lie in the set A is asymptotically equal to the measure $\mu(A)$ for almost all initial points $x \in M$. This can be formulated precisely as

$$\lim_{T\to\infty} \frac{\tau(T;A,x)}{T} \overset{a.e.}{=} \mu(A), \tag{1.6}$$

where $\tau(T;A,x) = m\left(\{t : t \in [0,T],\ \varphi^t x \in A\}\right)$, $x \in M$, where m is the Lebesgue measure on $\mathbb{R}$.

1.1.5 *The Birkhoff–Khinchin theorem for discrete dynamical systems*

Let us consider a discrete dynamical system, i.e. a dynamical system on M defined by iterations: $\varphi^n x = \varphi(\varphi^{n-1}x)$, $x \in M, n \in \mathbb{Z}_+$, where $\varphi : M \to M$ is an automorphism of M preserving the measure of any set $A \in \mathfrak{A}$.

For a discrete dynamical systems there is an analogous theorem of Birkhoff–Khinchin type. In this case the average value $\bar{f}(x)$ for almost any $x \in M$, $f \in L^1(M;\mu)$, is defined as follows:

$$\begin{aligned}\bar{f}(x) &\overset{a.e.}{=} \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} f(\varphi^j x) = \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} f(\varphi^{-j}x) \\ &= \lim_{n\to\infty} \frac{1}{2n+1}\sum_{j=-n}^{n-1} f(\varphi^j x).\end{aligned} \tag{1.7}$$

The function $\bar{f} \in L^1(M;\mu)$ therefore satisfies the equation

$$\int_M \bar{f}(x)\mu dx = \int_M f(x)\mu(dx)$$

which in the case of ergodicity of a dynamical system $\{\varphi^n\}$, $n \in \mathbb{Z}$, also leads to the equality of the time average $\bar{f}(x)$ and the space average for almost all $x \in M$.

Another equivalent definition of ergodicity of a discrete dynamical system $\{\varphi^n\}$, $n \in \mathbb{Z}$, is the following: as $n \to \infty$ the average frequency of hits of the orbit points $\{\varphi^n x : n \in \mathbb{Z}\}$ (with arbitrary initial point $x \in M$) in a fixed set $A \in \mathfrak{A}$ is equal to the measure of the set A, i.e.

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} X_A(\varphi^j x) = \lim_{n\to\infty} \frac{\tau(n;A,x)}{n} \overset{a.e.}{=} \mu(A).$$

Example 1.1. Consider the distribution of first digits of numbers 2^n, $n \in \mathbb{Z}_+$. The first digit of number 2^n is obviously equal to $k \in [1, 9]$ if

$$k \cdot 10^r \leq 2^n < (k+1)10^r, \tag{1.8}$$

where $r \in \mathbb{Z}_+$. Denoting by $\{s\}$ the fractional part of a number $s \in \mathbb{R}$ from (1.8) we get:

$$\log k \leq \{n\omega\} < \log(k+1), \tag{1.9}$$

where $\omega = \log 2$. Now consider a discrete dynamical system $\{\varphi^n\}$ on the one-dimensional torus $\mathbb{T}^1 \approx \mathbb{S}^1$ such that φ acts on $x \in \mathbb{T}^1$, $x \in \mathbb{R}$, as follows: $\varphi(x) = \{x + \omega\}$. As ω is irrational, the orbit $\{\varphi^n x : t \in \mathbb{Z}_+\}$ for any point $x \in \mathbb{T}^1$ is dense in torus $\mathbb{T}^1$, so the dynamical system $\{\varphi^n\}$ on $\mathbb{T}^1$ is ergodic. From the Birkhoff–Khinchin theorem applied to the set $A = [\log k, \log(k+1)]$ from (1.6) we obtain

$$\lim_{n\to\infty} \frac{\tau(n; A, 0)}{n} = \mu(A) = \log\left(1 + \frac{1}{k}\right), \tag{1.10}$$

where μ is the Lebesgue measure on $\mathbb{T}^1$. It is easy to see that formula (1.10) determines the average frequency of those sequence $\{2^n : n \in \mathbb{Z}_+\}$ elements, whose first digits are equal to $k \in \{1, \ldots, 9\}$. For example, from (1.10) it follows that the frequency of units as the first digits of the sequence $\{2^n, n \in \mathbb{Z}_+\}$ is far greater than that of the nines.

1.2 Poissonian and symplectic structures on manifolds

1.2.1 *Poisson brackets*

By requiring more (natural) structure on a dynamical system one generally is able to deduce many additional properties important for applications. One of the most ubiquitous and richest types of such structures is that associated with dynamical systems that are Hamiltonian.

Let M be a smooth manifold of even dimension $2n = \dim M < \infty$, $\Lambda(M) = \oplus_{p=1}^{2n} \Lambda^{(p)}(M)$ be the Grassmann algebra of differential forms on M [3, 144, 173]. A closed nonsingular differential 2-form $\omega^{(2)} \in \Lambda^{(2)}(M)$ induces a *symplectic structure* on M.

The pair (M, ω) is called a *symplectic manifold* and the 2-form $\omega^{(2)} \in \Lambda^{(2)}(M)$ is said to be symplectic.

Let a vector field $X \in \mathcal{F}(M)$ on manifold M be such that the 1-form $i_X\omega^{(2)} \in \Lambda^{(1)}(M)$, where i_X is the interior product of algebra the $\Lambda(M)$, is closed on M. Then the vector field X is called *quasi-Hamiltonian.*

If the 1-form $i_X\omega^2 \in \Lambda^{(1)}(M)$ is exact and a function $H \in \mathcal{D}(M)$ is such that $i_X\omega^{(2)} = -dH$ then the vector field $X_H := X$ is said to be *Hamiltonian*, and H is the corresponding *Hamiltonian function.*

Let the functions $F, G, \in \mathcal{D}(M)$ and the vector fields $X_F, X_G \in \mathcal{F}(M)$ be defined by the formulas $i_{X_F}\omega^{(2)} = -dF$, $i_{X_G}\omega^{(2)} = -dG$. The Poisson bracket $\{F, G\}_\omega$ of functions F, G is defined as follows:

$$\{F, G\}_\omega = \omega^{(2)}(X_F, X_G). \tag{1.11}$$

Let us consider on the manifold M the flow $\{\varphi^t : t \in \mathbb{R}\}$ generated by the vector field X_H. For any point $x \in M$,

$$\frac{d}{dt}\omega^{(2)}(\varphi^t x) = (i_{X_H}d + di_{X_H})\omega^{(2)}(\varphi^t x) = d(i_{X_H}\omega^{(2)})(\varphi^t x) = -d^2H = 0,$$

so $\mu(dx) = (\omega^{(2)})^n(x)$ is an invariant measure.

The rate of change of a function $F \in \mathcal{D}(M)$ along a trajectory of the dynamical system $\{\varphi^t\}$, by virtue of (1.11), can be written as follows:

$$dF/dt = \{H, F\}_\omega. \tag{1.12}$$

If $\{H, F\}_\omega = 0$, F is called a *first integral*, *invariant* or *constant of motion* of the Hamiltonian system X_H. Obviously, $H \in \mathcal{D}(M)$ is invariant owing to the antisymmetry of the Poisson bracket.

The important property of a Poisson bracket (1.11) introduced above is its closure with regard to the operation $\{\cdot,\ \cdot\}_\omega$. Namely, any three functions $F, G, H \in \mathcal{D}(M)$ satisfy the *Jacobi identity*

$$\{F, \{G, H\}_\omega\}_\omega + \{G, \{H, F\}_\omega\}_\omega + \{H, \{F, G\}_\omega\}_\omega = 0, \tag{1.13}$$

and, in addition, $\{F, G\}_\omega = -\{G, F\}_\omega$ as mentioned above.

It means that over the field $\mathbb{R}$ the set $\mathcal{D}(M)$ forms a Lie algebra with algebra operation $\{\cdot,\ \cdot\}_\omega$.

1.2.2 *The Liouville theorem and Hamilton–Jacobi method*

Let us consider in more detail a Hamiltonian dynamical system X_H on a symplectic manifold $(M, \omega^{(2)})$ having first integrals. The functions $F, G \in \mathcal{D}(M)$ are said to be *in involution* if $\{F, G\}_\omega = 0$ on M.

The following theorem of Liouville guarantees the ergodicity of Hamiltonian system X_H.

Theorem 1.3 (Liouville). *Let on the symplectic $2n$-dimensional manifold M the system of n functions $\mathcal{P} = \{P_j \in \mathcal{D}(M) : 1 \leq\ j \leq n\}$ which* *are in involution, i.e.* $\{P_j, P_k\}_\omega = 0,\ 1 \leq j, k \leq n$, *be given. Consider the level set of the system* $\mathcal{P} : M_p = \{x \in M : P_j(x) = p_j,\ j = 1, \ldots, n\ \}$. *Suppose that on* M_p *functions* $P_j,\ j = 1, \ldots, n$ *are independent, i.e.* 1*-forms* $dP_j,\ j = 1, \ldots, n$, *are linearly independent at every point* $x \in M_p$. *Then:*

(1) M_p *is a smooth submanifold, called Lagrangian, invariant with respect to flow* X_H *with Hamiltonian function* $H = P_1$*;*
(2) if manifold M_p *is compact and connected then it is diffeomorphic to the* n*-dimensional torus* $\mathbb{T}^n$*;*
(3) the phase flow X_H *with Hamiltonian function* H *determines on* M_p *the quasi-periodic motion;*
(4) the canonical equations of the field X_H *with Hamiltonian function* H *are integrable by quadratures.*

We omit a proof of this theorem because it is classic and is adduced in many books and papers [3, 14, 205]. In the sequel, our main interest in this result is primarily how it applies to a given dynamical system that is integrable by quadratures.

An involutive system $\mathcal{Q} = \{Q_k \in \mathcal{D}(M) :\ 1 \leq k \leq n\}$ of functionally independent functions on M is *canonically conjugated* to the initial involutive system $\mathcal{P} = \{P_j \in \mathcal{D}(M) : 1 \leq\ j \leq n\ \}$ if

$$\{P_j, Q_k\}_\omega \ = \ \delta_{jk}$$

for $1 \leq j, k \leq n$.

In terms of such systems of functions on M, the Liouville's theorem states that the system $\mathcal{Q} = \{Q_j \in \mathcal{D}(M) :\ \ 1 \leq j \leq n\}$, canonically conjugated to system P, can be constructed by quadratures.

Indeed, let us consider an arbitrary point $x \in M_p$ and its coordinate representation by the function system $\{P_j, S_j \in \mathcal{D}(M) : 1 \leq j \leq n\}$ where the choice of functions $\{P_j, S_j : 1 \leq j \leq n\}$ is not important.

Let the 1-form $\alpha^{(1)} \in \Lambda^{(1)}(M)$ be such that $d\alpha^{(1)} = \omega^{(2)}$, where the form $\omega^{(2)} \in \Lambda^{(2)}(M)$ is symplectic. It follows from Poincaré's theorem [3, 144, 173] that $\alpha^{(1)} \in \Lambda^{(1)}(M)$ always exists because the 2-form $\omega^{(2)}$ is closed.

The 1-form $\alpha^{(1)}$ can be written locally in coordinates $\{P_j, S_j : 1 \leq j \leq n\}$ as

$$\alpha^(1) = \sum_{j=1}^{n} a_j dP_j + \sum_{j=1}^{n} b_j dS_j,$$

where $a_j, b_j \in \mathcal{D}(M), 1 \leq j \leq n$, are some functions.

As the functions $P_j \in \mathcal{D}(M)$, $1 \le j \le n$, are involutive, we find that on the manifold $M_p \subset M$ the 2-form $\omega^{(2)}|_{M_p} = 0$. Hence, the 1-form $\alpha^{(1)} \in \Lambda^{(1)}(M)$ on the submanifold M_p will locally be a total differential of a function $\tilde{Q} \in \mathcal{D}(M)$, i.e. $\alpha^{(1)}|_{M_p} = d\tilde{Q}$. Consequently,

$$\alpha^{(1)} = d\tilde{Q} + \sum_{j=1}^{n} \left(a_j - \frac{\partial \tilde{Q}}{\partial P_j} \right) dP_j, \tag{1.14}$$

that is

$$\sum_{j=1}^{n} \frac{\partial \tilde{Q}}{\partial S_j}\, dS_j = \sum_{j=1}^{n} b_j dS_j.$$

When introducing the variables

$$Q_j = -a_j + \frac{\partial \widetilde{Q}}{\partial P_j},$$

$j = 1, \ldots, n$, the form $\alpha^{(1)}$ according to (1.14) can be written in the form

$$\alpha^(1) = -\sum_{j=1}^{n} Q_j dP_j + d\widetilde{Q}. \tag{1.15}$$

Thus, for the form $\omega^{(2)} = d\alpha^{(1)}$ from (1.15) we obtain

$$\omega^{(2)} = \sum_{j=1}^{n} dP_j \wedge dQ_j. \tag{1.16}$$

Because of the nondegeneracy of the symplectic form $\omega^{(2)} \in \Lambda^{(2)}(M)$ from (1.16), we find that all functions $P_j, Q_j \in \mathcal{D}(M)$, $j = 1, \ldots, n$, are functionally independent and obviously

$$\{P_j, Q_k\}_{\omega^{(2)}} = \delta_{jk},\ \{Q_j, Q_k\}_{\omega^{(2)}} = 0$$

for $j, k = 1, \ldots, n$.

The last relation means that the system of functions $\{Q_j \in \mathcal{D}(M) : 1 \le j \le n\}$ is canonically conjugated to the system $\{P_j \in \mathcal{D}(M) : 1 \le j \le n\}$. As the initial dynamical system $\{\varphi^t\}$ is generated by the Hamiltonian $H = P_1$, it follows from equations (1.12) that we have

$$\frac{dQ_j}{dt} = \frac{\partial H}{\partial P_j},\ \frac{dP_j}{dt} = -\frac{\partial H}{\partial Q_j}, \tag{1.17}$$

from which we find $Q_j = \delta_{j1} t + Q_{j0},\ P_j = p_j,\ j = 1, \ldots, n$. Thus, the initial dynamical system has been fully integrated in variables $P_j, Q_j \in \mathcal{D}(M)$, $j = 1, \ldots, n$. $\square$

Now we take a closer look at the invariant manifold M_p when it is compact. Then by virtue of Liouville's theorem $M_p \simeq \mathbb{T}^n$. Let $\sigma_j \in H^1(M_p;\mathbb{Z}), j = 1,\dots,n$, be a set of cycles forming a basis of the one-dimensional homology group $H^1(M_p;\mathbb{Z})$ of the manifold M_p. Consider the integrals:

$$I_j = \frac{1}{2\pi}\oint\limits_{\sigma_j} \alpha^{(1)}, \tag{1.18}$$

$j = 1,\dots,n$, where $d\alpha^{(1)} = \omega^{(2)} \in \Lambda^{(2)}(M)$ is the symplectic structure on M.

As the integrals $I_j, 1 \le j \le n$, are functionally independent, solving the equations $I_j = I_j(P_1, P_2, \dots, P_n), 1 \le j \le n$, with respect to variables $P_j \in \mathcal{D}(M)$, $1 \le j \le n$, we find that the torus M_p is diffeomorphically transformed by mapping $P_j = P_j(I_1, I_2, \dots, I_n)$, $1 \le j \le n$, into the torus M_I corresponding to the set of functionals $I_j \in \mathcal{D}(M)$, $j = 1,\dots,n$.

Let us introduce the following multi-valued mapping:

$$\widetilde{\Phi}(I,Q) = \int\limits_{\sigma(Q,Q^0)} \alpha^{(1)}, \tag{1.19}$$

where $I = (I_1, I_2, \dots, I_n),\ Q = (Q_1, Q_2, \dots, Q_n)$ and $\sigma(Q, Q_0)$ is a smooth path lying on the torus M_I with initial point Q^0 and end point Q.

Using the mapping $\widetilde{\Phi}(I,Q)$ we construct a canonical transformation $\Phi : (I,\varphi) \to (P,Q)$, where

$$\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n),\ P = (P_1, P_2, \dots, P_n),$$

and

$$P_j = \frac{\partial\widetilde{\Phi}}{\partial Q_j},\quad \varphi_j = \frac{\partial\widetilde{\Phi}}{\partial I_j}, \tag{1.20}$$

$1 \le j \le n$, under which the symplectic structure $\omega^{(2)} \in \Lambda^{(2)}(M)$ remains invariant, i.e. $\Phi^*\omega^{(2)} = \omega^{(2)}$.

Using equations (1.19) and (1.20) we estimate variation of the functions $\varphi_j \in \mathcal{D}(M),\ 1 \le j \le n$, when changing the point $Q \in M_I$ along the cycles $\sigma_k \in H^1(M_I;\mathbb{Z}),\ k = 1,\dots,n$.

We have:

$$\oint\limits_{\sigma_k} d\varphi_j = \frac{\partial}{\partial I_j}\oint\limits_{\sigma_k} d\Phi = 2\pi\frac{\partial}{\partial I_j}\oint\limits_{\sigma_k}\alpha = 2\pi\frac{\partial I_k}{\partial I_j} = 2\pi\delta_{kj}, \tag{1.21}$$

$j, k = 1,\dots,n$, i.e. the values $\{\varphi_j : 1 \le j \le n\}$ are the *angle* variables on torus M_I.

By virtue of the transformation $\tilde{\Phi}$ being canonical, the equations for the given dynamical system $\{\varphi^t\}$ can be written in terms of $\{I_j,\ \varphi_j : 1 \le j \le n\}$, according to (1.12), as follows:

$$\frac{dI_j}{dt} = 0, \quad \frac{d\varphi_j}{dt} = \frac{\partial H_1(I)}{\partial I_j}, \tag{1.22}$$

from which we obtain $I_j = I_j(P_1, P_2, \ldots, P_n),\ \varphi_j = \Omega_j t + \varphi_j^0, 1 \le j \le n$, where

$$\Omega_j = \frac{\partial H_1(I)}{\partial I_j}.$$

Taking into consideration the formula (1.21), from (1.22), we find that $Q_j = Q_j(t),\ j = 1, \ldots, n$, are quasi-periodic functions in the variable $t \in \mathbb{R}$ with the set of frequencies $\{\Omega_j : 1 \le j \le n\}$.

The canonical variables $\{I_j,\ \varphi_j :\ 1 \le j \le n\}$ introduced above are called the *action–angle variables.* In these variables the given dynamical system $\{\varphi^t\}$ generated by the vector field $X_H \in \mathcal{F}(M)$ with Hamiltonian function $H = P_1$ can be exactly integrated.

This integration procedure of a Hamiltonian dynamical system by means of canonical transformations is called the *Hamilton–Jacobi technique.*

1.2.3 *Dirac reduction: Symplectic and Poissonian structures on submanifolds*

We now consider some properties of Hamiltonian dynamical systems on the manifolds with phase space restrictions on certain submanifolds. Let $(M, \omega^{(2)})$ be a symplectic space with dynamical system X_H generated by the Hamiltonian function $H \in \mathcal{D}(M)$.

We suppose that on M $2m < 2n = \dim M$ functions $F_j \in \mathcal{D}(M)$, $j = 1, \ldots, 2m$, are determined to be functionally independent on the submanifold $M_F = \{x \in M : F_j(x) = 0,\ 1 \le j \le 2m\}$ which is smoothly imbedded into M. The set of functions $\{F_j \in \mathcal{D}(M) : 1 \le j \le 2m\}$ is called *the restrictions* or *the constraints.*

The construction of the regular Hamiltonian formalism with respect to the submanifold M_F is of great interest. In this connection we consider the restriction of the 2-form $\omega^{(2)} \in \Lambda^{(2)}(M)$ on the submanifold M_F and its properties. Let the mapping $\vartheta : T^*(M) \to T(M)$ be inverse to the mapping $\omega^{(2)} : T(M) \to T^*(M)$, naturally related with the symplectic structure $\omega^{(2)} \in \Lambda^2(M)$. Then, the Hamiltonian vector field $X_H = -\vartheta \operatorname{grad} H$ satisfies the condition $i_{X_H}\omega^{(2)} = -dH$. Correspondingly, the commutator

$\{H, F\}_\omega = \langle \text{grad } H,\ \vartheta \text{ grad } F\rangle$ for any $H, F \in \mathcal{D}(M)$. The following result [173, 326] holds.

Lemma 1.1. *The restriction $\omega_F^{(2)} \in \Lambda^{(2)}(M_F)$ of the symplectic form $\omega^{(2)}$ on the submanifold M_F is nonsingular 2-form if and only if the matrix $(\{F_j, F_k\}_\omega)_{j,k=1,\dots,2m}$ is nonsingular at every point $x \in M_F$.*

Proof. If the matrix $f = (\{F_j, F_k\}_\omega)_{1\le j,k\le 2m}$ is singular at point $x \in M_F$, then there exist numbers $a_j,\ j = 1, \dots, 2m$, not all equal to zero, and

$$\langle \text{ grad } F_j,\ \vartheta \sum_{k=1}^{2m} a_k \text{ grad } F_k \ \rangle = \sum_{k=1}^{2m} a_k \{F_j, F_k\}_\omega = 0 \tag{1.23}$$

for all $j = 1, \dots, 2m$, where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^{2n}$ and ϑ is a nonsingular skew-symmetric square matrix of dimension $2n$ which defines the initial Hamiltonian system by formula $X_H = -\vartheta \text{ grad}H$.

The equation (1.23) means that the vector

$$v = \vartheta \sum_{j=1}^{2m} a_j \text{ grad } F_j \in T_x(M_F),$$

where $T_x(M_F)$ is the tangent to the space M_F at $x \in M_F$. From this, we conclude that for all vectors $u, v \in T_x(M_F)$ the value

$$\omega^{(2)}(u, v) = -\langle u, \vartheta^{-1} v\rangle = -\langle\ u, \sum_{j=1}^{2m} a_j \text{ grad } F_j\ \rangle = 0,$$

i.e. the form $\omega^{(2)} \in \Lambda^{(2)}(M_F)$ is degenerate on M_F.

Conversely, let us suppose that for some nontrivial $v \in T_x(M_F)$ 2-form $\omega^{(2)}(u, v) = -\langle u, \vartheta^{-1} v\rangle = 0$ for all $u \in T_x(M_F)$. Then

$$\vartheta^{-1} v = \sum_{j=1}^{2m} a_j \text{ grad}F_j,$$

where $a_j, 1 \le j \le 2m$, are not all equal to zero. But in this case

$$\left\{F_j, \sum_{k=1}^{2m} a_k F_k\right\}_\omega = \langle \text{ grad}F_j, \vartheta\ \sum_{k=1}^{2m} a_j \text{ grad}F_k\ \rangle = \langle \text{grad } F_j, v\rangle = 0, \tag{1.24}$$

because of $v \in T_x(M_F)$. From here it directly follows that matrix $f = (\{F_j, F_k\}_\omega)_{1\le j,k\le 2m}$ is necessarily singular if $\omega^{(2)}$ is degenerate on M_F. The 2-form $\omega_F^{(2)}$ on M_F is also closed because of $d\omega_F^{(2)} = d\omega^{(2)}|_{M_F} = 0$.

Thus, if matrix $f = (\{F_j, F_k\}_\omega)_{1\le j,k\le 2m}$ is nonsingular, the 2-form $\omega_F^{(2)} \in \Lambda^{(2)}(M_F)$ determines a symplectic structure on M_F. $\square$

Let $\tilde{X}_H \in \mathcal{F}(M_F)$ be the vector field on M_F generated by the Hamiltonian function $H \in \mathcal{D}(M_F) \subset \mathcal{D}(M)$ with respect to the symplectic structure $\omega_F^{(2)}$ on M_F. Obviously, in general $\tilde{X}_H \neq X_H|_{M_F}$. The following lemma [262, 326] holds.

Lemma 1.2. *$\tilde{X}_H = X_H|_{M_F}$ if and only if $\{H, F_j\}_\omega = 0$ on M_F for all $1 \leq j \leq 2m$. Moreover, if $H_0 \in \mathcal{D}(M_F) \subset \mathcal{D}(M)$ and additionally*

$$a_j = \sum_{k=1}^{2m} (f^{-1})_{jk} \{H_0, F_k\}_\omega,$$

$1 \leq j \leq 2m$, then $\tilde{X}_{H_0} = X_H|_{M_F}$, where $H = H_0 + \sum\limits_{j=1}^{2m} a_j F_j$.

Proof. If

$$\{H, F_j\}_\omega = \langle \text{grad } H, \vartheta \text{ grad } F_j \rangle = 0$$

on M_F for all $1 \leq j \leq 2m$, then $X_H = -\vartheta \, \text{grad} H \in T_x(M_F)$ for all $x \in M_F$, so that $\tilde{X}_{H_0} = X_H|_{M_F}$.

Vice versa, if

$$-\vartheta \text{ grad } H = \tilde{X}_{H_0} = X_H|_{M_F} \in T_x(M_F), \; x \in M_F,$$

then

$$\{H, F_j\}_\omega = \langle \text{grad} H, \vartheta \text{ grad} F_j \rangle = 0,$$

$1 \leq j \leq 2m$, for all $x \in M_F$. Thus, the desired result follows from the equalities

$$\tilde{X}_{H_0} = \tilde{X}_H = X_H|_{M_F},$$

since $\{H, F_j\}_\omega = 0, \; j = 0$, and $H = H_0$. □

The Poisson bracket on M_F for functions $\Phi, G \in \mathcal{D}(M_F)$ with respect to the symplectic structure $\omega_F^{(2)}$ on M_F can be calculated using the following Dirac rule [3, 54, 100, 104, 173], given by the following lemma.

Lemma 1.3. *Let $\{\Phi, G\}_{\omega_F}$ be the Poisson bracket of functions $\Phi, G \in \mathcal{D}(M_F)$ with respect to the symplectic structure $\omega_F^{(2)}$ on M_F. Then*

$$\{\Phi, G\}_{\omega_F} = \{\Phi, G\}_{\omega^{(2)}} - \sum_{k,j=1}^{2m} \{\Phi, F_j\}_\omega (f^{-1})_{jk} \{F_k, G\}_\omega \tag{1.25}$$

for all points $x \in M_F$ and the right part in formula (1.25) is calculated for arbitrary smooth extension of functions $\Phi, G \in \mathcal{D}(M_F)$ to functions $\tilde{\Phi}, \tilde{G} \in \mathcal{D}(\tilde{M}_F) \subset \mathcal{D}(M)$ where $\tilde{M}_F$ is an open neighborhood of the submanifold M_F in M.

Proof. Suppose that $\{\tilde{\Phi}, \tilde{F}_j\}_\omega = 0 = \{\tilde{G}, \tilde{F}_j\}_\omega$, $j = 1, \ldots, 2m$. Then by Lemma 1.2

$$\{\Phi, G\}_{\omega_F} = \omega_F^{(2)}(\tilde{X}_\Phi, \tilde{X}_G) = \omega^2(\tilde{X}_\Phi, \tilde{X}_G) = \{\tilde{\Phi}, \tilde{G}\}_\omega.$$

In general,

$$\begin{aligned}
\{\Phi, G\}_{\omega_F} &= \left\{\Phi + \sum_{j,k=1}^{2m} (f^{-1})_{jk}\{\Phi, F_k\}_\omega F_j, G + \sum_{i,l=1}^{2m} (f^{-1})_{il}\{G, F_l\}_\omega F_i\right\}_{\omega_F} \\
&= \left\{\Phi + \sum_{j,k=1}^{2m} (f^{-1})_{jk}\{\Phi, F_k\}_\omega F_j, G + \sum_{i,l=1}^{2m} (f^{-1})_{il}\{G, F_l\}_\omega F_i\right\}_{\omega} \\
&= \{\Phi, G\}_\omega + \sum_{j,k=1}^{2m} (f^{-1})_{jk}\{\Phi, F_k\}_\omega\{F_j, G\}_\omega \\
&\qquad + \sum_{i,l=1}^{2m} (f^{-1})_{il}\{\Phi, F_i\}_\omega\{G, F_l\}_\omega \\
&\qquad + \sum_{i,j,k,l=1}^{2m} (f^{-1})_{jk}(f^{-1})_{il}\{\Phi, F_k\}_\omega\{G, F_l\}_\omega f_{ji} \\
&= \{F, G\}_\omega - \sum_{j,k=1}^{2m} \{\Phi, F_j\}(f^{-1})_{jk}\{F_j, G\}_\omega.
\end{aligned} \tag{1.26}$$

Thus, the proof is complete. □

Upon placing constraints on the submanifold M_F, formula (1.26) turns into formula (1.25) by means of calculating the Poisson bracket for the extended functions $\tilde{F}, \tilde{G} \in \mathcal{D}(\tilde{M}_F)$ with respect to the symplectic structure $\omega^{(2)} \in \Lambda^{(2)}(M)$, and this is easier to use on the whole phase space.

The constraints considered above are classified as *integrable* or *holonomic*. There is another wide class of integrable constraints which also has important applications. We shall deal with holonomic constraints only. As to the theory of non-integrable constraints and their applications, see [212]. Important problems related to studying the topological structure of integral submanifolds of a finite-dimensional Hamiltonian system are considered in general case in [40, 41, 82, 205].

Chapter 2

Geometric and Algebraic Properties of Nonlinear Dynamical Systems with Symmetry: Theory and Applications

2.1 The Poisson structures and Lie group actions on manifolds: Introduction

It has become increasingly clear in recent decades that many dynamical systems of classical physics and mechanics are endowed with symplectic structures [3, 14, 133, 134, 260] and associated Poisson brackets. In many such cases, the structure of the Poisson bracket proves to be canonical and is given on the dual space of the corresponding Lie algebra of symmetries, augmented in some cases with a 2-cocycle, and sometimes having a gauge nature. These observations give rise to a deep group-theoretical interpretation of these Poisson structures for many dynamical systems of mathematical physics, especially for those that are completely integrable.

The investigation of dynamical systems possessing a rich internal symmetry structure is usually carried out in three steps: 1) determining the symplectic structure (the Poisson bracket), and recasting the initial dynamical system into Hamiltonian form; 2) determining conservation laws (invariants or constants of the motion) in involution; 3) determining an additional set of variables and computing their evolution under the action of Hamiltonian flows associated with the invariants.

In many cases the above program is too difficult to realize because of the lack of regular methods for seeking both symplectic structures and a system of the related invariants. Consequently, one seeks additional structures that make the program more tractable; for example, of particular interest are those dynamical systems with a deep intrinsic group nature [60, 216, 250, 295–297, 326, 407] that allows investigation of their symmetry structure in exact form. The corresponding symplectic manifolds on which these systems lie, in general, are pull-backs of the corresponding group actions, related to the coadjont action [162, 164–167, 206, 336, 355] of Lie groups on the dual space $\mathfrak{g}^*$ to its Lie algebra $\mathfrak{g}$, together with the natural Poisson structure upon them. In many cases these spaces carry a principal fiber bundle structure and can be endowed with a variety of connections, which play a very important role in describing the symmetry structure of the related dynamical systems.

2.2 Lie group actions on Poisson manifolds and the orbit structure

Let us recall some definitions. A *Poisson structure* on a smooth manifold M is given by the pair $(M, \{\cdot,\cdot\})$, where

$$\{\cdot,\cdot\}: \quad \mathcal{D}(M) \times \mathcal{D}(M) \to \mathcal{D}(M), \tag{2.1}$$

is a Poisson bracket mapping onto the space of real-valued smooth functions on M, satisfying the conditions: 1) it is bilinear and skew-symmetric; 2) it is a differentiation with respect to each of the arguments; 3) it obeys the Jacobi identity. Any function $H \in \mathcal{D}(\mathcal{M})$ determines the vector field sgrad : H (*symplectic gradient*) for all $f \in \mathcal{D}(M)$ via the formula:

$$\text{sgrad} : H(f) := \{H, f\}. \tag{2.2}$$

The vector field sgrad : $H: \ M \to T(M)$ is called Hamiltonian, with Hamiltonian function $H \in \mathcal{D}(M)$.

A symplectic structure $\omega^{(2)}$ supplies the manifold M with a Poisson bracket in a natural manner. For any function $H \in \mathcal{D}(M)$, the vector field sgrad : $H: \ M \to T(M)$ is defined via the rule:

$$i_{\text{sgrad}:H}\,\omega^{(2)} := -dH, \tag{2.3}$$

whence

$$\{H, f\} := -\omega^{(2)}(\ \text{sgrad} : H,\ \text{sgrad} : f\ ) \tag{2.4}$$

for all $f \in \mathcal{D}(M)$, where $i_{\text{sgrad}:f}\,\omega^{(2)} := -df$.

The Poisson, or its associated Hamiltonian, structure comprises a wider class than the symplectic types. It is not hard to convince ourselves that any Poisson structure $\{\cdot,\cdot\}$ on a manifold M is stratified by symplectic structures.

The next class of Poisson structures is important for applications [3, 9, 141, 338, 381, 385]. Let G be a connected real Lie group, $\mathfrak{g}$ its Lie algebra over the field $\mathbf{R}$, and $\mathfrak{g}^*$ its linear space dual to $\mathfrak{g}$. To each element $x \in \mathfrak{g}^*$ there is associated an endomorphism $ad : x : \ \mathfrak{g} \to \mathfrak{g}$, $ad\ x(y) := [x, y]$, $y \in \mathfrak{g}$, where $[\cdot, \cdot]$ is the Lie structure of the Lie algebra $\mathfrak{g}$. To each element $X \in G$ there is the associated automorphism $Ad\ X :\ \mathfrak{g} \to \mathfrak{g}$ via the rule

$$Ad\ X :\ y \to dl_{X^*} \circ dr_X(y), \tag{2.5}$$

where $y \in \mathfrak{g}$, dl_X and dr_X are the tangent maps for left and right translations on the Lie group G, respectively.

Denote by ad^* and Ad^* adjoint mappings to ad and Ad, respectively, on $\mathfrak{g}$. Then for all $\alpha \in \mathfrak{g}^*$, $x, y \in \mathfrak{g}$ the following identity obtains

$$\langle\ ad^*\ x(\alpha), y\ \rangle := \langle\ \alpha, [x, y]\ \rangle, \tag{2.6}$$

where $\langle ., . \rangle$ is the convolution of $\mathfrak{g}^*$ with $\mathfrak{g}$.

The representation ad^* of the Lie algebra $\mathfrak{g}$ and Ad^* of the Lie group G in the space End $\mathfrak{g}^*$ are called *co-adjoint.*

Let $f \in \mathcal{D}(\mathfrak{g}^*)$; then one can determine the gradient $\nabla f :\ \mathfrak{g}^* \to \mathfrak{g}^*$ via the rule:

$$\langle\ m, \nabla f(\alpha)\ \rangle := df(\alpha; m) := \frac{d}{d\varepsilon} f(\alpha + \varepsilon m)|_{\varepsilon=0}, \tag{2.7}$$

where $(\alpha; m) \in \mathfrak{g}^* \times \mathfrak{g} \cong T(\mathfrak{g}^*)$. The structure of the Poisson bracket on $\mathfrak{g}^*$ is defined as

$$\{f, g\}(\alpha) := \langle\ \alpha,\ [\nabla f(\alpha), \nabla g(\alpha)]\ \rangle \tag{2.8}$$

for any $f, g \in \mathcal{D}(\mathfrak{g}^*)$. A proof that the bracket (2.8) is a Poisson bracket, is given below. The corresponding Hamiltonian vector field for a function $H \in \mathcal{D}(\mathfrak{g}^*)$ takes the form

$$\text{sgrad} : H(\alpha) = (\ \alpha; ad^* : \nabla H(\alpha)(\alpha)\ ) \in T(\mathfrak{g}^*), \tag{2.9}$$

where $\alpha \in \mathfrak{g}^*$ is arbitrary. The vector field sgrad : $H(\alpha)$ is tangent to the orbit $\mathcal{O}_\alpha(G)$ of the Lie group G through an element $\alpha \in \mathfrak{g}$ under the Ad^*-action. These orbits are symplectic strata of the manifold M. For each element $u_\nu := (\ \alpha; ad^*\ x_\nu(\alpha)) \in T_\alpha(O_\alpha) \equiv V_\alpha$, $\nu = 1, 2$, define

$$\omega_\alpha(u_1, u_2) := \langle\ \alpha,\ [x_1, x_2]\ \rangle. \tag{2.10}$$

Obviously ω_α is a symplectic structure on $\mathcal{O}_\alpha$ for all $\alpha \in \mathfrak{g}^*$. Thus, it is clear that the symplectic stratification of the Poisson structure (2.1) is realized by means of $Ad^*\ G$ - the expansion in orbits in the space $\mathfrak{g}^*$. Notice here that each orbit $\mathcal{O}_\alpha(G)$, $\alpha \in \mathfrak{g}^*$, is a uniform symplectic submanifold in $\mathfrak{g}^*$; i.e., the action Ad^* of the Lie group is symplectic and transitive. The restriction of the vector field sgrad : H to an orbit $\mathcal{O}_\alpha$ is defined uniquely via the restriction of the Hamiltonian function $H \in \mathcal{D}(\mathfrak{g}^*)$ to the orbit $\mathcal{O}_\alpha(G)$, $\alpha \in \mathfrak{g}^*$.

2.3 The canonical reduction method on symplectic spaces and related geometric structures on principal fiber bundles

The *canonical reduction method* applied to geometric objects on symplectic manifolds with symmetry turns out to be a very powerful tool; especially for finding the effective phase space variables [3, 216, 251, 326, 338, 385] on integral submanifolds of Hamiltonian dynamical systems on which they are integrable [3, 14, 304] via the Liouville-Arnold theorem, and for investigating related stability problems [14, 176, 177, 370] of Hamiltonian dynamical systems under small perturbations. Let G be a Lie group with the unity element $e \in G$ and with corresponding Lie algebra $\mathfrak{g} \simeq T_e(G)$. Consider a principal fiber bundle $M(N;G)$ with the projection $p : M \to N$, structure group G and base manifold N, on which the Lie group G acts [14, 3, 179, 326] by means of a smooth mapping $\varphi : M \times G \to M$. Namely, for each $g \in G$ there is a group of diffeomorphisms $\varphi_g : M \to M$, generating for any fixed $u \in M$ the induced mapping: $\hat{u} : G \to M$, where

$$\hat{u}(g) = \varphi_g(u). \tag{2.11}$$

On the principal fiber bundle $p : (M, \varphi) \to N$ there is a natural [179, 197, 198, 326] connection $\Gamma(\mathcal{A})$ associated to a morphism $\mathcal{A} : (T(M), \varphi_{g,*}) \to (\mathfrak{g}, Ad_{g^{-1}})$, such that for each $u \in M$ the map $\mathcal{A}(u) : T_u(M) \to \mathfrak{g}$ is a left inverse of the map $\hat{u}_*(e) : \mathfrak{g} \to T_u(M)$, and the map $\mathcal{A}^*(u) : \mathfrak{g}^* \to T_u^*(M)$ is a right inverse of the map $\hat{u}^*(e) : T_u^*(M) \to \mathfrak{g}^*$; that is,

$$\mathcal{A}(u)\hat{u}_*(e) = 1, \quad \hat{u}^*(e)\mathcal{A}^*(u) = 1. \tag{2.12}$$

As usual, let $\varphi_g^* : T^*(M) \to T^*(M)$ be the corresponding lift of the mapping $\varphi_g : M \to M$ at any $g \in G$. If $\alpha^{(1)} \in \Lambda^1(M)$ is the canonical G - invariant 1-form on M, the canonical symplectic structure $\omega^{(2)} \in \Lambda^2(T^*(M))$ given by

$$\omega^{(2)} := d\, pr_M^* \alpha^{(1)} \tag{2.13}$$

generates the corresponding momentum map $l : T^*(M) \to \mathfrak{g}^*$, where

$$l(\alpha^{(1)})(u) = \hat{u}^*(e)\alpha^{(1)}(u) \tag{2.14}$$

for all $u \in M$. Observe that the principal fiber bundle structure $p : M \to N$ subsumes, in part, the exactness of the following two adjoint sequences of mappings:

$$0 \leftarrow \mathfrak{g} \overset{\hat{u}^*(e)}{\longleftarrow} T_u^*(M) \overset{p^*(u)}{\longleftarrow} T_{p(u)}^*(N) \leftarrow 0,$$

$$0 \to \mathfrak{g} \overset{\hat{u}_*(e)}{\to} T_u(M) \overset{p_*(u)}{\to} T_{p(u)}(N) \to 0, \tag{2.15}$$

that is

$$p_*(u)\hat{u}_*(e) = 0, \quad \hat{u}^*(e)p^*(u) = 0 \tag{2.16}$$

for all $u \in M$. Combining (2.16) with (2.12) and (2.14), one obtains the embedding

$$[1 - \mathcal{A}^*(u)\hat{u}^*(e)]\alpha^{(1)}(u) \in range\ p^*(u) \tag{2.17}$$

for the canonical 1-form $\alpha^{(1)} \in \Lambda^1(M)$ at $u \in M$. The expression (2.17) means, of course, that

$$\hat{u}^*(e)[1 - \mathcal{A}^*(u)\hat{u}^*(e)]\alpha^{(1)}(u) = 0 \tag{2.18}$$

for all $u \in M$. Now, taking into account that the map $p^*(u) : T^*_{p(u)}(N) \to T^*_u(M)$ is injective for each $u \in M$, it has the unique inverse mapping $(p^*(u))^{-1}$ on its image $p^*(u)T^*_{p(u)}(N) \subset T^*_u(M)$. Thus, for each $u \in M$ one can define a morphism $p_{\mathcal{A}} : (T^*(M), \varphi^*_g) \to (T^*(N), id)$ as

$$p_{\mathcal{A}}(u) : \alpha^{(1)}(u) \to (p^*(u))^{-1}[1 - \mathcal{A}^*(u)\hat{u}^*(e)]\alpha^{(1)}(u). \tag{2.19}$$

It follows directly from (2.19) that the diagram

$$\begin{array}{ccc} T^*(M) & \overset{p_{\mathcal{A}}}{\to} & T^*(N) \\ pr_M \downarrow & & \downarrow pr_N \\ M & \overset{p}{\to} & N \end{array} \tag{2.20}$$

is commutative.

Let an element $\xi \in \mathfrak{g}^*$ be G-invariant, that is $Ad^*_{g^{-1}}\xi = \xi$ for all $g \in G$. Denote also by $p^\xi_{\mathcal{A}}$ the restriction of the map (2.19) to the subset $\mathcal{M}_\xi := l^{-1}(\xi) \in T^*(M)$, that is the mapping $p^\xi_{\mathcal{A}} : \mathcal{M}_\xi \to T^*(N)$, where for all $u \in M$

$$p^\xi_{\mathcal{A}}(u) : \mathcal{M}_\xi \to (p^*(u))^{-1}[1 - \mathcal{A}^*(u)\hat{u}^*(e)]\mathcal{M}_\xi. \tag{2.21}$$

Now, one can characterize the structure of the reduced phase space $\bar{\mathcal{M}}_\xi := \mathcal{M}_\xi/G$ by means of the following simple result.

Lemma 2.1. *The mapping $p^\xi_{\mathcal{A}}(u) : \mathcal{M}_\xi \to T^*(N)$ is a principal fiber G-bundle with reduced space $\bar{\mathcal{M}}_\xi = \mathcal{M}_\xi/G$.*

Denote by $< \cdot, \cdot >_{\mathfrak{g}}$ the standard Ad-invariant nondegenerate scalar product on $\mathfrak{g}^* \times \mathfrak{g}$. The next result follows directly from Lemma 2.1.

Theorem 2.1. *Given a principal fiber bundle G with a connection $\Gamma(\mathcal{A})$ and a G - invariant element $\xi \in \mathfrak{g}^*$, then every such connection $\Gamma(\mathcal{A})$ is* *a symplectomorphism $\nu_\xi : \bar{\mathcal{M}}_\xi \to T^*(N)$ between the reduced phase space $\bar{\mathcal{M}}_\xi$ and the cotangent bundle $T^*(N)$. Moreover, the following equality*

$$(p_{\mathcal{A}}^{\xi})(d\ pr_N^*\beta^{(1)} + pr_N^*\ \Omega_\xi^{(2)}) = d\ pr_{M_\xi}^*\alpha^{(1)} \tag{2.22}$$

holds for the canonical 1-forms $\beta^{(1)} \in \Lambda^{(1)}(N)$ and $\alpha^{(1)} \in \Lambda^{(1)}(M)$, where $M_\xi := pr_M \mathcal{M}_\xi \subset M$, and the 2-form $\Omega_\xi^{(2)} \in \Lambda^{(2)}(N)$ is the ξ - component of the corresponding curvature 2-form $\Omega^{(2)} \in \Lambda^{(2)}(M) \otimes \mathfrak{g}$.

Proof. In virtue of (2.19), one has on $\mathcal{M}_\xi \subset T^*(M)$ the following equation:

$$p^*(u)p_{\mathcal{A}}^{\xi}(\alpha^{(1)}(u)) := p^*(u)\beta^{(1)}(pr_N(u)) = \alpha^{(1)}(u) - \mathcal{A}^*(u)\hat{u}^*(e)\alpha^{(1)}(u)$$

for any $\beta^{(1)} \in T^*(N), \alpha^{(1)} \in \mathcal{M}_\xi$ and $u \in M_\xi$. Therefore, for such $\alpha^{(1)} \in \mathcal{M}_\xi$ one obtains

$$\alpha^{(1)}(u) = (p_{\mathcal{A}}^{\xi})^{-1}\beta^{(1)}(p_N(u)) = p^*(u)\beta^{(1)})(pr_N(u)) + < \mathcal{A}(u), \xi >_{\mathfrak{g}}$$

for all $u \in M_\xi$. Recall now that owing to (2.20) one obtains for M_ξ and $\mathcal{M}_\xi$ the relationships

$$p \cdot\ pr_{M_\xi} = pr_N \cdot\ p_{\mathcal{A}}^{\xi},\ \ pr_{M_\xi}^* \cdot\ p^* \ = \ (p_{\mathcal{A}}^{\xi})^* \cdot\ pr_N^*\ .$$

Consequently, for any $u \in M$,

$$\begin{aligned} pr_{M_\xi}^*\alpha^{(1)}(u) &= pr_{M_\xi}^* p^*(u)\beta^{(1)}(p(u)) + pr_{M_\xi}^* < \mathcal{A}(u), \xi > \\ &= (p_{\mathcal{A}}^{\xi})^* pr_N^*\beta^{(1)}(u) + pr_{M_\xi}^* < \mathcal{A}(u), \xi >, \end{aligned}$$

so upon taking the exterior derivative, one obtains

$$\begin{aligned} d\ pr_{M_\xi}^*\alpha^{(1)}(u) &= (p_{\mathcal{A}}^{\xi})^* d(pr_N^*\beta^{(1)})(u) + pr_{M_\xi}^* < d\ \mathcal{A}(u), \xi > \\ &= (p_{\mathcal{A}}^{\xi})^* d(pr_N^*\beta^{(1)})(u) + pr_{M_\xi}^* < \Omega^{(2)}(u)), \xi > \\ &= (p_{\mathcal{A}}^{\xi})^* d(pr_N^*\beta^{(1)})(u) + pr_{M_\xi}^* p^* < \Omega^{(2)}, \xi > (u) \\ &= (p_{\mathcal{A}}^{\xi})^* d(pr_N^*\beta^{(1)})(u) + (p_{\mathcal{A}}^{\xi})^* pr_N^* < \Omega^{(2)}, \xi > (u) \\ &= (p_{\mathcal{A}}^{\xi})^* [d(pr_N^*\beta^{(1)})(u) + pr_N^*\Omega_\xi^{(2)}(u)]. \end{aligned}$$

In deriving the above expression, we made use of the following property satisfied by the curvature 2-form $\Omega^{(2)} := d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} \in \Lambda^2(M) \otimes \mathfrak{g}$:

$$\begin{aligned} &< d\mathcal{A}(u), \xi >_{\mathfrak{g}} = < d\mathcal{A}(u) + \mathcal{A}(u) \wedge \mathcal{A}(u), \xi >_{\mathfrak{g}} \\ - &< \mathcal{A}(u) \wedge \mathcal{A}(u), \xi >_{\mathfrak{g}} = < \Omega^{(2)}(u), \xi >_{\mathfrak{g}} \\ &= < \Omega^{(2)}(u), Ad_g^*\xi >_{\mathfrak{g}} = < Ad_g\Omega^{(2)}(u), \xi >_{\mathfrak{g}} \\ &= < \Omega^{(2)}, \xi > (p(u))_{\mathfrak{g}} := p^*\Omega_\xi^{(2)}(u) \end{aligned}$$

at any $u \in M$, since for any $A, B \in \mathfrak{g}$, $< [A, B], \xi >_{\mathfrak{g}} = < B, ad_A^*\xi >_{\mathfrak{g}} = 0$ holds owing to the invariance condition $Ad_g^*\xi = \xi$ for any $g \in G$, which finishes the proof. $\square$

Remark 2.1. As the canonical 2-form $d\ pr_M^*\alpha^{(1)} \in \Lambda^{(2)}(T^*(M))$ is G-invariant on $T^*(M)$ due to its construction, it is evident that its restriction to the G-invariant submanifold $\mathcal{M}_\xi \subset T^*(M)$ is effectively defined only on the reduced space $\bar{\mathcal{M}}_\xi$, which ensures the validity of the equality sign in (2.22).

As a direct consequence of Theorem 2.1, one has the following result, which is useful for applications.

Theorem 2.2. *Let a momentum map value $l(\alpha^{(1)})(u) = \hat{u}^*(e)\alpha^{(1)}(u) = \xi \in \mathfrak{g}^*$ have the isotropy group G_ξ acting naturally on the subset $\mathcal{M}_\xi \subset T^*(M)$ invariantly, freely and properly, so that the reduced phase space $(\bar{\mathcal{M}}_\xi\ , \bar{\omega}_\xi^{(2)})$ is symplectic. By the definition* [14, 338]*, of the natural embedding mapping $\pi_\xi : \mathcal{M}_\xi \to T^*(M)$ and the reduction mapping $r_\xi : \mathcal{M}_\xi \to \bar{\mathcal{M}}_\xi$ the defining equality*

$$r_\xi^*\bar{\omega}_\xi^{(2)} := \pi_\xi^*(d\ pr_M^*\alpha^{(1)}) \tag{2.23}$$

holds on $\mathcal{M}_\xi$. If an associated principal fiber bundle $p : M \to N$ has a structure group G_ξ, then the reduced symplectic space $(\bar{\mathcal{M}}_\xi, \bar{\omega}_\xi^{(2)})$ is symplectomorphic to the cotangent symplectic space $(T^*(N), \sigma_\xi^{(2)})$, where

$$\sigma_\xi^{(2)} = d\ pr_N^*\beta^{(1)} + pr_N^*\Omega_\xi^{(2)}, \tag{2.24}$$

and the corresponding symplectomorphism is given by a relation of the form (2.22).

In the realm of applications, the following criterion can be useful when constructing associated fibre bundles with connections related to the symplectic structure reduced on the space $\bar{\mathcal{M}}_\xi$.

Theorem 2.3. *In order that two symplectic spaces $(\bar{\mathcal{M}}_\xi, \bar{\omega}_\xi^{(2)})$ and $(T^*(N), d\ pr_N^*\beta^{(1)})$ be symplectomorphic, it is necessary and sufficient that the element $\xi \in \ker h$, where for a G-invariant element $\xi \in \mathfrak{g}^*$ the map $h : \xi \to [\Omega_\xi^{(2)}] \in H^2(N;\mathbb{Z})$ with $H^2(N;\mathbb{Z})$ is the cohomology group of 2-forms on the manifold N.*

2.4 The form of reduced symplectic structures on cotangent spaces to Lie group manifolds and associated canonical connections

When one has a Lie group G, the tangent space $T(G)$ can be interpreted as a Lie group $\tilde{G}$ isomorphic to the semidirect product $G \otimes_{Ad} \bar{\mathcal{G}} \simeq \tilde{G}$ of the

Lie group G and its Lie algebra $\bar{\mathcal{G}}$ under the adjoint Ad-action of G on $\bar{\mathcal{G}}$. The Lie algebra $\tilde{\mathcal{G}}$ of $\tilde{G}$ is the semidirect product of $\mathfrak{g}$ with itself, regarded as a trivial abelian Lie algebra, under the adjoint ad-action; therefore, it has the induced bracket defined as $[(a_1, m_1), (a_2, m_2)] := ([a_1, a_2], [a_1, m_2] + [a_2, m_1])$ for all $(a_j, m_j) \in \mathfrak{g} \otimes_{ad} \bar{\mathcal{G}}$, $j = 1, 2$. Take now any element $\xi \in \mathfrak{g}^*$ and compute its isotropy group G_ξ under the co-adjoint action Ad^* of G on $\mathfrak{g}^*$, and let $\mathcal{G}_\xi$ denote its Lie algebra. The cotangent bundle $T^*(G)$ is obviously diffeomorphic to $M := G \times \mathfrak{g}^*$, on which the Lie group G_ξ acts freely and properly (due to its construction) by left translation on the first factor and Ad^*-action on the second. The corresponding momentum map $l : G \times \mathfrak{g}^* \to \mathfrak{g}_\xi^*$ can be obtained as

$$l(h, \alpha) = Ad^*_{h^{-1}} \alpha|_{\mathfrak{g}_\xi^*} \tag{2.25}$$

and has no critical points. Let $\eta \in \mathfrak{g}^*$ and $\eta(\xi) := \eta|_{\mathfrak{g}_\xi^*}$. Then the reduced space $(l^{-1}(\eta(\xi))/G_\xi^{\eta(\xi)}, \bar{\omega}_\xi^{(2)})$ has to be symplectic due to the Marsden-Weinstein reduction theorem [3, 14, 248–250, 338], where $G_\xi^{\eta(\xi)}$ is the isotropy subgroup of the G_ξ-co-adjoint action on $\eta(\xi) \in \mathfrak{g}_\xi^*$, and the condition $r_\xi^* \bar{\omega}_\xi^{(2)} := \pi_\xi^* d\ pr^* \alpha^{(1)}$ naturally induces the symplectic form $\bar{\omega}_\xi^{(2)}$ on the reduced space $\mathcal{M}_\xi := l^{-1}(\eta(\xi))/G_\xi^{\eta(\xi)}$ from the canonical symplectic structure on $T^*(G)$. Define for $\eta(\xi) \in \mathfrak{g}_\xi^*$ the 1-form $\alpha^{(1)}_{\eta(\xi)} \in \Lambda^1(G)$ as

$$\alpha^{(1)}_{\eta(\xi)}(h) := R_h^* \eta(\xi), \tag{2.26}$$

where $R_h : G \to G$ is right translation by an element $h \in G$. It is easy to check that the element (2.26) is right G-invariant and left $G_\xi^{\eta(\xi)}$-invariant, thus inducing a 1-form on the quotient $N_\xi := G/G_\xi^{\eta(\xi)}$. Denote its pullback to $T^*(N_\xi)$ by $pr^*_{N_\xi} \alpha^{(1)}_{\eta(\xi)}$, and form the symplectic manifold $(T^*(N_\xi), \sigma_\xi^{(2)})$, where $\sigma_\xi^{(2)} := (dpr^*_{N_\xi} \beta^{(1)} + dpr^*_{N_\xi} \alpha^{(1)}_{\eta(\xi)})$ and $dpr^*_{N_\xi} \beta^{(1)} \in \Lambda^{(2)}(T^*(N_\xi))$ is the canonical symplectic form. This leads directly to the next result.

Theorem 2.4. *Let $\xi, \eta \in \mathfrak{g}^*$ and $\eta(\xi) := \eta|_{\mathfrak{g}_\xi^*}$ be fixed. Then the reduced symplectic manifold $(\bar{\mathcal{M}}_\xi, \bar{\omega}_\xi^{(2)})$ is a symplectic covering of the co-adjoint orbit $Or(\xi, \eta(\xi); \tilde{G})$, which symplectically embeds into the subbundle over $N_\xi := G/G_\xi^{\eta(\xi)}$ of $(T^*(N_\xi),\ \sigma_\xi^{(2)})$, with $\omega_\xi^{(2)} := d\ \ pr^*_{N_\xi} \beta^{(1)} + dpr^*_{N_\xi} \alpha^{(1)}_{\eta(\xi)} \in \Lambda^2(T^*(N_\xi)$.*

This theorem follows directly from Theorem 2.3 if one defines a connection 1-form $\mathcal{A}(g) : T_g(G) \to \mathfrak{g}_\xi$ as follows:

$$< \mathcal{A}(g), \xi >_{\mathfrak{g}} := R_g^* \eta(\xi) \tag{2.27}$$

for any $g \in G$. The expression (2.27) generates a completely horizontal 2-form $d < \mathcal{A}(g), \xi >_{\mathfrak{g}}$ on the Lie group G, which gives rise immediately to the symplectic structure $\sigma_\xi^{(2)}$ on the phase space $T^*(N_\xi)$, where $N_\xi := G/G_\xi^{\eta(\xi)}$.

2.5 The geometric structure of abelian Yang–Mills type gauge field equations via the reduction method

If one studies the motion of a charged particle under an abelian Yang–Mills gauge, it is convenient [141, 163, 216, 217, 370] to introduce a special fiber bundle structure $p : M \to N$. Namely, one can take a fibered space M such that $M = N \times G \quad N := D \subset \mathbb{R}^n$, with $G := \mathbb{R}/\{0\}$ being the corresponding (abelian) structure Lie group. An analysis similar to that above yields a reduction on the symplectomorphic space $\mathcal{M}_\xi := l^{-1}(\xi)/G \simeq T^*(N)$, $\xi \in \mathfrak{g}$, with symplectic structure $\sigma_\xi^{(2)}(q,p) =< dp, \wedge dq > +d < \mathcal{A}(q,g), \xi >_{\mathfrak{g}}$, where $\mathcal{A}(q,g) := < A(q), dq > +g^{-1}dg$ is the usual connection 1-form on M, with $(q,p) \in T^*(N)$ and $g \in G$. The corresponding canonical Poisson brackets on $T^*(N)$ are easily found to be

$$\{q^i, q^j\}_\xi = 0, \quad \{p_j, q^i\}_\xi = \delta_j^i, \qquad \{p_i, p_j\}_\xi = F_{ji}(q) \tag{2.28}$$

for all $(q,p) \in T^*(N)$ and the curvature tensor is $F_{ij}(q) := \partial A_j/\partial q^i - \partial A_i/\partial q^j$, $1 \le i,j \le n$. If one introduces a new momentum variable $\tilde{p} := p + A(q)$ on $T^*(N) \ni (q,p)$, it is easy to verify that $\sigma_\xi^{(2)} \to \tilde{\sigma}_\xi^{(2)} :=< d\tilde{p}, \wedge dq > \in \Lambda^{(2)}(N)$, which gives rise to the following Poisson brackets [222] on $T^*(N)$:

$$\{q^i, q^j\}_\xi = 0, \qquad \{\tilde{p}_j, q^i\}_\xi = \delta_j^i, \qquad \{\tilde{p}_i, \tilde{p}_j\}_\xi = 0, \tag{2.29}$$

if and only if for all $1 \le i,j,k \le n$ the standard abelian Yang–Mills equations

$$\partial F_{ij}/\partial q_k + \partial F_{jk}/\partial q_i + \partial F_{ki}/\partial q_j = 0 \tag{2.30}$$

hold on N. This construction can be generalized to the case of a non-abelian structure Lie group yielding a description of non-abelian Yang–Mills type field equations covered by the reduction approach described above.

2.6 The geometric structure of non-abelian Yang–Mills gauge field equations via the reduction method

We start by defining a phase space M of a particle under a non-abelian Yang–Mills gauge field given on a region $D \subset \mathbb{R}^3$ as $M := D \times G$, where G

is a (not in general semi-simple) Lie group, acting on M from the right. Over the space M one can define a connection $\Gamma(\mathcal{A})$ by considering a trivial principal fiber bundle $p : M \to N$, where $N := D$, with structure group G. Namely, if $g \in G$, and $q \in N$, then a connection 1-form on $M \ni (q, g)$ can be written [14, 89, 173, 326, 407] as

$$\mathcal{A}(q; g) := g^{-1}(d + \sum_{i=1}^{n} a_i A^{(i)}(q))g, \tag{2.31}$$

where $\{a_i \in \mathfrak{g} : 1 \leq i \leq m\}$ is a basis for the Lie algebra $\mathfrak{g}$ of the Lie group G, and $A_i : D \to \Lambda^1(D)$, $1 \leq i \leq n$, are the Yang–Mills fields on the physical n-dimensional open space $D \subset \mathbb{R}^n$. One can define a natural left invariant Liouville form on M as

$$\alpha^{(1)}(q; g) :=< p, dq > + < y, g^{-1}dg >_{\mathfrak{g}}, \tag{2.32}$$

where $y \in T^*(G)$ and $< \cdot, \cdot >_{\mathfrak{g}}$ denotes, as above, the usual Ad-invariant nondegenerate bilinear form on $\mathfrak{g}^* \times \mathfrak{g}$, as obviously $g^{-1}dg \in \Lambda^1(G) \otimes \mathfrak{g}$. The main assumption we need in what follows is that the connection 1-form should be with the Lie group G action on M. Accordingly the condition

$$R_h^* \mathcal{A}(q; g) = Ad_{h^{-1}} \mathcal{A}(q; g) \tag{2.33}$$

is satisfied for all $(q, g) \in M$ and $h \in G$, where $R_h : G \to G$ is right translation by an element $h \in G$ on the Lie group G.

Now we are ready to apply the reduction Theorem 2.4 to our geometric model. Suppose that the Lie group G canonical action on M is naturally lifted to the cotangent space $T^*(M)$ endowed, owing to (2.32), with the following G-invariant canonical symplectic structure:

$$\begin{aligned} \omega^{(2)}(q, p; g, y) := d\, pr^* \alpha^{(1)}(q, p; g, y) =&< dp, \wedge dq > \\ &+ < dy, \wedge g^{-1}dg >_{\mathfrak{g}} + < ydg^{-1}, \wedge dg >_{\mathfrak{g}} \end{aligned} \tag{2.34}$$

for all $(q, p; g, y) \in T^*(M)$. Take an element $\xi \in \mathfrak{g}^*$ and assume that its isotropy subgroup $G_\xi = G$, that is $Ad_h^* \xi = \xi$ for all $h \in G$. In general, such an element $\xi \in \mathfrak{g}^*$ cannot exist unless it is trivial $\xi = 0$, as it is in the Lie group $G = SL_2(\mathbb{R})$. Then, owing to (2.22), one can construct the reduced phase space $\bar{\mathcal{M}}_\xi := \mathcal{M}_\xi / G$ symplectomorphic to $(T^*(N), \sigma_\xi^{(2)})$, $\mathcal{M}_\xi := l^{-1}(\xi) \in T^*(M)$, where for any $(q, p) \in T^*(N)$

$$\begin{aligned} \sigma_\xi^{(2)}(q, p) &=< dp, \wedge dq > + \Omega_\xi^{(2)}(q) \\ &=< dp, \wedge dq > + \sum_{s=1}^{m} \sum_{i,j=1}^{n} e_s F_{ij}^{(s)}(q) dq^i \wedge dq^j. \end{aligned} \tag{2.35}$$

In the above we expanded the element $\mathfrak{g}^* \ni \xi = \sum_{i=1}^n e_i a^i$ with respect to the bi-orthogonal basis $\{a^i \in \mathfrak{g}^* :< a^i, a_j >_{\mathfrak{g}} = \delta^i_j$, $1 \le i, j \le m$ with $e_i \in \mathbb{R}$, $1 \le i \le n$, just constants, and we denoted by $F^{(s)}_{ij}(q)$, $1 \le i, j \le m$, $s = 1, \ldots, m$, the (reduced on N) components of the curvature 2-form $\Omega^{(2)} \in \Lambda^2(N) \otimes \mathfrak{g}$ as well; that is

$$\Omega^{(2)}(q) := \sum_{s=1}^m \sum_{i,j=1}^n a_s \, F^{(s)}_{ij}(q) dq^i \wedge dq^j \tag{2.36}$$

at any point $q \in N$. Calculations similar to the above lead directly to the following result.

Theorem 2.5. *Suppose that a non-abelian Yang–Mills field (2.31) on the fiber bundle $p : M \to N$ with $M = D \times G$ is invariant with respect to the Lie group G action $G \times M \to M$. Suppose also that an element $\xi \in G^*$ is chosen so that $Ad^*_G \xi = \xi$. Then for the naturally constructed momentum mapping $l : T^*(M) \to G^*$ the reduced phase space $\bar{\mathcal{M}}_\xi := l^{-1}(\xi)/G$, is symplectomorphic to the space $(T^*(N), \sigma^{(2)})$, and is endowed with the symplectic structure (2.35) on $T^*(N)$, having the following componentwise Poisson brackets:*

$$\{p_i, q^j\}_\xi = \delta^j_i, \quad \{q^i, q^j\}_\xi = 0, \quad \{p_i, p_j\}_\xi = \sum_{s=1}^n e_s F^{(s)}_{ji}(q) \tag{2.37}$$

for all $1 \le i, j \le n$ and $(q, p) \in T^*(N)$.

Owing to (2.34), the extended Poisson bracket on the whole cotangent space $T^*(M)$ amounts to the following set of relationships:

$$\{y_s, y_k\}_\xi = \sum_{r=1}^n c^r_{sk} \, y_r, \qquad \{p_i, q^j\}_\xi = \ \delta^j_i \ , \tag{2.38}$$

$$\{y_s, p_j\}_\xi = 0 = \{q^i, q^j\}, \quad \{p_i, p_j\}_\xi = \sum_{s=1}^n y_s \, F^{(s)}_{ji}(q),$$

where $1 \le i, j \le n$, $c^r_{sk} \in \mathbb{R}$, $s, k, r = 1, \ldots, m$, are the structural constants of the Lie algebra $\mathfrak{g}$, and we made use of the expansion $A^{(s)}(q) = \sum_{j=1}^n A^{(s)}_j(q) \, dq^j$, as well as introducing alternative fixed values $e_i := y_i$, $1 \le i \le n$. The result (2.38) can easily be seen by making a shift $\sigma^{(2)} \to \sigma^{(2)}_{ext}$ within the expression (2.34), where $\sigma^{(2)}_{ext} := \sigma^{(2)}\big|_{\mathcal{A}_0 \to \mathcal{A}}$, $\mathcal{A}_0(g) := g^{-1}dg$, $g \in G$. Whence, one concludes, by virtue of the invariance properties of the connection $\Gamma(\mathcal{A})$, that

$$\sigma^{(2)}_{ext}(q, p; u, y) =< dp, \wedge dq > +d < y(g), Ad_{g^{-1}} \mathcal{A}(q; e) >_{\mathfrak{g}}$$

$$=< dp, \wedge dq > + < d\ Ad^*_{g^{-1}} y(g), \wedge \mathcal{A}(q; e) >_{\mathfrak{g}} = < dp, \wedge dq > + \sum_{s=1}^{m} dy_s \wedge du^s$$

$$+ \sum_{j=1}^{n} \sum_{s=1}^{m} A_j^{(s)}(q) dy_s \wedge dq - < Ad^*_{g^{-1}} y(g), \mathcal{A}(q, e) \wedge \mathcal{A}(q, e) >_{\mathfrak{g}}$$

$$+ \sum_{k \geq s=1}^{m} \sum_{l=1}^{m} y_l\ c^l_{sk}\ du^k \wedge du^s + \sum_{s=1}^{n} \sum_{i \geq j=1}^{3} y_s F_{ij}^{(s)}(q) dq^i \wedge dq^j, \tag{2.39}$$

where the coordinates $(q, p; u, y) \in T^*(M)$ are defined as follows: $\mathcal{A}_0(e) := \sum_{s=1}^{m} du^i\ a_i,\ Ad^*_{g^{-1}} y(g) = y(e) := \sum_{s=1}^{m} y_s\ a^s$ for any element $g \in G$. Hence, one immediately obtains the Poisson brackets (2.38) and additional brackets connected with conjugated sets of variables $\{u^s \in \mathbb{R} : 1 \leq s \leq m\}$ $\in \mathfrak{g}^*$ and $\{y_s \in \mathbb{R} : 1 \leq s \leq m\}$ in $\mathfrak{g}$:

$$\{y_s, u^k\}_\xi = \delta^k_s, \quad \{u^k, q^j\}_\xi = 0, \quad \{p_j, u^s\}_\xi = A_j^{(s)}(q), \quad \{u^s, u^k\}_\xi = 0, \tag{2.40}$$

where $1 \leq j \leq n$, $1 \leq k, s \leq m$, and $q \in N$. Note here that the above transition from the symplectic structure $\sigma^{(2)}$ on $T^*(N)$ to its extension $\sigma^{(2)}_{ext}$ on $T^*(M)$ just consists formally in adding an exact part to the symplectic structure $\sigma^{(2)}$, transforming it into its equivalent form.

Now we can immediately infer from expressions (2.39) that an element $\xi := \sum_{s=1}^{m} e_s a^s \in \mathfrak{g}^*$ will be invariant with respect to the Ad^*-action of the Lie group G if and only if

$$\{y_s, y_k\}_\xi|_{y_s = e_s} = \sum_{r=1}^{m} c^r_{sk}\ e_r\ \equiv 0 \tag{2.41}$$

holds identically for all $1 \leq s, k \leq m$, $1 \leq j \leq n$ and $q \in N$. In this and only this case, the reduction scheme elaborated above will go through. Returning our attention to expression (2.40), one easily obtains

$$\omega^{(2)}_{ext}(q, p; u, y) = \omega^{(2)}(q, p + \sum_{s=1}^{n} y_s\ A^{(s)}(q)\ ; u, y), \tag{2.42}$$

on the phase space $T^*(M) \ni (q, p; u, y)$, where for brevity we denoted $< A^{(s)}(q), dq >$ as $\sum_{j=1}^{n} A_j^{(s)}(q)\ dq^j$. A transformation like (2.42) was discussed in a somewhat different context in [222], which also contains a good background for the infinite-dimensional generalization of symplectic structure techniques. Having observed from (2.42) that the simple change of variable

$$\tilde{p} := p + \sum_{s=1}^{m} y_s\ A^{(s)}(q) \tag{2.43}$$

of the cotangent space $T^*(N)$ recasts our symplectic structure (2.39) into the old canonical form (2.34), one finds that the new set of Poisson brackets on $T^*(M) \ni (q, \tilde{p}; u, y)$ satisfies

$$\{y_s, y_k\}_\xi = \sum_{r=1}^{n} c_{sk}^r \; y_r, \qquad \{\tilde{p}_i, \tilde{p}_j\}_\xi = 0, \qquad \{\tilde{p}_i, q^j\} = \delta_i^j, \tag{2.44}$$
$$\{y_s, q^j\}_\xi = 0 \;= \{q^i, q^j\}_\xi, \; \{u^s, u^k\}_\xi = 0, \quad \{y_s, \tilde{p}_j\}_\xi = 0,$$
$$\{u^s, q^i\}_\xi = 0, \qquad \{y_s, u^k\}_\xi = \delta_s^k, \qquad \{u^s, \tilde{p}_j\}_\xi = 0,$$

where $1 \le k, s \le m$ and $1 \le i, j \le n$, if and only if the non-abelian Yang–Mills field equations

$$\partial F_{ij}^{(s)}/\partial q^l + \partial F_{jl}^{(s)}/\partial q^i + \partial F_{li}^{(s)}/\partial q^j \tag{2.45}$$
$$+ \sum_{k,r=1}^{m} c_{kr}^s (F_{ij}^{(k)} A_l^{(r)} + F_{jl}^{(k)} A_i^{(r)} + F_{li}^{(k)} A_j^{(r)}) = 0$$

obtain for all $s = 1, \ldots, m$ and $i, j, l = 1, \ldots, n$ on the base manifold N. This method for complete reduction of gauge Yang–Mills variables using the symplectic structure (2.39) is known in the literature [113, 141, 222] as the *principle of minimal interaction* and is useful for studying different interacting systems as in [251, 335]. In the sequel, we continue our investigation of the geometric properties of reduced symplectic structures connected with infinite-dimensional coupled dynamical systems such as those of Yang–Mills–Vlasov, Yang–Mills–Bogolubov and Yang–Mills–Josephson [251, 335] as well as their relationships with associated principal fiber bundles endowed with canonical connection structures.

2.7 Classical and quantum integrability

2.7.1 *The quantization scheme, observables and Poisson manifolds*

A quantum mechanical system is a triple consisting of an associative algebra with involution $\mathcal{A}$ called the *algebra of observables*, an irreducible $*$-representation π of $\mathcal{A}$ in a Hilbert space $\mathcal{H}$ and a distinguished self-adjoint observable $\hat{H}$ called the Hamiltonian. A typical question involves the description of the spectrum and the eigenvectors of $\pi(\mathcal{H})$. Quantum mechanical systems usually appear with their classical counterparts. A classical mechanical system is again specified by its algebra of observables

$\mathcal{A}_{cl}$ which is a commutative associative algebra equipped with a Poisson bracket $\{\cdot,\cdot\}$ (i.e., a Lie bracket which also satisfies the Leibniz rule

$$\{a, bc\} = \{a, b\}c + \{a, c\}b, \tag{2.46}$$

in other words, a Poisson bracket is a derivation of $\mathcal{A}^{cl}$ with respect to both of its arguments), and a Hamiltonian $H \in \mathcal{A}_{cl}$. A commutative algebra equipped with a Poisson bracket satisfying the Leibniz rule is called a Poisson algebra. Speaking informally, a quantum algebra of observables $\mathcal{A}$ arises as a deformation of the commutative algebra $\mathcal{A}_{cl}$ determined by the Poisson bracket. We shall not be concerned here with the quantization problem [207, 345] in its full generality. However, it will always be instructive to compare quantum systems with their classical counterparts.

The algebraic language that starts with Poisson algebras makes the gap between classical and quantum mechanics as narrow as possible; in practice, however, we also need the dual language based on the notion of the phase space. Roughly speaking, the phase space is the spectrum of the Poisson algebra. The accurate definition depends on the choice of a topology in the Poisson algebra. We shall not attempt to discuss these subtleties and shall always assume that the underlying phase space is a smooth manifold and that the Poisson algebra is realized as the algebra of functions on this manifold. In the examples that we have in mind, Poisson algebras always have an explicit geometric realization of this type. The Poisson bracket itself is then as usual represented by a bivector field on the phase space satisfying certain differential constraints which account for the Jacobi identity. This gives the definition of a Poisson manifold which is dual to the notion of a Poisson algebra. The geometry of Poisson manifolds has numerous obvious parallels with representation theory. Recall that the algebraic version of representation theory is based on the study of appropriate ideals in an associative algebra. For a Poisson algebra we have a natural notion of a Poisson ideal (i.e., a subalgebra which is an ideal with respect to both structures); the dual notion is that of a Poisson submanifold of a Poisson manifold.

The classical counterpart of Hilbert space representations of an associative algebra is the restriction of functions to various Poisson submanifolds. Poisson submanifolds are partially ordered by inclusion; minimal Poisson submanifolds are those for which the induced Poisson structure is nondegenerate. (This means that the center of the Lie algebra of functions contains only constants.) Minimal Poisson submanifolds always carry a symplectic structure and form a stratification of the Poisson manifold; they are called *symplectic leaves*. The restriction of functions to symplectic leaves gives a classical counterpart of the irreducible representations of associative algebras. Let M be a Poisson manifold, $H \in C^\infty(M)$. A classical system $\{M, H\}$ is *integrable* if the commutant of the Hamiltonian H in $\mathcal{A}_{cl}$ contains an abelian algebra of maximal possible rank. (A technical definition is provided by the well-known Liouville theorem.) Let us recall that the key idea that started the modern age in the study of classical integrable systems is how to bring them into Lax form. In the simplest case, the definition of a Lax representation may be given as follows. Let $\{\mathcal{A}; M, H\}$ be a classical mechanical system. Let $F^t : M \to M$, $t \in \mathbb{R}$, be the associated flow on M (defined at least locally). Suppose that $\mathfrak{g}$ is a Lie algebra, $\mathfrak{g}^*$ is its linear dual. A mapping $\ell : M \to \mathfrak{g}^*$ is called a *Lax representation* of $\{\mathcal{A}; M, H\}$, if the following conditions are satisfied:

(i) The flow F^t factorizes over $\mathfrak{g}^*$, i.e., there exists a (local) flow f^t $: \mathfrak{g}^* \to \mathfrak{g}^*$, such that the following diagram is commutative:

$$\begin{array}{ccc} M & \xrightarrow{F^t} & M \\ \ell \downarrow & & \downarrow \ell; \\ \mathfrak{g}^* & \xrightarrow{f^t} & \mathfrak{g}^* \end{array} \tag{2.47}$$

(ii) The quotient flow f^t on $\mathfrak{g}^*$ is isospectral, i.e., it is tangent to the co-adjoint orbits in $\mathfrak{g}^*$.

Remark 2.2. In the applications we have in mind, the Lie algebra $\mathfrak{g}$ is supposed to be endowed with a nondegenerate invariant inner product rendering its adjoint and coadjoint representations identical. In a more general way, we may assume that $\mathfrak{g}$ is arbitrary and replace the target space of the generalized Lax representation with its dual space $\mathfrak{g}^*$.

Consider $\ell(u) \in \mathfrak{g}^*$ for any $u \in M$; hence we may regard ℓ as a matrix with coefficients in $\mathcal{A} = C^\infty(M)$, i.e., as an element of $\mathfrak{g}^* \otimes \mathcal{A}$; the Poisson bracket on $\mathcal{A}$ extends to $\mathfrak{g}^* \otimes \mathcal{A}$ by linearity. Property (ii) means that there exists an element $p(\ell) \in \mathfrak{g} \otimes \mathcal{A}$ such that $\{H, \ell\} = ad^*_{p(\ell)} l$.

Let (Δ, V) be a (finite-dimensional) linear representation of $\mathfrak{g}$. Then $\ell_V = \Delta \otimes id\ (\ell) \in EndV \otimes A$ is a matrix-valued function on M; the coefficients of its characteristic polynomial $P(\mathfrak{g}) = det(\ell_V - \lambda I)$ are integrals of the motion. One may replace in the above definition a Lie algebra $\mathfrak{g}$ with a Lie group G. In that case iso-spectrality means that the flow preserves conjugate classes in G. More generally, the Lax operator may be a difference or a differential operator. The case of difference operator is of particular importance, since the quantization of difference Lax equations is the key element of the quantum inverse scattering transform [215] and a natural source of quantum groups.

Nonetheless, concerning Lax integrability, the following fact holds modulo various approaches based on the Cartan–Picard–Fuchs type separability equations and the gradient-holonomic algorithm to be discussed in later chapters. *There is no general way to find a Lax representation for a given system (even if it is known to be completely integrable).* However, there is a systematic way to produce examples of such representations. An ample source of such examples is provided by the general construction described in the next section.

2.7.2 *The Hopf and quantum algebras*

The basic construction outlined in this section goes back to [121, 373] and [1, 109] in some crucial cases; its relation with the $\mathcal{R}$-matrix method was established by [373]. We shall state it using the language of symmetric algebras, which simplifies its generalization to the quantum case. Let $\mathfrak{g}$ be a Lie algebra over k (where $k = \mathbb{R}$ or $\mathbb{C}$). Let $\mathcal{S}(\mathfrak{g})$ be the symmetric space of polynomial functions on $\mathfrak{g}$. Recall that there is a unique Poisson bracket on $\mathcal{S}(\mathfrak{g})$ (called the Lie–Poisson bracket), which extends the Lie bracket on $\mathfrak{g}$. Namely, the natural pairing $\mathfrak{g}^* \times \mathfrak{g} \to k$ extends to the evaluation map $\mathcal{S}(\mathfrak{g}) \times \mathcal{G}^* \to k$, which induces a canonical isomorphism of $\mathcal{S}(\mathfrak{g})$ with the space of polynomials $\mathcal{P}(\mathfrak{g}^*)$; thus, $\mathfrak{g}^*$ is a linear Poisson manifold and $\mathcal{S}(\mathfrak{g}) \simeq \mathcal{P}(\mathfrak{g}^*)$ is the corresponding algebra of observables. Linear functions on $\mathfrak{g}^*$ form a subspace in $\mathcal{P}(\mathfrak{g}^*)$, which may be identified with $\mathfrak{g}$. The Lie–Poisson bracket on $\mathfrak{g}^*$ is uniquely characterized by the following properties:

1. *The Poisson bracket of linear functions on $\mathfrak{g}^*$ is again a linear function.*

2. *The restriction of the Poisson bracket to $\mathfrak{g}^* \subset \mathcal{P}(\mathfrak{g}^*)$ coincides with the Lie bracket in $\mathfrak{g}$.*

Besides the Lie–Poisson structure on $\mathcal{S}(\mathfrak{g})$, we shall need its Hopf structure [1, 182]. In what follows we shall deal with other more complicated examples of Hopf algebras, so we recall the general definitions. A Hopf algebra is a set $(\mathcal{A}, m, \Delta, \epsilon, \nu)$, consisting of an associative algebra $\mathcal{A}$ over k with multiplication $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and the unit element 1, the coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, the counit $\epsilon : A \to k$ and the antipode $\nu : A \to A$, which satisfy the following axioms

α) mappings $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, $i : k \to A \quad (\lambda \to \lambda \cdot 1)$,

$\epsilon : A \to k$ *are homomorphisms of algebras.*

β) *The following diagrams are commutative:*

$$\begin{array}{ccc} \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} & \xrightarrow{id\otimes m} & \mathcal{A}\otimes\mathcal{A} \\ m\otimes id\downarrow & & \downarrow m \\ \mathcal{A}\otimes\mathcal{A} & \xrightarrow{m} & \mathcal{A} \end{array}, \quad \begin{array}{ccc} \mathcal{A} & \xrightarrow{\Delta} & \mathcal{A}\otimes\mathcal{A} \\ \Delta\downarrow & & \downarrow \Delta\otimes id \\ \mathcal{A}\otimes\mathcal{A} & \xrightarrow{id\otimes\Delta} & \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} \end{array}, \tag{2.48}$$

(these diagrams express the associativity of the product m and the coassociativity of the coproduct Δ, respectively),

$$\begin{array}{ccc} \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} & \xleftarrow{id\otimes i} & \mathcal{A}\otimes\mathcal{A} \\ m\downarrow & & \downarrow m \\ \mathcal{A} & \xleftarrow{id} & \mathcal{A} \end{array}, \quad \begin{array}{ccc} \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} & \xleftarrow{i\otimes id} & k\otimes\mathcal{A} \\ m\downarrow & & \downarrow m \\ \mathcal{A}\otimes\mathcal{A} & \xleftarrow{id} & \mathcal{A} \end{array},$$

$$\begin{array}{ccc} \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} & \xleftarrow{id\otimes i} & \mathcal{A}\otimes\mathcal{A} \\ m\downarrow & & \downarrow m \\ \mathcal{A} & \xleftarrow{id} & \mathcal{A} \end{array}, \quad \begin{array}{ccc} \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} & \xleftarrow{i\otimes id} & k\otimes\mathcal{A} \\ m\downarrow & & \downarrow m \\ \mathcal{A}\otimes\mathcal{A} & \xleftarrow{id} & \mathcal{A} \end{array}, \tag{2.49}$$

(these diagrams express, respectively, the properties of the unit element $1 \in \mathcal{A}$ and of the counit $\epsilon \in \mathcal{A}^*$).

γ) The antipode ν is an antihomomorphism of algebras and the following diagrams are commutative:

$$\begin{array}{c} \mathcal{A} \stackrel{\Delta}{\to} \mathcal{A}\otimes\mathcal{A} \stackrel{id\otimes\nu}{\to} \mathcal{A}\otimes\mathcal{A} \stackrel{m}{\to} \mathcal{A} \\ \epsilon\searrow\ \swarrow\epsilon \\ k \end{array}, \quad \begin{array}{c} \mathcal{A} \stackrel{\Delta}{\to} \mathcal{A}\otimes\mathcal{A} \stackrel{\nu\otimes id}{\to} \mathcal{A}\otimes\mathcal{A} \stackrel{m}{\to} \mathcal{A} \\ \epsilon\searrow\ \swarrow\epsilon \\ k \end{array}. \tag{2.50}$$

If $(\mathcal{A}, m, 1, \Delta, \epsilon, \nu)$ is a Hopf algebra [1, 180], its linear dual $\mathcal{A}^*$ is also a Hopf algebra; moreover, the coupling $< \cdot, \cdot >: \mathcal{A}\otimes\mathcal{A}^* \to k$ interchanges the roles of product and coproduct. Thus we have

$$< m(a\otimes b), \omega > = < a\otimes b, \Delta\omega > .$$

The Hopf structure on the symmetric algebra $\mathcal{S}(\mathfrak{g})$ is determined by the structure of the additive group on $\mathfrak{g}$. Namely, let

$$\mathcal{S}(\mathfrak{g}) \times \mathfrak{g}^* \to k, \qquad (a, X) \to a(X) \tag{2.51}$$

be the evaluation map. Since $\mathcal{S}(\mathfrak{g})\otimes\mathcal{S}(\mathfrak{g}) \simeq S(\mathfrak{g}\oplus\mathfrak{g}) \simeq P(\mathfrak{g}^*\oplus\mathfrak{g}^*)$, this map extends to $\mathcal{S}(\mathfrak{g})\otimes\mathcal{S}(\mathfrak{g})$. We have

$$\Delta a(X, Y) = a(X+Y), \quad \epsilon(a) = a(0), \quad \nu(a)(X) = a(-X) \tag{2.52}$$

(one can denote $\nu(a) = a'$ for brevity). It is important to notice that the Lie–Poisson bracket on $\mathcal{S}(\mathfrak{g})$ is compatible with the Hopf structure:

$$\{\Delta a, \Delta b\} = \Delta\{a, b\}. \tag{2.53}$$

Observe also that the coproducts in the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ and in $\mathcal{S}(\mathfrak{g})$ coincide (in other words, $\mathcal{U}(\mathfrak{g})$ is canonically isomorphic to $\mathcal{S}(\mathfrak{g})$ as a coalgebra).

2.7.3 *Integrable flows related to Hopf algebras and their Poissonian representations*

Consider, as above, a Hopf algebra $\mathcal{A}$ over $\mathbb{C}$ endowed with two special homomorphisms called the co-product $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and co-unit $\varepsilon : A \to \mathbb{C}$, as well as an anti-homomorphism (antipode) $\nu : A \to A$ such that for any $a \in A$

$$\begin{aligned}(id \otimes \Delta)\Delta(a) &= (\Delta \otimes id)\Delta(a), \\ (id \otimes \varepsilon)\Delta(a) &= (\varepsilon \otimes id)\Delta(a) = a, \\ m((id \otimes \nu)\Delta(a)) &= m((\nu \otimes id)\Delta(a)) = \varepsilon(a)I,\end{aligned} \tag{2.54}$$

where $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is the usual multiplication mapping, that is for any $a, b \in \mathcal{A}$ $m(a \otimes b) = ab$. The conditions (2.54) were introduced by Hopf [182] in a cohomological context. Since most Hopf algebras properties depend on the co-product operation $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and related Casimir elements, we shall focus on the objects called co-algebras endowed with this co-product.

The most interesting examples of co-algebras are provided by the universal enveloping algebras $\mathcal{U}(\mathfrak{g})$ of Lie algebras $\mathfrak{g}$. If, for instance, a Lie algebra $\mathfrak{g}$ possesses generators $X_i \in \mathfrak{g}$, $1 \leq i \leq n$, $n = \dim \mathfrak{g}$, the corresponding enveloping algebra $\mathcal{U}(\mathfrak{g})$ can be naturally endowed with a Hopf algebra structure by defining

$$\begin{gathered}\Delta(X_i) = I \otimes X_i + X_i \otimes I, \qquad \Delta(I) = I \otimes I, \\ \varepsilon(X_i) = 0, \quad \nu(I) = I, \quad \varepsilon(I) = 1, \quad \nu(X_i) = -X_i.\end{gathered} \tag{2.55}$$

These mappings acting only on the generators of $\mathfrak{g}$ can be readily extended to any monomial in $\mathcal{U}(\mathfrak{g})$ by means of the homomorphism condition $\Delta(XY) = \Delta(X)\Delta(Y)$ for any $X, Y \in \mathfrak{g} \subset \mathcal{U}(\mathfrak{g})$. In general an element $Y \in \mathcal{U}(\mathfrak{g})$ of a Hopf algebra such that $\Delta(Y) = I \otimes Y + Y \otimes I$ is called *primitive*, and the Friedrichs theorem [1, 307] ensures, that in $\mathcal{U}(\mathfrak{g})$ the only primitive elements are exactly generators $X_i \in \mathfrak{g}$, $1 \leq i \leq n$.

On the other hand, the homomorphism condition for the co-product $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ implies the compatibility of the coproduct with the Lie algebra commutator structure:

$$[\Delta(X_i), \Delta(X_j)]_{\mathcal{A} \otimes \mathcal{A}} = \Delta[X_i, X_j]_{\mathcal{A}} \tag{2.56}$$

for any $X_i, X_j \in \mathfrak{g}$, $1 \leq i, j \leq n$. After Drinfeld [109], the co-algebras defined above are often called *quantum groups* due to their importance [204] in studying many two-dimensional quantum models of modern field theory and statistical physics.

It has also been observed (see for instance [204]) that the standard co-algebra structure (2.55) of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ can be nontrivially extended using some of its infinitesimal deformations preserving the co-associativity (2.56) of the deformed co-product $\Delta : \mathcal{U}_z(\mathfrak{g}) \rightarrow \mathcal{U}_z(\mathfrak{g}) \otimes \mathcal{U}_z(\mathfrak{g})$. Here $\mathcal{U}_z(\mathfrak{g})$ is the corresponding universal enveloping algebra deformation by means of a parameter $z \in \mathbb{C}$, such that $\lim_{z\rightarrow 0}\mathcal{U}_z(\mathfrak{g}) = \mathcal{U}(\mathfrak{g})$ subject to some natural topology on $\mathcal{U}_z(\mathfrak{g})$.

2.7.4 *Casimir elements and their special properties*

Take any *Casimir element* $C \in \mathcal{U}_z(\mathfrak{g})$, that is an element satisfying the condition $[C, \mathcal{U}_z(\mathfrak{g})] = 0$, and consider the action on it of the co-product mapping Δ :

$$\Delta(C) = C(\{\Delta(X)\}), \tag{2.57}$$

where $C := C(\{X\})$ with a set $\{X\} \subset \mathfrak{g}$. It follows trivially that for $\mathcal{A} := \mathcal{U}_z(\mathfrak{g})$

$$[\Delta(C), \Delta(X_i)]_{\mathcal{A}\otimes\mathcal{A}} = \Delta([C, X_i]_{\mathcal{A}}) = 0 \tag{2.58}$$

for any $X_i \in \mathfrak{g}$, $1 \le i \le n$.

Now we inductively define the co-product $\Delta^{(N)} : \mathcal{A} \rightarrow \overset{(N+1)}{\otimes} \mathcal{A}$ for any $N \in \mathbb{Z}_+$, where $\Delta^{(2)} := \Delta$ and $\Delta^{(1)} := id$ and

$$\Delta^{(N)} := ((id\otimes)^{N-2} \otimes \Delta) \cdot \Delta^{(N-1)}, \tag{2.59}$$

or as

$$\Delta^{(N)} := (\Delta \otimes (id\otimes)^{N-2} \otimes id \otimes id) \cdot \Delta^{(N-1)}. \tag{2.60}$$

One can easily verify that

$$\Delta^{(N)} := (\Delta^{(m)} \otimes \Delta^{(N-m)}) \cdot \Delta \tag{2.61}$$

for any $0 \le m \le N$, and the mapping $\Delta^{(N)} : \mathcal{A} \rightarrow \overset{(N+1)}{\otimes} \mathcal{A}$ is an algebra homomorphism; that is,

$$[\Delta^{(N)}(X), \Delta^{(N)}(Y)]_{\overset{(N+1)}{\otimes} \mathcal{A}} = \Delta^{(N)}([X, Y]_{\mathcal{A}}) \tag{2.62}$$

for any $X, Y \in \mathcal{A}$. In particular, if $\mathcal{A} = \mathcal{U}(\mathfrak{g})$, the expression

$$\begin{aligned} \Delta^{(N)}(X) = X(\otimes id)^{N-1} \otimes id + id \otimes X(\otimes id)^{N-1} \otimes id + ... \\ ... + (\otimes id)^{N-1} \otimes id \otimes X \end{aligned} \tag{2.63}$$

holds for any $X \in \mathfrak{g}$.

2.7.5 *Poisson co-algebras and their realizations*

It is well known [304, 326] that a Poisson algebra P is a vector space endowed with a commutative multiplication and a Lie bracket $\{\cdot,\cdot\}$ including a derivation on P in the form

$$\{a, bc\} = b\{a, c\} + \{a, b\}c \tag{2.64}$$

for any a, b and $c \in P$. If P and Q are Poisson algebras one can naturally define the following Poisson structure on $P \otimes Q$:

$$\{a \otimes b, c \otimes d\}_{\mathcal{P}\otimes\mathcal{Q}} = \{a, c\}_{\mathcal{P}} \otimes (bd) + (ac) \otimes \{b, d\}_{\mathcal{Q}} \tag{2.65}$$

for any $a, c \in P$ and $b, d \in Q$. We shall also say that $(P\ ;\Delta)$ is a *Poisson co-algebra* if P is a Poisson algebra and $\Delta : P \to P \otimes P$ is a Poisson algebras homomorphism, that is

$$\{\Delta(a), \Delta(b)\}_{\mathcal{P}\otimes\mathcal{P}} = \Delta(\{a, b\}_{\mathcal{P}}) \tag{2.66}$$

for any $a, b \in P.$

It is useful to note that any Lie algebra $\mathfrak{g}$ generates a Poisson co-algebra $(P; \Delta)$ by defining a Poisson bracket on P via the following: for any $a, b \in P$

$$\{a, b\}_{\mathcal{P}} ;=< grad, \vartheta gradb > . \tag{2.67}$$

Here $P \simeq C^\infty(\mathbb{R}^n; \mathbb{R})$ is a space of smooth mappings linked with a base variables of the Lie algebra $\mathfrak{g}$, $n = \dim \mathfrak{g}$, and the implectic [137, 326] matrix $\vartheta : T^*(P) \to T(P)$ is given as

$$\vartheta(x) = \{\sum_{k=1}^{n} c_{ij}^k x_k : i, j = 1, \ldots, n\}, \tag{2.68}$$

where c_{ij}^k, $1 \le i, j, k \le n$, are the corresponding structural constants of the Lie algebra $\mathfrak{g}$ and $x \in \mathbb{R}^n$ are the corresponding linked coordinates. It is easy to check that the co-product (2.55) is a Poisson algebras homomorphism between P and $P \otimes P$. If one can find a *quantum deformation* $\mathcal{U}_z(\mathfrak{g})$, then the corresponding Poisson co-algebra P_z can be constructed making use of the naturally deformed implectic matrix $\vartheta_z : T^*(P_z) \to T(P_z)$. For instance, if $\mathfrak{g} = so(2, 1)$, there is a deformation $\mathcal{U}_z(so(2, 1))$ defined by the following deformed commutator relations with a parameter $z \in \mathbb{C}$:

$$\begin{aligned} &[\tilde{X}_2, \tilde{X}_1] = \tilde{X}_3, [\tilde{X}_2, \tilde{X}_3] = -\tilde{X}_1, \\ &[\tilde{X}_3, \tilde{X}_1] = \frac{1}{z}\sinh(z\tilde{X}_2), \end{aligned} \tag{2.69}$$

where at $z = 0$ elements $\tilde{X}_i\Big|_{z=0} = X_i \in so(2, 1)$, $i = 1, 2, 3$, comprise a base of generators of the Lie algebra $so(2, 1)$. Then, based on expressions

(2.69), one can easily construct the corresponding Poisson co-algebra P_z, endowed with the implectic matrix

$$\vartheta_z(\tilde{x}) = \begin{pmatrix} 0 & -\tilde{x}_3 & -\frac{1}{z}\sinh(z\tilde{x}_2) \\ \tilde{x}_3 & 0 & -\tilde{x}_1 \\ \frac{1}{z}\sinh(z\tilde{x}_2) & \tilde{x}_1 & 0 \end{pmatrix} \tag{2.70}$$

for any point $\tilde{x} \in \mathbb{R}^3$, linked naturally with the above deformed generators $\tilde{X}_i$, $i = 1, 2, 3$. Since the corresponding co-product on $\mathcal{U}_z(so(2,1))$ acts on this deformed base of generators as

$$\begin{aligned} \Delta(\tilde{X}_2) &= I \otimes \tilde{X}_2 + \tilde{X}_2 \otimes I, \qquad (2.71)\\ \Delta(\tilde{X}_1) &= \exp(-\frac{z}{2}\tilde{X}_2) \otimes \tilde{X}_1 + \tilde{X}_1 \otimes \exp(\frac{z}{2}\tilde{X}_2), \\ \Delta(\tilde{X}_2) &= \exp(-\frac{z}{2}\tilde{X}_2) \otimes \tilde{X}_3 + \tilde{X}_3 \otimes \exp(\frac{z}{2}\tilde{X}_2), \end{aligned}$$

thereby satisfying the main homomorphism property for the whole deformed universal enveloping algebra $\mathcal{U}_z(so(2,1))$.

We now consider a realization [21, 330] of the deformed generators $\tilde{X}_i \in \mathcal{U}_z(\mathfrak{g})$, $1 \le i \le n$, that is a homomorphism mapping $D_z : \mathcal{U}_z(\mathfrak{g}) \to P(M)$, such that

$$D_z(\tilde{X}_i) = \tilde{e}_i, \tag{2.72}$$

$1 \le i \le n$, are elements of a Poisson manifold $P(M)$ realized as a space of functions on a finite-dimensional manifold M, satisfying the deformed commutator relationships

$$\{\tilde{e}_i, \tilde{e}_j\}_{\mathcal{P}(M)} = \vartheta_{z,ij}(\tilde{e}), \tag{2.73}$$

where $[\tilde{X}_i, \tilde{X}_j] = \vartheta_{z,ij}(\tilde{X})$, $1 \le i, j \le n$, generate a Poisson co-algebra structure on the function space $P_z := P_z(\mathfrak{g})$ linked with a given Lie algebra G. Making use of the homomorphism property (2.66) for the co-product mapping $\Delta : P_z(\mathfrak{g}) \to P_z(\mathfrak{g}) \otimes P_z(\mathfrak{g})$, one finds that for all $1 \le i, j \le n$

$$\{\Delta(\tilde{x}_i), \Delta(\tilde{x}_j)\}_{\mathcal{P}_z(\mathcal{G})\otimes\mathcal{P}_z(\mathfrak{g})} = \Delta(\{\tilde{x}_i, \tilde{x}_j\}_{\mathcal{P}_z(\mathfrak{g})} = \vartheta_{z,ij}(\Delta(\tilde{x})) \tag{2.74}$$

and similarly for the corresponding co-product $\Delta : P(M) \to P(M) \otimes P(M)$ one has

$$\{\Delta(\tilde{e}_i), \Delta(\tilde{e}_j)\}_{\mathcal{P}(M)\otimes\mathcal{P}(M)} = \Delta(\{\tilde{e}_i, \tilde{e}_j\}_{\mathcal{P}(M)} = \vartheta_{z,ij}(\Delta(\tilde{e})), \tag{2.75}$$

where $\{\cdot,\cdot\}_{\mathcal{P}(M)}$ is an eventually canonical Poisson structure on the finite-dimensional manifold M.

Let $u \in M$ and consider its coordinates as elements of $P(M)$. Then one can define the following elements

$$u_j := (I\otimes)^{j-1} u(\otimes I)^{N-j} \in \overset{(N)}{\otimes} \mathcal{P}(M), \tag{2.76}$$

where $1 \leq j \leq N$ by means of which one can construct the corresponding N-tuple realization of the Poisson co-algebra structure (2.75) as follows:

$$\{\tilde{e}_i^{(N)}, \tilde{e}_j^{(N)}\}_{\overset{(N)}{\otimes}\mathcal{P}(M)} = \vartheta_{z,ij}(\tilde{e}^{(N)}), \tag{2.77}$$

with $1 \leq i, j \leq n$ and

$$(\overset{(N)}{\otimes} D_z)(\Delta^{(N)}(\tilde{e}_i) := \tilde{e}_i^{(N)}(u_1, u_2, ..., u_N). \tag{2.78}$$

For instance, for the $\mathcal{U}_z(so(2,1))$ case (2.69), one can take [21] the realization Poisson manifold $P(M) = P(R^2)$ with the standard canonical Heisenberg–Weil type Poissonian structure:

$$\{q, q\}_{\mathcal{P}(\mathbb{R}^2)} = 0 = \{p, p\}_{\mathcal{P}(\mathbb{R}^2)}, \qquad \{p, q\}_{\mathcal{P}(\mathbb{R}^2)} = 1, \tag{2.79}$$

where $(q, p) \in \mathbb{R}^2$. Then, expressions (2.78) for $N = 2$ give rise to

$$: \quad \begin{array}{ccc} \tilde{e}_1^{(2)}(q_1, q_2, p_1, p_2) & := & (D_z \otimes D_z)\Delta(\tilde{X}_1) \\ = 2\frac{\sinh(\frac{z}{2}p_1)}{z} \cos q_1 \exp(\frac{z}{2}p_1) & + & 2\exp(-\frac{z}{2}p_1)\frac{\sinh(\frac{z}{2}p_2)}{z} \cos q_2, \\ \tilde{e}_2^{(2)}(q_1, q_2, p_1, p_2) & := & (D_z \otimes D_z)\Delta(\tilde{X}_2) = p_1 + p_2, \\ \tilde{e}_3^{(2)}(q_1, q_2, p_1, p_2) & := & (D_z \otimes D_z)\Delta(\tilde{X}_3) \\ = 2\frac{\sinh(\frac{z}{2}p_1)}{z} \sin q_1 \exp(\frac{z}{2}p_2) & + & 2\exp(-\frac{z}{2}p_1)\frac{\sinh(\frac{z}{2}p_2)}{z} \sin q_2, \end{array} \tag{2.80}$$

where the elements $(q_1, q_2, p_1, p_2) \in \mathbb{R}^2 \otimes \mathbb{R}^2$ satisfy the Heisenberg–Weil commutator relations induced by (2.79):

$$\{q_i, q_j\}_{\mathcal{P}(\mathbb{R}^2)\otimes\mathcal{P}(\mathbb{R}^2)} = 0 = \{p_i, p_j\}_{\mathcal{P}(\mathbb{R}^2)\otimes\mathcal{P}(\mathbb{R}^2)}, \ \{p_i, q_j\}_{\mathcal{P}(\mathbb{R}^2)\otimes\mathcal{P}(\mathbb{R}^2)} = \delta_{ij} \tag{2.81}$$

for any $i, j = 1, 2$.

2.7.6 *Casimir elements and the Heisenberg–Weil algebra related structures*

Consider a Casimir element $\tilde{C} \in \mathcal{U}_z(\mathfrak{g})$ related to an $R \ni z$-deformed Lie algebra G structure of the type

$$[\tilde{X}_i, \tilde{X}_j] = \vartheta_{z,ij}(\tilde{X}), \tag{2.82}$$

where $1 \leq i, j \leq n$, $n = \dim \mathfrak{g}$, and, by definition, $[\tilde{C}, \tilde{X}_i] = 0$. The following general lemma holds.

Lemma 2.2. *Let $(\mathcal{U}_z(\mathfrak{g});\Delta)$ be a co-algebra with generators satisfying (2.82) and $\tilde{C}\in\mathcal{U}_z(\mathfrak{g})$ be its Casimir element; then*

$$[\Delta^{(m)}(\tilde{C}),\Delta^{(N)}(\tilde{X}_i)]_{\overset{(N+1)}{\otimes}\mathcal{U}_z(\mathfrak{g})}=0 \tag{2.83}$$

for any $1\leq i\leq n$ and $1\leq m\leq N$.

As a simple corollary of this lemma, one finds from (2.83) that

$$[\Delta^{(m)}(\tilde{C}),\Delta^{(N)}(\tilde{C})]_{\overset{(N)}{\otimes}\mathcal{U}_z(\mathfrak{g})}=0$$

for any $k,m\in\mathbb{Z}_+$.

Consider now a realization (2.72) of our deformed Poisson co-algebra structure (2.82) and check that the expression

$$[\Delta^{(m)}(C(\tilde{e}),\Delta^{(N)}(\mathcal{H}(\tilde{e}))]_{\overset{(N)}{\otimes}\mathcal{P}(M)}=0 \tag{2.84}$$

too for any $m=1,\ldots,N$, $N\in\mathbb{Z}_+$, if $C(\tilde{e})\in I(P(M))$, that is $\{C(\tilde{e}),q\}_{\mathcal{P}(M)}=0$ for any $u\in M$. Since

$$\mathcal{H}^{(N)}(q):=\Delta^{(N)}(\mathcal{H}(\tilde{e})) \tag{2.85}$$

are in general smooth functions on $\overset{(N)}{\otimes}M$, which can be used as Hamiltonians subject to the Poisson structure on $\overset{(N)}{\otimes}P(M)$; the expressions (2.85) mean that

$$\gamma^{(m)}(u):=\Delta^{(m)}(C(\tilde{e})) \tag{2.86}$$

are their invariants, that is

$$\{\gamma^{(m)}(u),\mathcal{H}^{(N)}(u)\}_{\overset{(N+1)}{\otimes}\mathcal{P}(M)}=0 \tag{2.87}$$

for any $1\leq m\leq N$. Thus, the functions (2.85) and (2.86) generate, under some additional but natural conditions, a hierarchy of Liouville integrable Hamiltonian flows on the Poisson manifold $\overset{(N)}{\otimes}P(M)$.

Suppose now that the Poisson manifold $P(M)$ possesses its co-algebra deformation $P_z(\mathfrak{g})$. Then for any coordinate points $x_i\in P(\mathfrak{g})$, the following relationships

$$\{x_i,x_j\}=\sum_{k=1}^{n}c_{ij}^k x_k:=\vartheta_{ij}(x) \tag{2.88}$$

define a Poisson structure on $P(\mathfrak{g})$, related to the corresponding Lie algebra structure of G, and there exists a representation (2.72), such that the elements $\tilde{e}_i := D_z(\tilde{X}_i) = \tilde{e}_i(x)$ satisfy the relationships $\{\tilde{e}_i, \tilde{e}_j\}_{\mathcal{P}_z(\mathfrak{g})} = \vartheta_{z,ij}(\tilde{e})$ for which the limiting conditions

$$\lim_{z \to 0} \vartheta_{z,ij}(\tilde{e}) = \sum_{k=1}^{n} c_{ij}^k x_k, \quad \lim_{z \to 0} \tilde{e}_i(x) = x_i \tag{2.89}$$

hold for all $1 \leq i, j \leq n$. For instance, take the Poisson co-algebra $P_z(so(2,1))$ for which there is a realization (2.72) in the form

$$\begin{aligned} \tilde{e}_1 &:= D_z(\tilde{X}_1) = \frac{\sinh(\frac{z}{2} x_2)}{z x_2} x_1, \ \tilde{e}_2 := D_z(\tilde{X}_2) = x_2, \\ \tilde{e}_3 &:= D_z(\tilde{X}_3) = \frac{\sinh(\frac{z}{2} x_2)}{z x_2} x_3, \end{aligned} \tag{2.90}$$

where $x_i \in P(so(2,1))$, $i = 1, 2, 3$, satisfy the $so(2,1)$-commutator relationships

$$\begin{aligned} &\{x_2, x_1\}_{\mathcal{P}(so(2,1))} = x_3, \quad \{x_2, x_3\}_{\mathcal{P}(so(2,1))} = -x_1, \\ &\{x_3, x_1\}_{\mathcal{P}(so(2,1))} = x_2, \end{aligned} \tag{2.91}$$

with the co-product operator $\Delta : \mathcal{U}_z(so(2,1)) \to \mathcal{U}_z(so(2,1)) \otimes \mathcal{U}_z(so(2,1))$ given as (2.71). It is easy to check that conditions (2.88) and (2.89) hold.

The next example is related to the co-algebra $\mathcal{U}_z(\pi(1,1))$ of the Poincaré algebra $\pi(1,1)$ for which the following non-deformed relationships

$$[X_1, X_2] = X_3, \ [X_1, X_3] = X_2, \ [X_3, X_2] = 0 \tag{2.92}$$

hold. The corresponding co-product $\Delta : \mathcal{U}_z(\pi(1,1)) \to \mathcal{U}_z(\pi(1,1)) \otimes \mathcal{U}_z(\pi(1,1))$ is given by the Woronowicz [402] expressions

$$\begin{aligned} \Delta(\tilde{X}_1) &= I \otimes \tilde{X}_1 + \tilde{X}_1 \otimes I, \\ \Delta(\tilde{X}_2) &= \exp(-\frac{z}{2} \tilde{X}_1) \otimes \tilde{X}_1 + \tilde{X}_1 \otimes \exp(\frac{z}{2} \tilde{X}_1), \\ \Delta(\tilde{X}_3) &= \exp(-\frac{z}{2} \tilde{X}_1) \otimes \tilde{X}_3 + \tilde{X}_3 \otimes \exp(\frac{z}{2} \tilde{X}_1), \end{aligned} \tag{2.93}$$

where $z \in \mathbb{R}$ is a parameter. Within the deformed expressions (2.93), the elements $\tilde{X}_j \in \mathcal{U}_z(\pi(1,1))$, $j = 1, 2, 3$, still satisfy non-deformed commutator relationships, that is $\vartheta_{z,ij}(\tilde{X}) = \vartheta_{ij}(X)|_{X \Rightarrow \tilde{X}}$ for any $z \in \mathbb{R}$, $i, j = 1, 2, 3$, given by (2.92). Accordingly we find that $\tilde{e}_i := D_z(\tilde{X}_i) = \tilde{e}_i(x) = x_i$, where for $x_i \in P(\pi(1,1))$, $i = 1, 2, 3$, the following Poisson structure

$$\begin{aligned} &\{x_1, x_2\}_{\mathcal{P}(\pi(1,1))} = x_3, \quad \{x_1, x_3\}_{\mathcal{P}(\pi(1,1))} = x_2, \\ &\{x_3, x_2\}_{\mathcal{P}(\pi(1,1))} = 0 \end{aligned} \tag{2.94}$$

holds. Moreover, since $C = x_2^2 - x_3^2 \in I(P(\pi(1,1)))$, that is $\{C, x_i\}_{\mathcal{P}(\pi(1,1))} = 0$ for any $i = 1,2,3$, one can construct using (2.85) and (2.86), integrable Hamiltonian systems on $\overset{(N)}{\otimes} P(\pi(1,1))$. The same can be done subject to the above Poisson co-algebra $P_z(so(2,1))$ realized by means of the Poisson manifold $P(so(2,1))$, by taking into account that $C = x_2^2 - x_1^2 - x_3^2 \in I(P(so(2,1)))$ is a Casimir.

Now we consider a special extended *Heisenberg–Weil co-algebra* $\mathcal{U}_z(h_4)$, also called the oscillator co-algebra. The nondeformed Lie algebra h_4 commutator relationships take the form

$$\begin{aligned} [n, a_+] &= a_+, \quad [n, a_-] = -a_-, \\ [a_-, a_+] &= m, \quad [m, \cdot] = 0, \end{aligned} \tag{2.95}$$

where $\{n, a_\pm, m\} \subset h_4$ comprise a basis of $h_4, \dim h_4 = 4$. The Poisson co-algebra $P(h_4)$ is naturally endowed with the Poisson structure similar to (2.95) and admits its realization (2.72) on the Poisson manifold $P(\mathbb{R}^2)$. Namely, on $P(\mathbb{R}^2)$ one has

$$\begin{aligned} e_\pm &= D(a_\pm) = \sqrt{p}\exp(\mp q), \\ e_1 &= D(m) = 1, \ e_0 = D(n) = p, \end{aligned} \tag{2.96}$$

where $(q,p) \in \mathbb{R}^2$ and the Poisson structure on $P(\mathbb{R}^2)$ is canonical and is the same as (2.79).

Closely related to the relationships (2.95) is a generalized $\mathcal{U}_z(su(2))$ co-algebra, for which

$$\begin{aligned} &[x_3, x_\pm] = \pm x_\pm, \qquad [y_\pm, \cdot] = 0, \\ &[x_+, x_-] = y_+ \sin(2zx_3) + y_- \cos(2zx_3)\frac{1}{\sin z}, \end{aligned} \tag{2.97}$$

where $z \in \mathbb{C}$ is an arbitrary parameter. The co-algebra structure is now given as

$$\begin{aligned} \Delta(x_\pm) &= c^\pm_{1(2)} e^{izx_3} \otimes x_\pm + x_\pm \otimes c^\pm_{2(1)} e^{-izx_3}, \\ \Delta(x_3) &= I \otimes x_3 + x_3 \otimes I, \ \Delta(c_i^\pm) = c_i^\pm \otimes c_i^\pm, \\ \nu(x_\mp) &= -(c^\pm_{1(2)})^{-1} e^{-izx_3} x_\mp e^{izx_3} (c^\pm_{2(1)})^{-1}, \\ \nu(c_i^\pm) &= (c_i^\pm)^{-1}, \ \ \nu(e^{\pm izx_3}) = e^{\mp izx_3} \end{aligned} \tag{2.98}$$

with $c_i^\pm \in \mathcal{U}_z(su(2))$, $i = 1,2$, being fixed elements. One can check that the Poisson structure on $P_z(su(2))$ corresponding to (2.97) can be realized by means of the canonical Poisson structure on the phase space $P(\mathbb{R}^2)$ as follows:

$$[q,p]=i, \qquad D_z(x_3)=q, \qquad D_z(x_\mp)=e^{\pm ip}g_z(q), \tag{2.99}$$
$$g_z(q)=(k+\sin[z(s-q)])(y_+\sin[(q+s+1)]+y_-\cos[z(q+s+1)])^{1/2}\frac{1}{\sin z},$$

where $k,s\in\mathbb{C}$ are constant parameters. Whence, by making use of (2.86) and (2.87) one can construct a new class of Liouville integrable Hamiltonian flows.

2.7.7 *The Heisenberg–Weil co-algebra structure and related integrable flows*

Consider the Heisenberg–Weil algebra commutator equations (2.95) and their related homogeneous quadratic expressions

$$R(x)=\begin{cases} x_1x_2-x_2x_1-\alpha x_3^2=0, \\ x_1x_3-x_3x_1=0, \quad x_2x_3-x_3x_2=0, \end{cases} \tag{2.100}$$

where $\alpha\in\mathbb{C}$, $x_i\in\mathcal{A}$, $i=1,2,3$, are elements of a free associative algebra $\mathcal{A}$. The quadratic algebra $\mathcal{A}/R(x)$ can be deformed via

$$R_z(x)=\begin{cases} x_1x_2-z_1x_2x_1-\alpha x_3^2=0, \\ x_1x_3-z_2x_3x_1=0, \quad x_2x_3-z_2^{-1}x_3x_2=0, \end{cases} \tag{2.101}$$

where $z_1,z_2\in\mathbb{C}\backslash\{0\}$ are parameters.

Let V be the vector space of columns $X:=(x_1,x_2,x_3)^\intercal$ and define the following action

$$h_T: V\to(V\otimes V^*)\otimes V, \tag{2.102}$$

where for any $X\in V$ and $T\in V\otimes V$,

$$h_T(X)=T\otimes X. \tag{2.103}$$

It is easy to check that conditions (2.101) will be satisfied if the following relations [33]

$$\begin{aligned} &T_{11}T_{33}=T_{33}T_{11}, \quad T_{12}T_{33}=z_2^{-2}T_{33}T_{12}, \; T_{21}T_{33}=z_1^2T_{33}T_{21}, \\ &T_{22}T_{33}=T_{33}T_{22}, \quad T_{31}T_{33}=z_2T_{33}T_{31}, \quad T_{32}T_{33}=z_1^{-1}T_{33}T_{32}, \\ &T_{11}T_{12}=z_1T_{12}T_{11}, \; T_{21}T_{22}=z_1T_{22}T_{21}, \; z_2T_{11}T_{32}-z_2T_{32}T_{11} \\ &\qquad =z_1z_2T_{12}T_{31}-T_{31}T_{12}, \; T_{21}T_{32}-z_1z_2T_{32}T_{21} \\ &\qquad =z_1T_{22}T_{31}-z_2T_{31}T_{22}, \quad T_{11}T_{22}-T_{22}T_{11} \\ &\qquad =z_1T_{12}T_{21}-z_1^{-1}T_{21}T_{12}, \quad (T_{11}T_{22}-z_1T_{12}T_{21}) \\ &\qquad =\alpha T_{33}^2-T_{31}T_{32}+z_1T_{32}T_{31} \end{aligned} \tag{2.104}$$

hold. For convenience, we set $z_1 = z_2^2 := z^2 \in \mathbb{C}$ and compute the "quantum determinant" $D(T)$ of the matrix $T : (\mathcal{A}/R_z(x))^3 \to (\mathcal{A}/R_z(x))^3$:

$$D(T) = (T_{11}T_{22} - z^{-2}T_{21}T_{12})T_{33}. \tag{2.105}$$

Note here that the determinant (2.105) is not central, that is

$$\begin{aligned}
&D^{-1}T_{11} = T_{11}D^{-1}, \quad D^{-1}T_{12} = z^{-6}T_{12}D^{-1}, \\
&D^{-1}T_{33} = T_{33}D^{-1}, \quad z^{-6}D^{-1}T_{21} = T_{12}D^{-1}, \\
&D^{-1}T_{22} = T_{22}D^{-1}, \quad z^{-3}D^{-1}T_{31} = T_{31}D^{-1}, \\
&D^{-1}T_{32} = z^{-3}T_{32}D^{-1}.
\end{aligned} \tag{2.106}$$

Taking into account properties (2.104) - (2.106), one can construct the Heisenberg–Weil related co-algebra $\mathcal{U}_z(h)$ as a Hopf algebra with the following coproduct Δ, counit ε and antipode ν :

$$\begin{aligned}
&\Delta(T) := T \otimes T, \quad \Delta(D^{-1}) := D^{-1} \otimes D^{-1}, \\
&\varepsilon(T) := I, \quad \varepsilon(D^{-1}) := I, \quad \nu(T) := T^{-1}, \\
&\nu(D) := D^{-1}.
\end{aligned} \tag{2.107}$$

Using relationships (2.104), one can easily construct the Poisson tensor

$$\{\Delta(\tilde{T}), \Delta(\tilde{T})\}_{\mathcal{P}_z(h)\otimes\mathcal{P}_z(h)} = \Delta(\{\tilde{T}, \tilde{T}\}_{\mathcal{P}_z(h)}) := \vartheta_z(\Delta(\tilde{T})), \tag{2.108}$$

subject to which all of functionals (2.86) are mutually commuting and Casimirs. Choosing appropriate Hamiltonian functions $H^{(N)}(\tilde{T}) := \Delta^{(N)}(H(\tilde{T}))$ for $N \in \mathbb{Z}_+$ yields nontrivial *a priori* integrable Hamiltonian systems. On the other hand, the co-algebra $\mathcal{U}_z(h)$ constructed from (2.106) and (2.107) possesses the following fundamental $\mathcal{R}$-matrix [204] property:

$$\mathcal{R}(z)(T \otimes I)(I \otimes T) = (I \otimes T)(T \otimes I)\mathcal{R}(z) \tag{2.109}$$

for a complex-valued matrix $\mathcal{R}(z) \in Aut(\mathbb{C}^3 \otimes \mathbb{C}^3)$, $z \in \mathbb{C}$. This property is well known [204, 373], and gives rise to a regular procedure for constructing an infinite hierarchy of Liouville integrable operator (quantum) Hamiltonian systems on related quantum Poissonian phase spaces.

This page intentionally left blank

Chapter 3

Integrability by Quadratures of Hamiltonian and Picard–Fuchs Equations: Modern Differential-Geometric Aspects

3.1 Introduction

Integration by quadratures [3, 14, 140, 310, 383] of a differential equation in $\mathbb{R}^n$ is a method of seeking its solutions by means of a finite number of algebraic operations (together with inversion of functions) and *quadratures*, which are calculations of integrals of known functions.

Assume the differential equation is given as a Hamiltonian dynamical system on a symplectic manifold $(M^{2n}, \omega^{(2)})$, $n \in \mathbb{Z}_+$, in the form

$$du/dt = \{H, u\}, \tag{3.1}$$

where $u \in M^{2n}$, $H : M^{2n} \to \mathbb{R}$ is a sufficiently smooth Hamiltonian function [14, 3, 326] with respect to the Poisson bracket $\{\cdot, \cdot\}$ on $\mathcal{D}(M^{2n})$, dual to the symplectic structure $\omega^{(2)} \in \Lambda^2(M^{2n})$, and $t \in \mathbb{R}$ is the evolution parameter. To set the stage for the investigation in this chapter, we first summarize some fundamental results on integrability by quadratures.

More than 150 years ago French mathematicians and physicists, starting with Bour and then Liouville, proved the first integrability by quadratures theorem, which in modern terms [212] can be stated as follows.

Theorem 3.1 (Bour–Liouville). *Let $M^{2n} \simeq T^*(\mathbb{R}^n)$ be a canonically symplectic phase space for a Hamiltonian dynamical system (3.1) with a Hamiltonian function $H : M^{2n} \times \mathbb{R} \to \mathbb{R}$ possessing a Poissonian Lie algebra $\mathfrak{g}$ of $n \in \mathbb{Z}_+$ invariants $H_j : M^{2n} \times \mathbb{R} \to \mathbb{R}$ such that*

$$\{H_i, H_j\} = \sum_{s=1}^{n} c_{ij}^s H_s,$$

and for all $1 \leq i, j, k \leq n$ the c_{ij}^s are real constants on $M^{2n} \times \mathbb{R}$. Suppose further that

$$M_h^{n+1} := \left\{(u,t) \in M^{2n} \times \mathbb{R} : h(H_j) = h_j, 1 \le j \le n, h \in \mathfrak{g}^*\right\},$$

the integral submanifold of the set $\mathfrak{g}$ *of invariants at a regular* $h \in \mathfrak{g}^*$, *is a well-defined connected submanifold of* $M^{2n} \times \mathbb{R}$. *Then if*

i) *all functions of* $\mathfrak{g}$ *are functionally independent on* M_h^{n+1};
ii) $\sum_{s=1}^{n} c_{ij}^s h_s = 0$ *for all* i, j $=1, \ldots, n$;
iii) *the Lie algebra* $g = span_{\mathbb{R}}\{H_j : M^{2n} \times \mathbb{R} \to \mathbb{R} : 1 \le j \le n\}$ *is solvable,*

the system (3.1) on M^{2n} *is integrable by quadratures.*

As a simple corollary of the Bour–Liouville theorem, one has the following result.

Corollary 3.1. *If a Hamiltonian system on* $M^{2n} = T^*(\mathbb{R}^n)$ *has just* n *functionally independent invariants in involution, that is the Lie algebra* $\mathfrak{g}$ *is abelian, then it is integrable by quadratures.*

In the autonomous case when the Hamiltonian $H = H_1$ is a constant of motion, and the invariants $H_j : M^{2n} \to \mathbb{R}$, $j = 1, \ldots, n$, are independent of the evolution parameter $t \in \mathbb{R}$, the involutivity condition $\{H_i, H_j\} = 0$, $1 \le i, j \le n$, can be replaced by the weaker requirement $\{H, H_j\} = c_j H$ for some constants $c_j \in \mathbb{R}$, $j = 1, \ldots, n$.

The first proof of Theorem 3.1 was based on a result of S. Lie that follows.

Theorem 3.2 (Lie). *Let the vector fields* $K_j \in \Gamma(M^{2n})$ *be independent in an open set* $U_h \in M^{2n}$, *generate a solvable Lie algebra* $\mathfrak{g}$ *with respect to the usual commutator* $[\cdot,\cdot]$ *on* $\Gamma(M^{2n})$ *and* $[K_j, K] = c_j K$ *for all* $1 \le j \le n$, *where the* c_j *are real constants. Then the dynamical system*

$$\dot{u} := du/dt = K(u),$$

where $u \in U_h \subset M^{2n}$ *is integrable by quadratures.*

One of the first systems to be shown to be integrable by quadratures is that of the governing equations for three point particles moving along a line in the presence of a uniform potential field.

Theorem 3.3. *The dynamics of three particles on a line* $\mathbb{R}$ *under a uniform potential field is integrable by quadratures.*

The motion of three particles on an axis $\mathbb{R}$ in a uniform potential field $Q(\|\cdot\|)$ is described as a Hamiltonian system on the canonical symplectic phase space $M = T^*(\mathbb{R}^3)$ with the following Lie algebra $\mathfrak{g}$ of invariants on M^{2n}:

$$H = H_1 = \sum_{j=1}^{3} \left(p_j^2/2m_j\right) + \sum_{i<j=1}^{3} Q(\|q_i - q_j\|), \tag{3.2}$$

$$H_2 = \sum_{j=1}^{3} q_j p_j, \qquad\qquad H_3 = \sum_{j=1}^{3} p_j,$$

where $(q_j, p_j) \in T^*(\mathbb{R})$, $j = 1, 2, 3$, are the canonical coordinates and momenta, respectively. The commutation relations for the Lie algebra $\mathfrak{g}$ are

$$\{H_1, H_3\} = 0, \qquad \{H_2, H_3\} = H_3, \quad \{H_1, H_2\} = 2H_1, \tag{3.3}$$

hence it is clearly solvable. Taking a regular element $h \in \mathfrak{g}^*$ such that $h(H_j) = h_j = 0$, for $j = 1$ and 3, and $h(H_2) = h_2 \in \mathbb{R}$ arbitrary, one obtains the integrability in quadratures.

In 1974, V. Arnold proved [14] the following important result known as the commutative (abelian) Liouville–Arnold theorem.

Theorem 3.4 (Liouville–Arnold). *Suppose a set $\mathfrak{g}$ of functions $H_j : M^{2n} \to \mathbb{R}$ on a symplectic manifold M^{2n} is abelian, that is*

$$\{H_i, H_j\} = 0 \tag{3.4}$$

$i, j = 1, \ldots, n$. If on the compact and connected integral submanifold $M_h^n = \{u \in M^{2n} : h(H_j) = h_j \in \mathbb{R}, j = 1, \ldots, n, h \in \mathfrak{g}^*\}$ with $h \in \mathfrak{g}$ regular, all functions $H : M^{2n} \in \mathbb{R},\ 1 \le j \le n$ are functionally independent, then M_h^n is diffeomorphic to the n-dimensional torus $\mathbb{T}^n \simeq M^{2n}$ and the motion on it with respect to the Hamiltonian $H = H_1 \in \mathfrak{g}$ is a quasiperiodic function of the evolution parameter $t \in \mathbb{R}$.

A dynamical system satisfying the hypotheses of Theorem 3.4 is called *completely integrable.*

Mishchenko and Fomenko [259] proved the following generalization of the Liouville–Arnold Theorem 3.4 in 1978.

Theorem 3.5 (Mischenko–Fomenko). *Assume that on a symplectic manifold $(M^{2n}, \omega^{(2)})$ there is a nonabelian Lie algebra $\mathfrak{g}$ of invariants $H_j : M \to \mathbb{R}$, $1 \le j \le k$ with respect to the dual Poisson bracket on M^{2n}, that is*

$$\{H_i, H_j\} = \sum_{s=1}^{k} c_{ij}^s H_s, \tag{3.5}$$

where all values $c_{ij}^s \in \mathbb{R}$, $1 \leq i, j, s \leq k$, *are constants, and the following conditions are satisfied:*

i) *the integral submanifold* $M_h^r := \left\{u \in M^{2n} : h(H)_j = h \in \mathfrak{g}^*\right\}$ *is compact and connected at a regular element* $h \in \mathfrak{g}^*$;

ii) *all functions* $H_j : M^{2n} \to \mathbb{R}$, $1 \leq j \leq k$, *are functionally independent on* M^{2n};

iii) *the Lie algebra* $\mathfrak{g}$ *of invariants satisfies*

$$\dim \mathfrak{g} + \mathrm{rank}\mathfrak{g} = \dim M^{2n}, \tag{3.6}$$

where $\mathrm{rank}\mathfrak{g} = \dim \mathfrak{g}_h$ *is the dimension of the Cartan subalgebra* $\mathfrak{g}_h \subset \mathfrak{g}$. *Then the submanifold* $M_h^r \subset M^{2n}$ *is* $r = \mathrm{rank}\mathfrak{g}$*-dimensional, invariant with respect to each vector field* $K \in \Gamma(M^{2n})$ *generated by an element* $H \in \mathfrak{g}_h$, *and diffeomorphic to the* r*-dimensional torus* $\mathbb{T}^r \simeq M_h^r$, *on which the motion is a quasiperiodic function of the evolution parameter* $t \in \mathbb{R}$.

The simplest proof of the Mishchenko–Fomenko Theorem 3.5 can be obtained from the classical Lie–Cartan theorem [3, 85, 173, 344].

Theorem 3.6 (Lie–Cartan). *Suppose that a point* $h \in \mathfrak{g}^*$ *for a Lie algebra* $\mathfrak{g}$ *of invariants* $H_j : M^{2n} \to \mathbb{R}$, $1 \leq j \leq k$, *is not critical, and the rank of the matrix* $(\{H_i, H_j\} : 1 \leq i, j \leq k) = 2(n - r)$ *is constant in an open neighborhood* $U_h \in \mathbb{R}^n$ *of* $\{h(H_j) = h_j \in \mathbb{R} : j = 1, \ldots, k\} \subset \mathbb{R}^k$. *Then in* $\left(h \circ H^{-1}\right)(U)_h \subset M^{2n}$ *there exist* k *independent functions* $f_s : \mathfrak{g} \to \mathbb{R}$ *such that* $F_s := (f_s \circ H) : M^{2n} \to \mathbb{R}$, $1 \leq s \leq k$, *satisfy*

$$\{F_1, F_2\} = \{F_3, F_4\} = \ldots = \{F_{2(n-r)-1}, F_{2(n-r)}\} = 1, \tag{3.7}$$

with all other brackets $\{F_i, F_j\} = 0$, *when* $(i, j) \neq (2s-1, 2s)$, $1 \leq s \leq n-r$. *In particular,* $(k + r - n)$ *functions* $F_j : M^{2n} \to \mathbb{R}$, $1 \leq j \leq n - r$, *and* $F_s : M^{2n} \to \mathbb{R}$, $1 \leq s \leq k - 2(n - r)$, *comprise an abelian algebra* $\mathfrak{g}_\tau$ *of new invariants on* M^{2n}, *independent on* $(h \circ H)^{-1}(U_h) \subset M^{2n}$.

As a simple corollary of the Lie–Cartan Theorem 3.6, one obtains the following : in the case of the Mishchenko–Fomenko theorem when $\mathrm{rank}\mathfrak{g} + \dim \mathfrak{g} = \dim M^{2n}$, that is $r + k = 2n$, the abelian algebra $\mathfrak{g}_\tau$

(not a subalgebra of $\mathfrak{g}$) of invariants on M^{2n} is just $n = (1/2)\dim M^{2n}$-dimensional, which implies its complete integrability in $(h \circ H)^{-1}(U_h) \subset M^{2n}$ owing to Theorem 3.4. It is also evident that the Mishchenko–Fomenko nonabelian integrability theorem 3.5 reduces to the commutative (abelian) Liouville–Arnold case when a Lie algebra $\mathfrak{g}$ of invariants is abelian, since then $\operatorname{rank}\mathfrak{g} = \dim \mathfrak{g} = (1/2)\dim M^{2n} = n \in \mathbb{Z}_+$ - the standard complete integrability condition.

All the cases of integrability by quadratures described above suggest the following fundamental question: How can algebraic-analytical methods be used to construct the corresponding integral submanifold embedding

$$\pi_h : M_h^r \to M^{2n}, \tag{3.8}$$

where $r = \dim \mathfrak{g} = \operatorname{rank}\mathfrak{g}$, thereby making it possible to express the solutions of an integrable flow on M_h^r as quasiperiodic functions on the torus $\mathbb{T}^r \simeq M_h^r$?

In the remainder of this chapter, we shall describe an algebraic-analytical algorithm for resolving this question for the case when a symplectic manifold M^{2n} is diffeomorphic to the canonical symplectic cotangent phase space $T^*(\mathbb{R}) \simeq M^{2n}$.

3.2 Preliminaries

Our focus will be differential systems of vector fields on the cotangent phase space $M^{2n} = T^*(\mathbb{R}^n)$, endowed with the canonical symplectic structure $\omega^{(2)} \in \Lambda^2(M^{2n})$, where $\omega^{(2)} = d(pr^*\alpha^{(1)})$, and

$$\alpha^{(1)} :=< p, dq >= \sum_{j=1}^{n} p_j dq_j, \tag{3.9}$$

is the canonical 1-form on the base space $\mathbb{R}^n$. If lifted naturally to the space $\Lambda^1(M^{2n})$, $(q,p) \in M^{2n}$ become canonical coordinates on $T^*(\mathbb{R}^n)$, $pr : T^*(\mathbb{R}^n) \to \mathbb{R}^n$ is the canonical projection, and $< \cdot, \cdot >$ is the usual scalar product in $\mathbb{R}^n$.

Assume further that there is a Lie subgroup G (not necessary compact), acting symplectically via the mapping $\varphi : G \times M^{2n} \to M^{2n}$ on M^{2n}, generating a Lie algebra homomorphism $\varphi_* : T(\mathfrak{g}) \to \Gamma(M^{2n})$ from the diagram

$$\begin{array}{ccc} \mathfrak{g}\times\mathfrak{g} \simeq T(\mathfrak{g}) & \stackrel{\varphi_*(u)}{\to} & T(M^{2n}) \\ \downarrow & & \downarrow \\ \mathfrak{g} & \stackrel{\varphi(u)}{\to} & M^{2n} \end{array} \tag{3.10}$$

where $u \in M^{2n}$. Thus, for any $a \in \mathfrak{g}$ we can define a vector field $K_a \in \Gamma(M^{2n})$ as

$$K_a = \varphi_* \cdot a. \tag{3.11}$$

Since the manifold M^{2n} is symplectic, one has for any $a \in \mathfrak{g}$ a function $H_a \in \mathcal{D}(M^{2n})$ as defined by

$$-i_{K_a}\omega^{(2)} = dH_a, \tag{3.12}$$

whose existence follows from the invariance property

$$L_{K_a}\omega^{(2)} = 0 \tag{3.13}$$

for all $a \in \mathfrak{g}$. The following lemma [3] is useful in many applications.

Lemma 3.1. *If the first homology group $H_1(\mathfrak{g};\mathbb{R})$ of the Lie algebra $\mathfrak{g}$ vanishes, then the map $\Phi : \mathfrak{g} \to \mathcal{D}(M^{2n})$ defined as*

$$\Phi(a) := H_a \tag{3.14}$$

*for any $a \in \mathfrak{g}$, is a Lie algebra homomorphism of $\mathfrak{g}$ and $\mathcal{D}(M^{2n})$ endowed with the Lie structure induced by the symplectic structure $\omega^{(2)} \in \Lambda^2(M^{2n})$. In this case $\mathfrak{g}$ is said to be **Poissonian**.*

As the mapping $\Phi : \mathfrak{g} \to \mathcal{D}(M^{2n})$ is linear in $\mathfrak{g}$, the expression (13.29) defines a momentum map $l : M^{2n} \to \mathfrak{g}^*$ as follows: for any $u \in M^{2n}$ and all $a \in \mathfrak{g}$

$$(l(u), a)_{\mathfrak{g}} := H_a(u), \tag{3.15}$$

where $(\cdot,\cdot)_{\mathfrak{g}}$ is the standard scalar product on the dual pair $\mathfrak{g}^* \times \mathfrak{g}$. The following characteristic equivariance [3, 326] result holds.

Lemma 3.2. *The diagram*

$$\begin{array}{ccc} M^{2n} & \xrightarrow{l} & \mathfrak{g}^* \\ \varphi_g \downarrow & & \downarrow Ad^*_{g^{-1}} \\ M^{2n} & \xrightarrow{l} & \mathfrak{g}^* \end{array} \tag{3.16}$$

commutes for all $g \in G$, where $Ad^*_{g^{-1}} : \mathfrak{g}^* \to \mathfrak{g}^*$ is the corresponding co-adjoint action of the Lie group G on the dual space $\mathfrak{g}^*$.

Take now any vector $h \in \mathfrak{g}^*$ and consider a subspace $\mathfrak{g}_h \subset \mathfrak{g}$ consisting of elements $a \in \mathfrak{g}$ such that $ad_a^* h = 0$, where $ad_a^* : \mathfrak{g}^* \to \mathfrak{g}^*$ is the corresponding Lie algebra $\mathfrak{g}$ representation in the dual space $\mathfrak{g}^*$.

The following lemmas can easily be verified.

Lemma 3.3. *The subspace $\mathfrak{g}_h \subset \mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g}$, called a **Cartan subalgebra**.*

Lemma 3.4. *Assume a vector $h \in \mathfrak{g}^*$ is chosen such that $r = \dim \mathfrak{g}_h$ is minimal. Then the Cartan Lie subalgebra $\mathfrak{g}_h \subset \mathfrak{g}$ is abelian.*

In Lemma 3.4 the corresponding element $h \in \mathfrak{g}^*$ is called regular and the number $r = \dim\, \mathfrak{g}_h$ is called the rank$\mathfrak{g}$ of the Lie algebra $\mathfrak{g}$.

Some 20 years ago, Mishchenko and Fomenko [259] proved the following important noncommutative (nonabelian) Liouville–Arnold type theorem.

Theorem 3.7. *On a symplectic space $(M^{2n}, \omega^{(2)})$ suppose $H_j \in \mathcal{D}(M^{2n})$, $j = 1, \ldots, k$, are smooth functions whose linear span over $\mathbb{R}$ comprises a Lie algebra $\mathfrak{g}$ with respect to the corresponding Poisson bracket on M^{2n}. Suppose also that the set*

$$M_h^{2n-k} := \{u \in M^{2n} : h(H_j) = h_j \in \mathbb{R}, \;\; j = 1, \ldots, k, \quad h \in \mathfrak{g}^*\}$$

with $h \in \mathfrak{g}^*$ regular, is a submanifold of M^{2n}, and on M_h^{2n-k} all the functions $H_j \in \mathcal{D}(M^{2n})$, $j = 1, \ldots, k$, are functionally independent. Assume also that the Lie algebra $\mathfrak{g}$ satisfies the following condition:

$$\dim \mathfrak{g} + \mathrm{rank}\mathfrak{g} = \dim M^{2n}. \tag{3.17}$$

Then the submanifold $M_h^r := M_h^{2n-k}$ is rank$\mathfrak{g}$ $= r$*-dimensional and invariant with respect to each vector field $K_{\overline{a}} \in \Gamma(M^{2n})$ with $\overline{a} \in \mathfrak{g}_h \subset \mathfrak{g}$. Given a vector field $K = K_{\overline{a}} \in \Gamma(M^{2n})$ with $\overline{a} \in \mathfrak{g}_h$ or $K \in \Gamma(M^{2n})$ such that $[K, K_a] = 0$ for all $a \in \mathfrak{g}$, then if the submanifold M_h^r is connected and compact, it is diffeomorphic to the r-dimensional torus $\mathbb{T}^r \simeq M_h^r$ and the motion on it induced by the vector field $K \in \Gamma(M^{2n})$ is quasiperiodic.*

The easiest proof of this result employs the [85, 173, 212] classical Lie–Cartan theorem described above. Here we shall only sketch the original Mishchenko–Fomenko proof that is based on symplectic theory techniques, some of which we have already discussed.

Sketch of proof. Define a Lie group G as $G = \exp \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of functions $H_j \in \mathcal{D}(M^{2n})$, $1 \leq j \leq k$, in the theorem, with respect to the Poisson bracket $\{\cdot,\cdot\}$ on M^{2n}. Then, for an element $h \in \mathfrak{g}^*$ and any $a = \sum_{j=1}^{k} c_j H_j \in \mathfrak{g}$, where $c_j \in \mathbb{R}$, $1 \leq j \leq k$, the formula

$$(h, a)_{\mathfrak{g}} := \sum_{j=1}^{k} c_j h(H_j) = \sum_{j=1}^{k} c_j h_j \tag{3.18}$$

holds. Since the set of functions $H_j \in \mathcal{D}(M^{2n})$, $1 \leq j \leq k$, is independent on the level submanifold $M_h^r \subset M^{2n}$, it follows that $h \in \mathfrak{g}^*$ is regular for the Lie algebra $\mathfrak{g}$. Consequently, the Cartan Lie subalgebra $\mathfrak{g}_h \subset \mathfrak{g}$ is abelian. This can be proved by means of simple straightforward calculations. Moreover, the corresponding momentum map $l : M^{2n} \to \mathfrak{g}^*$ is constant on M_h^r and satisfies

$$l(M_h^r) = h \in \mathfrak{g}^*. \tag{3.19}$$

Therefore, all the vector fields $K_{\overline{a}} \in \Gamma(M^{2n})$, $\overline{a} \in \mathfrak{g}_h$, are tangent to the submanifold $M_h^r \subset M^{2n}$. Thus, the corresponding Lie subgroup $G_h := \exp \mathfrak{g}_h$ acts naturally and invariantly on M_h^r. If the submanifold $M_h^r \subset M^{2n}$ is connected and compact, it follows from (3.17) that $\dim M_h^r = \dim M^{2n} - \dim \mathfrak{g} = \operatorname{rank}\mathfrak{g} = r$, and it follows from the Arnold theorem [14, 3, 173] that $M_h^r \simeq \mathbb{T}^r$ and the motion induced of the vector field $K \in \Gamma(M^{2n})$ is a quasiperiodic function of the evolution parameter $t \in \mathbb{R}$, thus proving the theorem. $\square$

As a nontrivial consequence of the Lie–Cartan theorem and Theorem 3.7, one can prove the following dual theorem concerning abelian Liouville–Arnold integrability.

Theorem 3.8. *Let a vector field $K \in \Gamma(M^{2n})$ be completely integrable via the nonabelian scheme of Theorem 3.7. Then it is also Liouville–Arnold integrable on M^{2n} and possesses, under some additional conditions, yet another abelian Lie algebra $\mathfrak{g}_h$ of functionally independent invariants on M^{2n} for which* $\dim \mathfrak{g}_h = n = (1/2) \dim M^{2n}$.

The available proof of the theorem above is quite complicated, and we shall comment on it in detail in the sequel. We mention here only some analogs of the reduction Theorem 3.7 for the case where $M^{2n} \simeq \mathfrak{g}^*$, so that an arbitrary Lie group G acts symplectically on the manifold, as also proved in [153, 216, 250, 326, 370, 381, 407]. Notice here, that if the equality (3.17) is not satisfied, one can, in the usual way, construct the reduced manifold $\overline{M}_h^{2n-k-r} := M_h^{2n-k}/G_h$ on which there exists a symplectic structure $\overline{\omega}_h^{(2)} \in \Lambda^2(\overline{M}_h^{2n-k-r})$, defined as

$$r_h^* \overline{\omega}_h^{(2)} = \pi_h^* \omega^{(2)} \tag{3.20}$$

with respect to the following compatible reduction-embedding diagram:

$$\overline{M}_h^{2n-k-r} \xleftarrow{r_h} M_h^{2n-k} \xrightarrow{\pi_h} M^{2n}, \tag{3.21}$$

where $r_h : M_h^{2n-k} \to \overline{M}_h^{2n-k-r}$ and $\pi_h : M_h^{2n-k} \to M^{2n}$ are, respectively, the corresponding reductions and embedding maps. The nondegeneracy of the 2-form $\overline{\omega}_h^{(2)} \in \Lambda^2(\overline{M}_h)$, defined by (3.20), follows simply from the expression

$$\ker(\pi_h^* \omega^{(2)}(u)) = T_u(M_h^{2n-k}) \cap T_u^{\perp}(M_h^{2n-k}) \tag{3.22}$$

$$= \mathrm{span}_{\mathbb{R}}\{K_{\overline{a}}(u) \in T_u(\overline{M}_h^{2n-k-r}) := M_h^{2n-k}/G_h : \overline{a} \in \mathfrak{g}_h\}$$

for any $u \in M_h^{2n-k}$, since all the vector fields $K_{\overline{a}} \in \Gamma(M^{2n})$, $\overline{a} \in \mathfrak{g}_h$, are tangent to $\overline{M}_h^{2n-k-r} := M_h^{2n-k}/G_h$. Thus, the reduced space is $\overline{M}_h^{2n-k-r} := M_h^{2n-k}/G_h$ with respect to the orbits of the Lie subgroup G_h action on M_h^{2n-k} will be a $(2n-k-r)$-dimensional symplectic manifold. Whence, it is clear that the number $2n-k-r = 2s \in \mathbb{Z}_+$ is even as there is no symplectic structure on odd-dimensional manifolds. This obviously is closely connected with the problem of existence of a symplectic group action of a Lie group G on a given symplectic manifold $(M^{2n}, \omega^{(2)})$ with a symplectic structure $\omega^{(2)} \in \Lambda^{(2)}(M^{2n})$ that is fixed.

From this point of view, one can consider the inverse problem of constructing symplectic structures on a manifold M^{2n} that admits a Lie group G action. Namely, owing to the equivariance property (3.16) of the momentum map $l : M^{2n} \to \mathfrak{g}^*$, one can obtain the induced symplectic structure $l^*\Omega_h^{(2)} \in \Lambda^2(\overline{M}_h^{2n-k-r})$ on $\overline{M}_h^{2n-k-r}$ from the canonical symplectic structure $\Omega_h^{(2)} \in \Lambda^{(2)}(Or(h;G))$ on the orbit $Or(h;G) \subset \mathfrak{g}^*$ of a regular element $h \in \mathfrak{g}^*$. Since the symplectic structure $l^*\Omega_h^{(2)} \in \Lambda^2(\overline{M}_h)$ can be naturally lifted to the 2-form $\widetilde{\omega}^{(2)} = (r_h^* \circ l^*)\Omega_h^{(2)} \in \Lambda^2(M_h^{2n-k})$, which is degenerate, M_h^{2n-k} can be non-uniquely extended on the whole manifold M^{2n} to a symplectic structure $\omega^{(2)} \in \Lambda^2(M^{2n})$, for which the action of the Lie group G is symplectic. Thus, many properties of a given dynamical system with a Lie algebra $\mathfrak{g}$ of invariants on M^{2n} are intrinsically connected with the symplectic structure $\omega^{(2)} \in \Lambda^2(M^{2n})$ with which the manifold M^{2n} is endowed, and in particular, with the corresponding integral submanifold embedding mapping $\pi_h : M_h^{2n-k} \to M^{2n}$ at a regular element $h \in \mathfrak{g}^*$. The problem of direct algebraic-analytical construction of this map is partially solved in [310, 329] when $n = 2$ for an abelian algebra $\mathfrak{g}$ on the manifold $M^4 = T^*(\mathbb{R}^2)$. The treatment of this problem in [329] has been extensively based both on the classical studies of Cartan on integral submanifolds of ideals in Grassmann algebras and on the recent Galissot–Reeb–Francoise results for a symplectic manifold $(M^{2n}, \omega^{(2)})$ structure, on which there exists an involutive set $\mathfrak{g}$ of functionally independent invariants $H_j \in \mathcal{D}(M^{2n})$, $1 \leq j \leq n$. In what follows, we generalize the Galissot–Reeb–Francoise results [135, 136] to the case of a nonabelian set of functionally independent functions $H_j \in \mathcal{D}(M^{2n})$, $1 \leq j \leq k$, comprising a Lie algebra $\mathfrak{g}$ and satisfying the Mishchenko–Fomenko condition (3.17): $\dim \mathfrak{g} + \operatorname{rank} \mathfrak{g} = \dim M^{2n}$. This makes it possible to devise an effective algebraic-analytical method of constructing the corresponding integral submanifold embedding and reduction mappings, giving rise to a wide class of exact, integrable by quadratures solutions of a given integrable vector field on M^{2n}.

3.3 Integral submanifold embedding problem for an abelian Lie algebra of invariants

We shall consider here only a set $\mathfrak{g}$ of commuting polynomial functions $H_j \in \mathcal{D}(M^{2n})$, $1 \leq j \leq n$, on the canonically symplectic phase space $M^{2n} = T^*(\mathbb{R}^n)$. The Liouville–Arnold theorem [14] guarantees that any dynamical system $K \in \Gamma(M^{2n})$ commuting with corresponding Hamiltonian vector fields K_a for all $a \in \mathfrak{g}$, is integrable by quadratures in case of a regular element $h \in \mathfrak{g}^*$, which defines the corresponding integral submanifold $M_h^n := \{u \in M^{2n} : h(H_j) = h_j \in \mathbb{R},\ 1 \leq j \leq n\}$, which is diffeomorphic (when compact and connected) to the n-dimensional torus $\mathbb{T}^n \simeq M_h^n$. This, in particular, means that there exists an algebraic-analytical expression for the integral submanifold embedding mapping $\pi_h : M_h^n \to M^{2n}$ into the ambient phase space M^{2n}, which one needs to find in order to properly demonstrate integrability by quadratures.

The problem formulated above was posed and partially solved (as already mentioned) for $n = 2$ in [329] and in [203] for a Henon–Heiles dynamical system, which had previously been integrated [13] using other tools. Here we generalize the approach of [329] for the general case $n \in \mathbb{Z}_+$ and proceed to solve this problem in the case of a nonabelian Lie algebra $\mathfrak{g}$ of polynomial invariants on $M^{2n} = T^*(\mathbb{R}^n)$, satisfying all the conditions of the Mishchenko–Fomenko Theorem 3.4.

Define now the basic vector fields $K_j \in \Gamma(M^{2n})$, $1 \leq j \leq n$, generated by elements $H_j \in \mathfrak{g}$ of an abelian Lie algebra $\mathfrak{g}$ of invariants on M^{2n}, as follows:

$$-i_{K_j}\omega^{(2)} = dH_j \tag{3.23}$$

for all $j = 1, \dots, n$. It is easy to see that the condition $\{H_j, H_i\} = 0$ for all $1 \le i, j \le n$, yields also $[K_i, K_j] = 0$ for all $1 \le i, j \le n$. Taking into account that $\dim M^{2n} = 2n$ one obtains the equality $(\omega^{(2)})^n = 0$ on M^{2n}. This leads directly to the following result due to Galissot and Reeb.

Theorem 3.9. *Assume that an element $h \in \mathfrak{g}^*$ is regular and a Lie algebra $\mathfrak{g}$ of invariants on M^{2n} is abelian. Then there exist differential 1-forms $h_j^{(1)} \in \Lambda^1(U(M_h^n))$, $1 \le j \le n$, where $U(M_h^n)$ is a neighborhood of the integral submanifold $M_h^n \subset M^{2n}$ satisfying the following properties:*

i) $\omega^{(2)}\big|_{U(M_h^n)} = \sum_{j=1}^n dH_j \wedge h_j^{(1)}$;

ii) the exterior differentials $dh_j^{(1)} \in \Lambda^2(U(M_h^n))$ belong to the ideal $\mathcal{I}(\mathfrak{g})$ in the Grassmann algebra $\Lambda(U(M_h^n))$ generated by 1-forms $dH_j \in \Lambda^1(U(M_h^n))$, $j = 1, \dots, n$.

Proof. Consider the following identity on M^{2n}:

$$(\otimes_{j=1}^n i_{K_j})(\omega^{(2)})^{n+1} = 0 = \pm(n+1)!(\wedge_{j=1}^n dH_j) \wedge \omega^{(2)}, \tag{3.24}$$

which implies that the 2-form $\omega^{(2)} \in \mathcal{I}(\mathfrak{g})$. Whence, one can find 1-forms $h_j^{(1)} \in \Lambda^1(U(M_h^n))$, $j = 1, \dots, n$, satisfying the condition

$$\omega^{(2)}\Big|_{U(M_h^n)} = \sum_{j=1}^n dH_j \wedge h_j^{(1)}. \tag{3.25}$$

Since $\omega^{(2)} \in \Lambda^2(U(M_h^n))$ is nondegenerate on M^{2n}, it follows that all 1-forms $h_j^{(1)}$, $j = 1, \dots, n$, in (3.25) are independent on $U(M_h^n)$, proving part $i)$ of the theorem. As $d\omega^{(2)} = 0$ on M^{2n}, from (3.25) one obtains

$$\sum_{j=1}^n dH_j \wedge dh_j^{(1)} = 0 \tag{3.26}$$

on $U(M_h^n)$, hence it is obvious that $dh_j^{(1)} \in \mathcal{I}(\mathfrak{g}) \subset \Lambda(U(M_h^n))$ for all $j = 1, \dots, n$, proving part $ii)$ of the theorem. □

Now we proceed to study properties of the integral submanifold $M_h^n \subset M^{2n}$ of the ideal $\mathcal{I}(\mathfrak{g})$ in the Grassmann algebra $\Lambda(U(M_h^n))$. In general, the integral submanifold M_h^n is completely determined [173, 344] by the embedding

$$\pi_h : M_h^n \to M^{2n} \tag{3.27}$$

and using this, one can reduce all vector fields $K_j \in \Gamma(M^{2n})$, $1 \leq j \leq n$, on the submanifold $M_h^n \subset M^{2n}$, since they are clearly in its tangent space. If $\overline{K}_j \in \Gamma(M_h^n)$ are the corresponding pullbacks of the vector fields $K_j \in \Gamma(M^{2n})$, then by definition, the equality

$$\pi_{h*} \circ \overline{K}_j = K_j \circ \pi_h \tag{3.28}$$

holds for all $1 \leq j \leq n$. Similarly one can construct 1-forms $\overline{h}_j^{(1)} := \pi_h^* \circ h_j^{(1)} \in \Lambda^1(M_h^n)$, $1 \leq j \leq n$, which are characterized by the following Cartan–Jost [173, 344] theorem.

Theorem 3.10. *The following properties obtain:*

i) the 1-forms $\overline{h}_j^{(1)} \in \Lambda^1(M_h^n)$, $j = 1, \ldots, n$, are independent on M_h^n;

ii) the 1-forms $\overline{h}_j^{(1)} \in \Lambda^1(M_h^n)$, $j = 1, \ldots, n$, are exact on M_h^n and satisfy $\overline{h}_j^{(1)}(\overline{K}_j) = \delta_{ij}$, $1 \leq i, j \leq n$.

Proof. As the ideal $\mathcal{I}(\mathfrak{g})$ vanishes on $M_h^n \subset M^{2n}$ and is closed on $U(M_h^n)$, the integral submanifold M_h^n is well-defined in the case of a regular element $h \in \mathfrak{g}^*$. This implies that the embedding (3.27) is nondegenerate on $M_h^n \subset M^{2n}$, or the 1-forms $\overline{h}_j^{(1)} := \pi_h^* \circ h_j^{(1)}$, $1 \leq j \leq n$, persist in being independent if they are 1-forms $h_j^{(1)} \in \Lambda^1(U(M_h^n))$, $1 \leq j \leq n$, proving part i) of the theorem. Using property ii) of Theorem 3.9, one sees that on the integral submanifold $M_h^n \subset M^{2n}$, all 2-forms $d\overline{h}_j^{(1)} = 0$, $1 \leq j \leq n$. Consequently, owing to the Poincaré lemma [144, 173], the 1-forms $\overline{h}_j^{(1)} = d\overline{t}_j \in \Lambda^1(M_h^n)$ for some maps $\overline{t}_j : M_h^n \to \mathbb{R}$, $1 \leq j \leq n$, defining global coordinates on a universal covering of M_h^n. Consider now the following identity based on the representation (3.25):

$$i_{K_j}\, \omega^{(2)}\Big|_{U(M_h^n)} = -\sum_{i=1}^{n} h_i^{(1)}(K_j) \quad dH_i := -dH_j, \tag{3.29}$$

which holds for all $1 \leq j \leq n$. As all $dH_j \in \Lambda^1(U(M_h^n))$, are independent, from (3.29) one infers that $h_i^{(1)}(K_j) = \delta_{ij}$ for all $1 \leq i, j \leq n$. Recalling now that $K_i \circ \pi_h = \pi_{h*} \circ K_i$ for every $1 \leq i \leq n$, one readily computes that $\overline{h}_i^{(1)}(\overline{K}_j) = \pi_h^* h_i^{(1)}(\overline{K}_j) := h_i^{(1)}(\pi_{h*} \circ K_j) := h_i^{(1)}(K_j \circ \pi_h) = \delta_{ij}$ for all $1 \leq i, j \leq n$, which proves ii). $\square$

The following is a simple consequence of Theorem 3.10:

Corollary 3.2. *Suppose that the vector fields $K_j \in \Gamma(M^{2n})$, $1 \leq j \leq n$, are parametrized globally along their trajectories by corresponding parameters $t_j : M^{2n} \to \mathbb{R}$; that is, on the phase space M^{2n}*

$$d/dt_j := K_j \tag{3.30}$$

for all $1 \leq j \leq n$. Then the following equalities hold (up to constant normalizations) on the integral submanifold $M_h^n \subset M^{2n}$:

$$t_j|_{M_h^n} = \bar{t}_j, \tag{3.31}$$

for $1 \leq j \leq n$.

Consider a Liouville–Arnold integrable Hamiltonian system on the cotangent canonically symplectic manifold $(T^*(\mathbb{R}^n), \omega^{(2)})$, $n \in \mathbb{Z}_+$, possessing exactly n functionally independent and Poisson commuting algebraic polynomial invariants $H_j : T^*(\mathbb{R}^n) \to \mathbb{R}$, $1 \leq j \leq n$. Due to the Liouville–Arnold theorem, this Hamiltonian system can be completely integrated by quadratures in quasiperiodic functions on its integral submanifold when taken to be compact. Equivalently, this compact integral submanifold is diffeomorphic to a torus $\mathbb{T}^n$, which makes it possible to formulate the problem of integrating the system by quadratures in terms of the corresponding integral submanifold embedding $\pi_h : M_h^n \longrightarrow T^*(\mathbb{R}^n)$, where

$$M_h^n := \{(q,p) \in T^*(\mathbb{R}^n) : H_j(q,p) = h_j \in \mathbb{R},\ 1 \leq j \leq n\}. \tag{3.32}$$

Since $M_h^n \simeq \mathbb{T}^n$, and the integral submanifold (3.32) is invariant for all Hamiltonian flows $K_j : T^*(\mathbb{R}^n) \to T(T^*(\mathbb{R}^n))$, $1 \leq j \leq n$, where

$$i_{K_j}\omega^{(2)} = -dH_j, \tag{3.33}$$

and there exist corresponding "action-angle"- coordinates $(\varphi, \gamma) \in (\mathbb{T}_\gamma^n, \mathbb{R}^n)$ on the torus $\mathbb{T}_\gamma^n \backsimeq M_h^n$, specifying the embedding $\pi_\gamma : \mathbb{T}_\gamma^n \to T^*(\mathbb{R}^n)$ by means of a set of smooth functions $\gamma \in \mathcal{D}(\mathbb{R}^n)$, where

$$\mathbb{T}_\gamma^n := \{(q,p) \in T^*(\mathbb{R}^n) : \quad \gamma_j(H) = \gamma_j \in \mathbb{R},\ 1 \leq j \leq n\}. \tag{3.34}$$

The map induced by (3.34), $\gamma : \mathbb{R}^n \ni h \to \mathbb{R}^n$, has many applications and was studied by Picard and Fuchs subject to the corresponding differential equations it satisfies:

$$\partial\gamma_j(h)/\partial h_i = F_{ij}(\gamma; h), \tag{3.35}$$

where $h \in \mathbb{R}^n$ and $F_{ij} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $1 \leq i,j \leq n$, are almost everywhere smooth functions. When the right-hand side of (3.35) is a set of algebraic functions on $\mathbb{C}^n \times \mathbb{C}^n \ni (\gamma; h)$, all Hamiltonian flows

$K_j : T^*(\mathbb{R}^n) \to T(T^*(\mathbb{R}^n))$, $1 \le j \le n$, are said to be *algebraically completely integrable in quadratures.* In general, equations like (3.35) were studied in [93, 181]; a recent example can be found in [135, 136]. It is clear that Picard–Fuchs equations (3.35) are related to the associated canonical transformation of the symplectic 2-form $\omega^{(2)} \in \Lambda^2(T^*(\mathbb{R}^n))$ in a neighborhood $U(M_h^n)$ of the integral submanifold $M_h^n \subset T^*(\mathbb{R}^n)$. More precisely, set $\omega^{(2)}(q,p) = dpr^*\alpha^{(1)}(q;p)$, where for $(q,p) \in T^*(\mathbb{R}^n)$

$$\alpha^{(1)}(q;p) := \sum_{j=1}^{n} p_j dq_j =< p, dq >\in \Lambda^1(\mathbb{R}^n) \tag{3.36}$$

is the canonical Liouville 1-form on $\mathbb{R}^n$, $< \cdot, \cdot >$ is the usual scalar product in $\mathbb{R}^n$, $pr : T^*(\mathbb{R}^n) \to \mathbb{R}^n$ is the bundle projection. One can now define a mapping

$$dS_q :\ \mathbb{R}^n \to T_q^*(\mathbb{R}^n), \tag{3.37}$$

such that $dS_q(h) \in T_q^*(\mathbb{R}^n)$ is an exact 1-form for all $q \in M_h^n$ and $h \in \mathbb{R}^n$, yielding

$$(dS_q)^*(dpr^*\alpha^{(1)}) = (dS_q)^*\omega^{(2)} := d^2S_q \equiv 0. \tag{3.38}$$

Thus the mapping (3.37) is a *generating function* [3, 14, 132, 173] $S_q : \mathbb{R}^n \to \mathbb{R}$, satisfying

$$pr^*\alpha^{(1)}(q;p)+ < t, dh >= dS_q(h), \tag{3.39}$$

where $t \in \mathbb{R}^n$ is the set of evolution parameters. It follows immediately from (3.39) that

$$S_q(h) = \int_{q^{(0)}}^{q} < p, dq > \Big|_{M_h^n} \tag{3.40}$$

holds for any $q, q^{(0)} \in M_h^n$. One can define another generating function $S_\mu : \mathbb{R}^n \to \mathbb{R}$ such that

$$dS_\mu : \mathbb{R}^n \to T_\mu^*(M_h^n), \tag{3.41}$$

where $\mu \in M_h^n \backsimeq \otimes_{j=1}^n \mathbb{S}_j^1$ are global separable coordinates on M_h^n owing to the Liouville–Arnold theorem. Thus, we have the following canonical relationship:

$$< w, d\mu > + < t, dh >= dS_\mu(h), \tag{3.42}$$

where $w_j := w_j(\mu_j; h) \in T_{\mu_j}^*(\mathbb{S}_j^1)$ for every $1 \le j \le n$. Consequently, it follows readily that

$$S_\mu(h) = \sum_{j=1}^{n} \int_{\mu_j^{(0)}}^{\mu_j} w(\lambda; h) d\lambda, \tag{3.43}$$

satisfying the following relationship on $M_h^n \subset T^*(\mathbb{R}^n)$

$$dS_\mu + d\mathcal{L}_\mu = dS_q|_{q=q(\mu;h)} \tag{3.44}$$

for a mapping $\mathcal{L}_\mu : \mathbb{R}^n \to \mathbb{R}$. As a result of (3.43) and (3.44), one readily finds that

$$t_i = \partial S_\mu(h)/\partial h_i, \quad < p, \partial q/\partial \mu_i >= w_i + \partial \mathcal{L}_\mu/\partial \mu_i \tag{3.45}$$

for all $i = 1, \ldots, n$. A construction similar to the above can be performed subject on the embedded torus $\mathbb{T}_\gamma^n \subset T^*(\mathbb{R}^n)$:

$$d\tilde{S}_q(\gamma) := \sum_{j=1}^{n} p_j dq_j + \sum_{i=1}^{n} \varphi_i d\gamma_i, \tag{3.46}$$

where owing to (3.37) $\tilde{S}_q(\gamma) := S_q(\xi \cdot \gamma)$, $\xi \cdot \gamma(h) = h$, for all $(q; \gamma) \in U(M_h^n)$. For angle coordinates $\varphi \in \mathbb{T}_\gamma^n$, one obtains from (3.46) that

$$\varphi_i = \partial \tilde{S}_q(\gamma)/\partial \gamma_i \tag{3.47}$$

for all $1 \leq i \leq n$. As $\varphi_i \in \mathbb{R}/2\pi\mathbb{Z}$, $1 \leq i \leq n$, from (3.48) one computes that

$$\frac{1}{2\pi} \oint_{\sigma_j^{(h)}} d\varphi_i = \delta_{ij} = \frac{1}{2\pi} \frac{\partial}{\partial \gamma_i} \oint_{\sigma_j^{(h)}} d\tilde{S}_q(\gamma) = \frac{1}{2\pi} \frac{\partial}{\partial \gamma_i} \oint_{\sigma_j^{(h)}} < p, dq > \tag{3.48}$$

for all canonical cycles $\sigma_j^{(h)} \subset M_h^n$, $1 \leq j \leq n$, constituting a basis of the one-dimensional (integral) homology group $H_1(M_h^n; \mathbb{Z})$. Accordingly it follows from (3.48) that for all $1 \leq i \leq n$ "action" variables can be defined as

$$\gamma_i = \frac{1}{2\pi} \oint_{\sigma_i^{(h)}} < p, dq > . \tag{3.49}$$

Recall now that the $M_h^n \simeq \mathbb{T}_\gamma^n$ are also diffeomorphic to $\otimes_{j=1}^n \mathbb{S}_j^1$, where the factors $\mathbb{S}_j^1$ are all circles. The evolution along any of the vector fields $K_j : T^*(\mathbb{R}^n) \to T(T^*(\mathbb{R}^n)$ on $M_h^n \subset T^*(\mathbb{R}^n)$ is known [3, 132] to be linear and winding around the torus $\mathbb{T}_\gamma^n$, and can be interpreted as follows: these independent global coordinates on circles $\mathbb{S}_j^1$, $1 \leq j \leq n$, evolve quasiperiodically. Moreover, these (Hamilton–Jacobi) coordinates prove to be very important for establishing complete integrability by quadratures by solving the corresponding Picard–Fuchs equations.

Let us denote these separable coordinates on the integral submanifold $M_h^n \simeq \otimes_{j=1}^n \mathbb{S}_j^1$ by $\mu_j \in \mathbb{S}_j^1$, $1 \leq j \leq n$, and define the corresponding embedding map $\pi_h : M_h^n \to T^*(\mathbb{R}^n)$ as

$$q = q(\mu; h), \ p = p(\mu; h). \tag{3.50}$$

There exist two important cases to consider for this embedding (3.50).

First case: This is when the integral submanifold $M_h^n \subset T^*(\mathbb{R}^n)$ can be parametrized as a manifold by means of the base coordinates $q \in \mathbb{R}^n$ of the cotangent bundle $T^*(\mathbb{R}^n)$, and can be explained as follows: the canonical Liouville 1-form $\alpha^{(1)} \in \Lambda^1(\mathbb{R}^n)$, in accordance with the diagram

$$\begin{array}{lll} T^*(\ M_h^n) \simeq T^*(\otimes_{j=1}^n \mathbb{S}_j^1) & \xleftarrow{\pi^*} & T^*(\mathbb{R}^n) \\ pr \downarrow \qquad\quad pr \downarrow & & pr \downarrow \\ M_h^n \qquad\quad \simeq \otimes_{j=1}^n \mathbb{S}_j^1 & \xrightarrow{\pi} & \mathbb{R}^n \end{array} \tag{3.51}$$

is mapped by the embedding $\pi = pr \cdot \pi_h : M_h^n \to \mathbb{R}^n$, independent of the parameters $h \in \mathbb{R}^n$, into the 1-form

$$\alpha_h^{(1)} = \pi^* \alpha^{(1)} = \sum_{j=1}^n w_j(\mu_j; h) d\mu_j, \tag{3.52}$$

where $(\mu, w) \in T^*(\otimes_{j=1}^n \mathbb{S}_j^1) \simeq \otimes_{j=1}^n T^*(\mathbb{S}_j^1)$. The embedding $\pi : M_h^n \to \mathbb{R}^n$ sends the function $\mathcal{L}_\mu : \mathbb{R}^n \to \mathbb{R}$ to zero, due to the equality (3.52), giving rise to the generating function $S_\mu : \mathbb{R}^n \to \mathbb{R}$, which satisfies

$$dS_\mu = \left. dS_q \right|_{q=q(\mu;h)}, \tag{3.53}$$

where as before

$$S_\mu(h) = \sum_{j=1}^n p_j dq_j + \sum_{j=1}^n t_j dh_j \tag{3.54}$$

and $\det(\partial q(\mu; h)/\partial \mu) \neq 0$ almost everywhere on M_h^n for all $h \in \mathbb{R}^n$. As with (3.45), one obtains from (3.54) that

$$t_j = \partial S_\mu(h)/\partial h_j \tag{3.55}$$

for $1 \leq j \leq n$. Concerning the second part of the embedding map (3.50) we arrive, due to the equality (3.52), at the following simple result:

$$p_i = \sum_{j=1}^n w_j(\mu_j; h) \partial \mu_j / \partial q_i, \tag{3.56}$$

where $i = 1, \ldots, n$ and the matrix $(\partial \mu/\partial q)$ is invertible almost everywhere on $\pi(M_h^n)$ due to the local invertibility of the embedding map $\pi : M_h^n \to \mathbb{R}^n$.

Thus, we claim that the problem of complete integrability is solved if and only if such an embedding $\pi : M_h^n \to \mathbb{R}^n \subset T^*(\mathbb{R}^n)$ is constructed. This case was considered in detail in [329], where the corresponding Picard–Fuchs equations were constructed using an extension of results of Galissot–Reeb and Francois [135, 136]. More precisely, as in (3.35) the equations are defined as

$$\partial w_j(\mu_j; h)/\partial h_k = P_{kj}(\mu_j, w_j; h), \tag{3.57}$$

where $P_{kj} : T^*(\otimes_{j=1}^n \mathbb{S}_j^1) \times \mathbb{C}^n \to \mathbb{C}$, $1 \leq k, j \leq n$, are algebraic functions.

Second case: This is when the integral submanifold $M_h^n \subset T^*(\mathbb{R}^n)$ *cannot be* embedded almost everywhere in $\mathbb{R}^n \subset T^*(\mathbb{R}^n)$, and one cannot use the relationship (3.54). Thus, we are forced to consider the usual canonical transformation from $T^*(\mathbb{R}^n)$ to $T^*(\mathbb{R}^n)$ based on a map $d\mathcal{L}_q : \otimes_{j=1}^n \mathbb{S}_j^1 \to T^*(\mathbb{R}^n)$, where $\mathcal{L}_q : \otimes_{j=1}^n \mathbb{S}_j^1 \to \mathbb{R}$ satisfies for all $\mu \in \otimes_{j=1}^n \mathbb{S}_j^1 \backsimeq M_h^n \ni q$ the following relationship :

$$pr^*\alpha^{(1)}(q; p) = \sum_{j=1}^n w_j \ d\mu_j + d\mathcal{L}_q(\mu). \tag{3.58}$$

In this case, for any $\mu \in \otimes_{j=1}^n \mathbb{S}_j^1$ the hereditary generating function $\mathcal{L}_\mu : \mathbb{R}^n \to T^*(\otimes_{j=1}^n \mathbb{S}_j^1)$ can be derived in the form

$$d\mathcal{L}_\mu = d\mathcal{L}_q|_{q=q(\mu;h)}, \tag{3.59}$$

which clearly satisfies the canonical transformation condition

$$dS_q(h) = \sum_{j=1}^n w_j(\ \mu_j; h) \ d\mu_j + \sum_{j=1}^n t_j dh_j + d\mathcal{L}_\mu(h), \tag{3.60}$$

for almost all $\mu \in \otimes_{j=1}^n \mathbb{S}_j^1$ and $h \in \mathbb{R}^n$. Then, using (3.60) we derive

$$\partial\mathcal{L}_\mu(h)/\partial h_j = \ < p, \partial q/\partial h_j >|_{M_h^n} \tag{3.61}$$

for all $j = 1, 2,\ \ \mu \in \otimes_{j=1}^n \mathbb{S}_j^1$ and $h \in \mathbb{R}^n$. Whence, the following important analytical result

$$t_s = \sum_{j=1}^n \int_{\mu_j^{(0)}}^{\mu_j} (\partial w_j(\lambda; h)/\partial h_s) d\lambda,$$

$$\sum_{j=1}^n p_j(\mu; h)(\partial q_j/\partial \mu_s) = w_s + \partial\mathcal{L}_\mu(h)/\partial \mu_s \tag{3.62}$$

holds for all $s = 1, 2$ and $\mu, \mu^{(0)} \in \otimes_{j=1}^n \mathbb{S}_j^1$ with fixed parameters $h \in \mathbb{R}^n$. Thus, we have found a natural generalization of the relationships (3.56) subject to the extended integral submanifold embedding map $\pi_h : M_h^n \to T^*(\mathbb{R}^n)$ in the form (3.50).

Assume now that functions $w_j : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}$, $1 \leq j \leq n$, satisfy Picard–Fuchs equations like (3.57), which have [93, 182, 181] algebraic solutions of the form

$$w_j^{n_j} + \sum_{k=0}^{n_j-1} c_{j,k}(\lambda; h) w_j^k = 0, \tag{3.63}$$

where $c_{j,k} : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}$, $0 \leq k \leq n_j - 1$, $1 \leq j \leq n$, are polynomials in $\lambda \in \mathbb{C}$. Each algebraic curve of (3.63) is known to be topologically equivalent, owing to Riemann's theorem [409], to a Riemannian surface $\Gamma_h^{(j)}$ of genus $g_j \in \mathbb{Z}_+$, $1 \leq j \leq n$. Accordingly there exists a local diffeomorphism $\rho : M_h^n \to \otimes_{j=1}^n \Gamma_h^{(j)}$ mapping homology group basis cycles $\sigma_j^{(h)} \subset M_h^n$ into homology subgroup $H_1(\otimes_{j=1}^n \Gamma_h^{(j)}; \mathbb{Z})$ basis cycles $\sigma_j(\Gamma_h) \subset \Gamma_h^{(j)}$ satisfying

$$\rho(\ \sigma_j^{(h)}) = \sum_{k=1}^{n} n_{jk}\ \sigma_k(\Gamma_h), \tag{3.64}$$

where $n_{jk} \in \mathbb{Z}$, $1 \leq k \leq j$ and $1 \leq j \leq n$, are fixed integer coefficients. Hence, it follows from (3.64) and (3.59) that, for instance, we have expressions (3.49) such as

$$\gamma_i = \frac{1}{2\pi} \sum_{j=1}^{n} n_{ij} \oint_{\sigma_j(\Gamma_h)} w_j(\lambda; h) d\lambda, \tag{3.65}$$

where $i = 1, \ldots, n$. Subject to the evolution on $M_h^n \subset T^*(\mathbb{R}^n)$, we readily infer from (3.62) that

$$dt_i = \sum_{j=1}^{n} (\partial w_j(\mu_j; h)/\partial h_i) d\mu_j \tag{3.66}$$

at $dh_i = 0$ for all $i = 1, \ldots, n$, which clearly gives rise to a global τ-parametrization of the set of circles $\otimes_{j=1}^n \mathbb{S}_j^1 \subset \otimes_{j=1}^n \Gamma_h^{(j)}$; that is, one can define inverse algebraic functions to the Abelian integrals (3.59) in the form

$$\mu = \mu(\tau; h), \tag{3.67}$$

where, as above, $\tau = (t_1, t_2, ..., t_n) \in \mathbb{R}^n$ is a vector of evolution parameters. Recalling now expressions (3.50) for the integral submanifold map $\pi_h : M_h^n \to T^*(\mathbb{R}^n)$, we can finally write the expressions in quadrature for the motions on $M_h^n \subset T^*(\mathbb{R}^n)$ as

$$q = q(\mu(\tau; h)) = \tilde{q}(\tau; h), \quad p = p(\mu(\tau; h)) = \tilde{p}(\tau; h), \tag{3.68}$$

where it is obvious now that the solutions $(\tilde{q}, \tilde{p}) \in T^*(\mathbb{R}^n)$ are quasiperiodic in each variable $t_i \in \tau$, $1 \leq i \leq n$. Thus, we have essentially proved the following result.

Theorem 3.11. *Every completely integrable Hamiltonian system admitting an algebraic submanifold $M_h^n \subset T^*(\mathbb{R}^n)$ has a separable canonical transformation (3.60) that is described by differential algebraic Picard–Fuchs type equations whose solution is a set of algebraic curves (3.63).*

Therefore, the main ingredient of this scheme of integrability by quadratures is finding the Picard–Fuchs equations (3.57) corresponding to the integral submanifold embedding (3.50), which depends, in general, on $\mathbb{R}^n \ni h$-parameters for the case when the integral submanifold $M_h^n \subset T^*(\mathbb{R}^n)$ cannot be embedded in the base space $\mathbb{R}^n \subset T^*(\mathbb{R}^n)$ of the phase space $T^*(\mathbb{R}^n)$.

It follows from Theorem 3.9 that we can find 1-forms $h_j^{(1)} \in \Lambda^1(T^*(\mathbb{R}^n))$ satisfying the following identity on $T^*(\mathbb{R}^n)$:

$$\omega^{(2)}(q,p) := \sum_{j=1}^{n} dp_j \wedge dq_j = \sum_{j=1}^{n} dH_j \wedge h_j^{(1)}. \tag{3.69}$$

The 1-forms $h_j^{(1)} \in \Lambda^1(T^*(\mathbb{R}^n))$, $1 \leq j \leq n$, have the property described above: their pullbacks to the integral submanifold (3.9) gives rise to the global linearization

$$\pi_h^* h_j^{(1)} := \bar{h}_j^{(1)} = dt_j \tag{3.70}$$

where $\bar{h}_j^{(1)} \in \Lambda^1(M_h^n)$ and $\pi_{h*} d/dt_j = K_j \cdot \pi_h$ for all $1 \leq j \leq n$. The expressions (3.70) combined with those of (3.66) lead directly to the formulas

$$\bar{h}_j^{(1)} = \sum_{j=1}^{n} (\partial w_j(\mu_j; h)/\partial h_i) d\mu_j \tag{3.71}$$

at $dh_j = 0$ on $M_h^n \backsimeq \otimes_{j=1}^n \mathbb{S}_j^1 \subset \otimes_{j=1}^n \Gamma_h^{(j)}$ for all $1 \leq j \leq n$. Since we require that the integral submanifold embedding (3.47) be a local diffeomorphism in a neighborhood $U(M_h^n) \subset T^*(\mathbb{R}^n)$, the Jacobian $\det(\partial q(\mu, h)/\partial \mu)$ must be nonzero almost everywhere in $U(M_h^n)$. On the other hand, as proved in [14], the set of 1-forms $\bar{h}_j^{(1)} \in \Lambda^1(M_h^n)$, $1 \leq j \leq n$, can be represented in $U(M_h^n)$ as

$$\bar{h}_j^{(1)} = \sum_{k=1}^{n} \bar{h}_{jk}^{(1)}(q,p) dq_k \Big|_{M_h^n}, \tag{3.72}$$

where $\bar{h}_{jk}^{(1)} : T^*(\mathbb{R}^n) \to \mathbb{R},\ \ 1 \leq k, j \leq n,$ are algebraic functions of their arguments. Consequently, it follows easily from (3.72) and (3.71) that

$$\partial w_i(\mu_i; h)/\partial h_j = \sum_{k=1}^{n} \bar{h}_{jk}^{(1)}(q(\mu; h), p(\mu; h))(\partial q_k(\mu; h)/\partial \mu_i) \tag{3.73}$$

for all $1 \leq i, j \leq n$. Subject to p-variables in (3.73) and owing to (3.62), we must use the expressions

$$\sum_{j=1}^{n} p_j(\mu; h)(\partial q_j/\partial \mu_s) = w_s + \partial \mathcal{L}_\mu(h)/\partial \mu_s, \tag{3.74}$$

$$\partial \mathcal{L}_\mu(h)/\partial h_j = < p, \partial q/\partial h_j >|_{M_h^n},$$

which are valid for $s = 1, \ldots, n$ and all $\mu \in \otimes_{j=1}^{n} \mathbb{S}_j$, $h \in \mathbb{R}^n$ in the neighborhood $U(M_h^n) \subset T^*(\mathbb{R}^n)$ chosen above. Whence, we have arrived at the following form of equations (3.73):

$$\partial w_i(\mu_i; h)/\partial h_j = \bar{P}_{ji}(\mu, w; h), \tag{3.75}$$

where for all $1 \leq i, j \leq n$ each expression

$$\bar{P}_{ji}(\mu, w; h) := \sum_{k=1}^{n} \bar{h}_{jk}^{(1)}(q(\mu; h), p(\mu; h))\partial q_k/\partial \mu_i \tag{3.76}$$

depends respectively only on $\Gamma_h^{(i)} \ni (\mu_i, w_i)$-variables for each $1 \leq i \leq n$ and all $1 \leq j \leq n$. This condition can obviously be written as

$$\partial \bar{P}_{ji}(\mu, w; h)/\partial \mu_k = 0, \quad \partial \bar{P}_{ji}(\mu, w; h)/\partial w_k = 0 \tag{3.77}$$

for $j, i \neq k \in \{1, \ldots, n\}$ at almost all $\mu \in \otimes_{j=1}^{n} \mathbb{S}_j^1$ and $h \in \mathbb{R}^n$. The set of conditions (3.73) induces a system of algebraic-differential equations subject to the embedding $pr \cdot \pi_h : M_h^n \to \mathbb{R}^n$ defined analytically by (3.50) and the generating function (3.59). As a result of solving these equations, we obtain, by virtue of (3.75) and (3.77), the system of Picard–Fuchs equations

$$\partial w_i(\mu_i; h)/\partial h_j = P_{ji}(\mu_i, w_i; h) \tag{3.78}$$

where the maps

$$P_{ji} : \Gamma_h^{(i)} \times \mathbb{R}^n \to \mathbb{C} \tag{3.79}$$

are all algebraic expressions. Since the set of algebraic curves (3.63) must satisfy (3.78), we can retrieve this set by solving the Picard–Fuchs equations (3.78). From this, (3.61) and (3.50), the integrability by quadratures of all flows on $M_h^n \subset T^*(\mathbb{R}^n)$ follows immediately, and this yields an important theorem.

Theorem 3.12. *Let there be given a completely integrable Hamiltonian system on the co-adjoint manifold $T^*(\mathbb{R}^n)$ whose integral submanifold $M_h^n \subset T^*(\mathbb{R}^n)$ is described by Picard–Fuchs algebraic equations (3.78). The corresponding embedding $\pi_h : M_h^n \to T^*(\mathbb{R}^n)$ (3.50) is a solution of a compatibility condition subject to the differential-algebraic relationships (3.77) on the generating function of the canonical transformations (3.59).*

To show that the scheme described above leads to an effective algorithmic procedure for constructing the Picard–Fuchs equations (3.78) and the corresponding integral submanifold embedding $\pi_h : M_h^n \to T^*(\mathbb{R}^n)$ in the form (3.50), we shall apply it in the sequel to some Hamiltonian systems including a truncated Fokker–Planck Hamiltonian system on the canonical symplectic cotangent space $T^*(\mathbb{R}^n)$. Making use of the representations (3.73) and (3.74) and equation (3.77), we have shown above that the set of 1-forms (3.72) is reduced to the following purely differential-algebraic relationships on $M_{h,\tau}^{2n}$:

$$\partial w_i(\mu_i; h)/\partial h_j = \mathbf{P}_{ji}(\mu_i, w_i; h), \tag{3.80}$$

generalizing similar ones of [135, 136, 182, 181], where the characteristic functions $\mathbf{P}_{ji} : T^*(M_h^n) \to \mathbb{R}$, $1 \le i, j \le n$, are defined as

$$\mathbf{P}_{ji}(\mu_i, w_i; h) := \bar{\mathbf{P}}_{ji}(\mu, w; h)\big|_{M_h^n} . \tag{3.81}$$

It is clear that this set of purely differential-algebraic relationships (3.80) and (3.81) makes it possible to explicitly express first order compatible differential-algebraic equations, whose solution yields the first half of the desired embedding (3.73) for the integral submanifold $M_h^n \subset M^{2n}$ in an open neighborhood $M_{h,\tau}^{2n} \subset M^{2n}$. As a result of the above computations, we have the following main theorem.

Theorem 3.13. *The embedding (3.73) for the integral submanifold $M_h^n \subset M^{2n}$ (compact and connected), parametrized by a regular parameter $h \in \mathfrak{g}^*$, is an algebraic solution (up to diffeomorphism) to the set of characteristic Picard–Fuchs equations (3.80) on $T^*(M_h^n)$, and can be represented* [93] *in the following algebraic-geometric form:*

$$w_j^{n_j} + \sum_{s=1}^{n} c_{js}(\lambda; h) w_j^{n_j - s} = 0, \tag{3.82}$$

where $c_{js} : \mathbb{R} \times \mathfrak{g}^* \to \mathbb{R}$, $1 \le s, j \le n$ are algebraic expressions, depending only on the functional structure of the original abelian Lie algebra $\mathfrak{g}$ of invariants on M^{2n}. In particular, if the right-hand side of the characteristic equations (3.80) is independent of $h \in \mathfrak{g}^*$, then this dependence will be linear in $h \in \mathfrak{g}^*$.

It should be noted here that an attempt was made in [93, 135, 136] to describe the explicit algebraic form of the Picard–Fuchs equations (3.80) by means of straightforward calculations for the well-known completely integrable Kowalewskaya top. The idea suggested in [93, 135, 136] is in some aspects very close to that devised independently in [329], which did not consider the explicit form of the algebraic curves (3.82) starting from an abelian Lie algebra $\mathfrak{g}$ of invariants on a canonically symplectic phase space M^{2n}.

As is well-known [158, 157], a set of algebraic curves (3.82), prescribed, via the algorithm devised above, to an abelian Lie algebra $\mathfrak{g}$ of invariants on the canonically symplectic phase space $M^{2n} = T^*(\mathbb{R}^n)$ can be realized by means of a set of n_j-sheeted Riemannian surfaces $\Gamma_h^{n_j}$ covering the associated real-valued cycles $\mathbb{S}_j^1$, $1 \leq j \leq n$, which generate the corresponding homology group $H_1(\mathbb{T}^n;\mathbb{Z})$ of the Arnold torus $\mathbb{T}^n \simeq \otimes_{j=1}^n \mathbb{S}_j^1$, which is diffeomorphic to the integral submanifold $M_h^n \subset M^{2n}$.

Thus, by solving the set of algebraic equations (3.80) with respect to functions $w_j : \mathbb{S}_j^1 \times \mathfrak{g}^* \to \mathbb{R}$, $1 \leq j \leq n$, from (3.62) one obtains a vector parameter $\tau = (t_1, ..., t_n) \in \mathbb{R}^n$ on M_h^n explicitly described by means of the following Abelian equations:

$$t_j = \sum_{s=1}^n \int_{\mu_s^0}^{\mu_s} d\lambda \ \ \partial w_s(\lambda; h)/\partial h_j$$

$$= \sum_{s=1}^n \int_{\mu_s^0}^{\mu_s} d\lambda \ \bar{\mathbf{P}}_{js}(\lambda, w_s; h), \tag{3.83}$$

where $1 \leq j \leq n$, and $(\mu^0; h) \in (\otimes_{j=1}^n \Gamma_h^{n_j}) \times \mathfrak{g}^*$. Using the expression (3.83) and recalling that the generating function $S : M_h^n \times \mathbb{R}^n \to \mathbb{R}$ is a one-valued mapping on an appropriate covering space $(\bar{M}_h^n; H_1(M_h^n;\mathbb{Z}))$, one can construct via the method of Arnold [14] the action-angle coordinates on M_h^n. Denote the basic oriented cycles on M_h^n by $\sigma_j \subset M_h^n$, $1 \leq j \leq n$. These cycles together with their duals generate the homology group $H_1(M_h^n;\mathbb{Z}) \simeq H_1(\mathbb{T}^n;\mathbb{Z}) = \oplus_{j=1}^n \mathbb{Z}_j$ and corresponding cohomology group $H^1(M_h^n;\mathbb{Z})$. By virtue of the diffeomorphism $M_h^n \simeq \otimes_{j=1}^n \mathbb{S}_j^1$ described above, there is a one-to-one correspondence between the basic cycles of $H_1(M_h^n;\mathbb{Z})$ and those on the algebraic curves $\Gamma_h^{n_j}$, $1 \leq j \leq n$, given by (3.82):

$$\rho : H_1(M_h^n;\mathbb{Z}) \to \oplus_{j=1}^n \mathbb{Z}_j \sigma_{h,j} \ , \tag{3.84}$$

where $\sigma_{h,j} \subset \Gamma_h^{n_j}$ are the corresponding real-valued cycles on the Riemann surfaces $\Gamma_h^{n_j}$, $1 \leq j \leq n$.

Assume that the mapping (3.84) is prescribed by

$$\rho(\sigma_i) := \oplus_{j=1}^{n} n_{ij}\ \sigma_{h,j} \tag{3.85}$$

where $n_{ij} \in \mathbb{Z}$, $1 \leq i, j \leq n$, are integers. Then following the Arnold construction [14, 135, 136, 173], one obtains the action-variables on $M_h^n \subset M^{2n}$ in the form

$$\gamma_{j:} := \frac{1}{2\pi} \oint_{\sigma_j} dS = \sum_{s=1}^{n} n_{js} \oint_{\sigma_{h,s}} d\lambda\ w_s(\lambda; h), \tag{3.86}$$

where $1 \leq j \leq n$. It is easy to show [14, 173, 344], that the expressions (3.86) naturally define an almost everywhere differentiable invertible map

$$\xi : \mathfrak{g}^* \to \mathbb{R}^n, \tag{3.87}$$

enabling one to treat the integral submanifold M_h^n as a submanifold $M_\gamma^n \subset M^{2n}$, where

$$M_\gamma^n := \{u \in M^{2n} : \xi(h) = \gamma \in \mathbb{R}^n\}. \tag{3.88}$$

But, as was demonstrated in [114, 135, 136], the functions (3.86) do not in general generate a global foliation of the phase space M^{2n}, as they are subject to both topological and analytical constraints. Since the functions (3.86) are also commuting invariants on M^{2n}, one can define a further canonical transformation of the phase space M^{2n} generated by the following relationship on $M_{h,\tau}^{2n}$:

$$\sum_{j=1}^{n} w_j\ d\mu_j + \sum_{j=1}^{n} \varphi_j\ d\gamma_j = dS(\mu; \gamma), \tag{3.89}$$

where $\varphi = (\varphi_1, ..., \varphi_n) \in \mathbb{T}^n$ are the angle-variables on the torus $\mathbb{T}^n \simeq M_h^n$ and $S : M_\gamma^n \times \mathbb{R}^n \to \mathbb{R}$ is the corresponding generating function. Therefore, it follows easily from (3.86) and (3.89) that

$$\varphi_j\ := \partial S(\mu; \gamma)/\partial \gamma_j = \sum_{s=1}^{n} \partial S(\mu; \gamma(h))/\partial h_s\ \partial h_s/\partial \gamma_j \tag{3.90}$$

$$= \sum_{s=1}^{n} t_s \omega_{sj}(\gamma), \qquad \frac{1}{2\pi} \oint_{\sigma_j} d\varphi_k\ = \delta_{jk},$$

where $\Omega := \{\omega_{sj} : \mathbb{R}^n \to \mathbb{R},\ 1 \leq s, j \leq n\}$ is the so-called [3, 14, 262] frequency matrix, which is almost everywhere invertible on the integral submanifold $M_\gamma^n \subset M^{2n}$. An obvious consequence of (3.90) is that the evolution of any vector field $K_a \in \Gamma(M^{2n})$ for $a \in \mathfrak{g}$ on the integral submanifold $M_\gamma^n \subset M^{2n}$ is quasiperiodic, with a set of frequencies generated by the matrix $\Omega \overset{a.e.}{\in} Aut(\mathbb{R}^n)$. As examples showing the effectiveness of the proposed method of construction of integral submanifold embedding for abelian integrable Hamiltonian systems, one can check the Liouville–Arnold integrability of many finite-dimensional Hamiltonian systems, amongst which we mention the Henon–Heiles and Neumann systems described in detail in [54, 48–53, 262, 326].

3.4 Integral submanifold embedding problem for a nonabelian Lie algebra of invariants

We shall assume in this section that $K \in \Gamma(M^{2n})$ is a Hamiltonian vector field on the canonical phase space $M^{2n} = T^*(\mathbb{R}^n)$, which is endowed with a nonabelian Lie algebra $\mathfrak{g}$ of invariants satisfying the hypotheses of the Mishchenko–Fomenko Theorem 3.7, that is

$$\dim \mathfrak{g} + \mathrm{rank}\mathfrak{g} = \dim M^{2n}. \tag{3.91}$$

Then, an integral submanifold $M_h^r \subset M^{2n}$ at a regular element $h \in \mathfrak{g}^*$ is $\mathrm{rank}\mathfrak{g} = r$-dimensional and diffeomorphic (when compact and connected) to the standard r-dimensional torus $\mathbb{T}^r \simeq \otimes_{j=1}^r \mathbb{S}_j^1$. It is natural to ask the following question: How does one construct the corresponding integral submanifold embedding

$$\pi_h : M_h^r \to M^{2n} \quad , \tag{3.92}$$

which characterizes all possible orbits of the dynamical system $K \in \Gamma(M^{2n})$?

Using the experience gained in constructing the embedding in the case of the abelian Liouville–Arnold theorem, we proceed to study the integral submanifold $M_h^r \subset M^{2n}$ by means of Cartan's theory [26, 44, 85, 173, 326, 344] of integrable ideals in the Grassmann algebra $\Lambda(M^{2n})$. Let $\mathcal{I}(\mathfrak{g}^*)$ be an ideal in $\Lambda(M^{2n})$ generated by independent differentials $dH_j \in \Lambda^1(M^{2n})$, $1 \leq j \leq k$, on an open neighborhood $U(M_h^r)$ of M_h^r, where by definition, $r = \dim \mathfrak{g}$. The ideal $\mathcal{I}(\mathfrak{g}^*)$ is obviously Cartan integrable [44, 173, 344] with the integral submanifold $M_h^r \subset M^{2n}$ (at a regular element $h \in \mathfrak{g}^*$), on which it vanishes, that is $\pi_h^*\mathcal{I}(\mathfrak{g}^*) = 0$. The dimension $\dim M_h^r = \dim M^{2n} - \dim \mathfrak{g} = r = \mathrm{rank}\mathfrak{g}$ due to the condition (3.91) imposed on the Lie algebra $\mathfrak{g}$. It is useful to note here that owing to the inequality $r \leq k$ for rank $\mathfrak{g}$, one readily obtains from (3.91) that $\dim\mathfrak{g} = k \geq n$. Since each base element $H_j \in \mathfrak{g}$ generates a symplectically dual vector field $K_j \in \Gamma(M^{2n})$, $1 \le j \le k$, one can study the corresponding differential system $K(\mathfrak{g})$ which is also Cartan integrable on the entire open neighborhood $U(M_h^r) \subset M^{2n}$. Denote the corresponding dimension of the integral submanifold by $\dim M_h^k = \dim K(\mathfrak{g}) = k$. Consider now an abelian differential system $K(\mathfrak{g}_h) \subset K(\mathfrak{g})$ generated by the Cartan subalgebra $\mathfrak{g}_h \subset \mathfrak{g}$ and its integral submanifold $\bar{M}_h^r \subset U(M_h^r)$. Since the Lie subgroup $G_h = \exp \mathfrak{g}_h$ acts on the integral submanifold M_h^r invariantly and $\dim \bar{M}_h^r = \operatorname{rank}\mathfrak{g} = r$, it follows that $\bar{M}_h^r = M_h^r$. On the other hand, the system $K(\mathfrak{g}_h) \subset K(\mathfrak{g})$ by definition, so the integral submanifold M_h^r is an invariant part of the integral submanifold $M_h^k \subset U(M_h^r)$ with respect to the Lie group $G = \exp \mathfrak{g}$ - action on M_h^k. In this case one has the following result.

Lemma 3.5. *There exist exactly $(n-r)$ vector fields $\tilde{F}_j \in K(\mathfrak{g})/K(\mathfrak{g}_h)$, $1 \le j \le n-r$ for which*

$$\omega^{(2)}(\tilde{F}_i, \tilde{F}_j) = 0 \tag{3.93}$$

on $U(M_h^r)$ for all $1 \le i, j \le n-r$.

Proof. Obviously, the matrix $\omega(\tilde{K}) := \{\omega^{(2)}(\tilde{K}_i, \tilde{K}_j)\colon\ 1 \le i, j \le k\}$ is of rank $\omega(\tilde{K}) = k - r$ on $U(M_h^r)$, since $\dim_{\mathbb{R}} \ker(\pi_h^* \omega^{(2)}) = \dim_{\mathbb{R}}(\pi_{h*} K(\mathfrak{g}_h)) = r$ on M_h^r at the regular element $h \in \mathfrak{g}^*$. Now we complexify the tangent space $T(U(M_h^r))$ using its even dimensionality. Whence, one easily deduces that on $U(M_h^r)$ there exist just $(n-r)$ vectors (not vector fields!) $\tilde{K}_j^{\mathbb{C}} \in K^{\mathbb{C}}(\mathfrak{g})/K^{\mathbb{C}}(\mathfrak{g}_h)$ from the complexified [396] factor space $K^{\mathbb{C}}(\mathfrak{g})/K^{\mathbb{C}}(\mathfrak{g}_h)$. To show this, let us reduce the skew-symmetric matrix $\omega(\tilde{K}) \in Hom(\mathbb{R}^{k-r})$ to its selfadjoint equivalent $\omega(\tilde{K}^{\mathbb{C}}) \in Hom(\mathbb{C}^{n-r})$, having taken into account that $\dim_{\mathbb{R}} \mathbb{R}^{k-r} = \dim_{\mathbb{R}} \mathbb{R}^{k+r-2r} = \dim_{\mathbb{R}} \mathbb{R}^{2(n-r)}$ $= \dim_{\mathbb{C}} \mathbb{C}^{n-r}$. To this end, let $f_j^{\mathbb{C}} \in \mathbb{C}^{n-r}$, $1 \le j \le n-r$, be eigenvectors of the nondegenerate self-adjoint matrix $\omega(\tilde{K}^{\mathbb{C}}) \in Hom(\mathbb{C}^{n-r})$, that is

$$\omega(\tilde{K}^{\mathbb{C}})\ f_j^{\mathbb{C}} = \tilde{\lambda}_j f_j^{\mathbb{C}}, \tag{3.94}$$

where $\tilde{\lambda}_j \in \mathbb{R}$, $1 \le j \le n-r$, and for all $1 \le i, j \le n-r$, $< f_i^{\mathbb{C}}, f_j^{\mathbb{C}} >= \delta_{i,j}$. Thus, in the basis $\{f_j^{\mathbb{C}} \in K^{\mathbb{C}}(\mathfrak{g})/K^{\mathbb{C}}(\mathfrak{g}_h) : 1 \le j \le n-r\}$, the matrix $\omega(\tilde{K}^{\mathbb{C}}) \in Hom(\mathbb{C}^{n-r})$ is diagonal and representable as

$$\omega(\tilde{K}^{\mathbb{C}}) = \sum_{j=1}^{n-r} \tilde{\lambda}_j f_j^{\mathbb{C}} \otimes_{\mathbb{C}} f_j^{\mathbb{C}}\ \ , \tag{3.95}$$

where $\otimes_{\mathbb{C}}$ is the usual Kronecker tensor product of vectors from $\mathbb{C}^{n-r}$. Owing to the construction of the complexified matrix $\omega(\tilde{K}^{\mathbb{C}}) \in Hom(\mathbb{C}^{n-r})$, we see that the space $K^{\mathbb{C}}(\mathfrak{g})/K^{\mathbb{C}}(\mathfrak{g}_h) \simeq \mathbb{C}^{n-r}$ carries a Kähler structure [396] with respect to which the following expressions

$$\omega(\tilde{K}) = \mathrm{Im}\omega(\tilde{K}), \quad <\cdot,\cdot>_{\mathbb{R}} = \mathrm{Re} <\cdot,\cdot> \tag{3.96}$$

hold. Making use of the representation (3.95) and expressions (3.96), one can find vector fields $\tilde{F}_j \in K(\mathfrak{g})/K(\mathfrak{g}_h)$, $1 \leq j \leq n-r$, such that

$$\omega(\tilde{F}) = \mathrm{Im}\,\omega(\tilde{F}^{\mathbb{C}}) = J, \tag{3.97}$$

holds on $U(M_h^r)$, where Im and Re denote, respectively the imaginary and real parts of a complex expression and $J \in Sp(\mathbb{C}^{n-r})$ is the standard symplectic matrix satisfying the complex structure [396] identity $J^2 = -I$. By virtue of the normalization conditions $< f_j^{\mathbb{C}}, f_j^{\mathbb{C}} >= \delta_{i,j}$, for all $1 \leq i$, $j \leq n-r$, it follows immediately from (3.97) that $\omega^{(2)}(\tilde{F}_i, \tilde{F}_j) = 0$ for all $1 \leq i,\, j \leq n-r$, where

$$\tilde{F}_j := \mathrm{Re}\tilde{F}_j^{\mathbb{C}} \tag{3.98}$$

for all $1 \leq j \leq n-r$, and this proves the lemma. □

Assume now that the Lie algebra $\mathfrak{g}$ of invariants on M^{2n} has been split into a direct sum of subspaces as

$$\mathfrak{g} = \mathfrak{g}_h \oplus \tilde{\mathfrak{g}}_h, \tag{3.99}$$

where $\mathfrak{g}_h$ is the Cartan subalgebra at a regular element $h \in \mathfrak{g}^*$ (which is abelian) and $\tilde{\mathfrak{g}}_h \simeq \mathfrak{g}\,/\mathfrak{g}_h$ is the corresponding complement to $\mathfrak{g}_h$. Let $\{\bar{H}_i \in \mathfrak{g}_h : 1 \leq i \leq r\}$ be a basis for $\mathfrak{g}_h$, where $\dim \mathfrak{g}_h = \mathrm{rank}\mathfrak{g} = k$, and let $\{\tilde{H}_j \in \tilde{\mathfrak{g}}_h \simeq \mathfrak{g}/\mathfrak{g}_h : 1 \leq j \leq k-r\}$ be the corresponding dual basis for $\tilde{\mathfrak{g}}_h$. Then, owing to the results obtained above, the following relationships hold:

$$\{\bar{H}_i, \bar{H}_j\} = 0, \quad h(\{\bar{H}_i, \tilde{H}_s\}) = 0 \tag{3.100}$$

on the open neighborhood $U(M_h^r) \subset M^{2n}$ for all $1 \leq i,\ j \leq r$ and $1 \leq s \leq k-r$. We have as yet had nothing to say of the expressions $h(\{\tilde{H}_s, \tilde{H}_m\})$ for $1 \leq s, m \leq k-r$. Making use of the representation (3.98) for our vector fields (if they exist) $\tilde{F}_j \in K(\mathfrak{g})/K(\mathfrak{g}_h)$, $1 \leq j \leq n-r$, one obtains the expansion

$$\tilde{F}_i = \sum_{j=1}^{k-r} c_{ji}(h)\tilde{K}_j \quad , \tag{3.101}$$

where $i_{\tilde{K}_j}\omega^{(2)} := -d\tilde{H}_j$, $c_{ji} : \mathfrak{g}^* \to \mathbb{R}$, $1 \leq i \leq n-r$, $1 \leq j \leq k-r$, are real-valued functions on $\mathfrak{g}^*$, which are uniquely defined as a result of (3.101).

Whence, it follows directly that there exist invariants $\tilde{f}_s : U(M_h^r) \to \mathbb{R}$ such that

$$-i_{\tilde{F}_s}\omega^{(2)} = \sum_{j=1}^{k-r} c_{js}(h)\ d\tilde{H}_j := d\tilde{f}_s \quad , \tag{3.102}$$

where $\tilde{f}_s = \sum_{j=1}^{k-r} c_{js}(h)\tilde{H}_j$, $1 \le s \le n-r$, holds on $U(M_h^r)$.

To proceed further, let us look at the following identity which is similar to (3.24):

$$(\otimes_{j=1}^{r} i_{\bar{K}_j})(\otimes_{s=1}^{n-r} i_{\tilde{F}_s})(\omega^{(2)})^{n+1} = 0 = \pm(n+1)!(\wedge_{j=1}^{r} d\bar{H}_j)(\wedge_{s=1}^{n-r} d\ \tilde{f}_s) \wedge \omega^{(2)}, \tag{3.103}$$

on $U(M_h^r)$. Then, the following result is easily proved using Cartan theory [85, 173, 344].

Lemma 3.6. *The symplectic structure $\omega^{(2)} \in \Lambda^2(U(M_h^r))$ has the canonical representation*

$$\omega^{(2)}\Big|_{U(M_h^r)} = \sum_{j=1}^{r} d\bar{H}_j \wedge\ \bar{h}_j^{(1)} + \sum_{s=1}^{n-r} d\tilde{f}_s \wedge\ \tilde{h}_s^{(1)} \quad , \tag{3.104}$$

where $\bar{h}_j^{(1)}, \tilde{h}_s^{(1)} \in \Lambda^1(U(M_h^r))$, $1 \le j \le r$, $1 \le s \le n-r$.

The expression (3.104) obviously means that on $U(M_h^r) \subset M^{2n}$ the differential 1-forms $\bar{h}_j^{(1)}$, $\tilde{h}_s^{(1)}$, $\in \Lambda^1(U(M_h^r))$ are independent together with exact 1-forms $d\bar{H}_j$ and $d\tilde{f}_s$, for $1 \le j \le r,\ 1 \le s \le n-r$. As $d\omega^{(2)} = 0$ on M^{2n}, it follows from (3.104) that the differentials $d\bar{h}_j^{(1)}$, $d\tilde{h}_s^{(1)} \in \Lambda^2(U(M_h^r))$, $1 \le j \le r,\ 1 \le s \le n-r$ belong to the ideal $\mathcal{I}(\widetilde{\mathfrak{g}}_h) \subset \mathcal{I}(\mathfrak{g}^*)$ generated by exact forms $d\tilde{f}_s$ and $d\bar{H}_j$ for all regular $h \in \mathfrak{g}^*$. Thus, we have the elements of a proof of the Galissot–Reeb Theorem 3.9 discussed above.

Theorem 3.14. *Let a Lie algebra $\mathfrak{g}$ of invariants on the symplectic space M^{2n} be nonabelian and satisfy the Mishchenko–Fomenko condition (3.91). Then, at a regular element $h \in \mathfrak{g}^*$ on an open neighborhood $U(M_h^r)$ of the integral submanifold $M_h^r \subset M^{2n}$ there exist differential 1-forms $\bar{h}_j^{(1)}$, $1 \le j \le n$, and $\tilde{h}_s^{(1)}$, $1 \le s \le n-r$, such that*

i) $\omega^{(2)}|_{U(M_h^r)} = \sum_{j=1}^{r} d\bar{H}_j \wedge \bar{h}_j^{(1)} + \sum_{s=1}^{n-r} d\tilde{f}_s \wedge \tilde{h}_s^{(1)}$, *where $\bar{H}_j \in \mathfrak{g}$, $1 \le j \le r$, is a basis of the (abelian) Cartan subalgebra $\mathfrak{g}_h \subset \mathfrak{g}$, and $\tilde{f}_s \in \mathfrak{g}$, $1 \le s \le n-r$, are invariants from the complementary space $\widetilde{\mathfrak{g}}_h \simeq \mathfrak{g}/\mathfrak{g}_h$;*

ii) 1-forms $\bar{h}_j^{(1)} \in \Lambda^1(U(M_h^r))$, $1 \leq j \leq r$, and $\tilde{h}_s^{(1)} \in \Lambda^1(U(M_h^r))$, $1 \leq s \leq n-r$, are exact on M_h^r and satisfy the equations: $\bar{h}_j^{(1)}(\bar{K}_i) = \delta_{i,j}$ for all $1 \leq i, j \leq r$, $\bar{h}_j^{(1)}(\tilde{F}_s) = 0$ and $\tilde{h}_s^{(1)}(\bar{K}_j) = 0$ for all $1 \leq j \leq r$, $1 \leq s \leq n-r$, and $\tilde{h}_s^{(1)}(\tilde{F}_m) = \delta_{s,m}$ for all $1 \leq s, m \leq n-r$.

Proof. Obviously we need only prove statement ii). Making use of Theorem 3.14, one finds that the differential 2-forms $d\bar{h}_j^{(1)} \in \Lambda^2(U(M_h^r))$, $1 \leq j \leq r$, and $d\tilde{h}_s^{(1)} \in \Lambda^2(U(M_h^r))$, $1 \leq s \leq n-r$, vanish on the integral submanifold $M_h^r \subset M^{2n}$. Thus, in particular, owing to the Poincaré lemma [3, 14, 173, 344], there exist exact 1-forms $d\bar{t}_{h,j} \in \Lambda^1(U(M_h^r))$, $1 \leq j \leq r$, and $d\tilde{t}_{h,s} \in \Lambda^1(U(M_h^r))$, $1 \leq s \leq n-r$, where $\bar{t}_{h,j} : M_h^r \to \mathbb{R}$ and $\tilde{t}_{h,s} : M_h^r \to \mathbb{R}$ are smooth almost everywhere independent functions on M_h^r; they are single-valued on an appropriate covering of the manifold $M_h^r \subset M^{2n}$ and supply global coordinates on the integral submanifold M_h^r. Using the representation (3.104), one easily computes that

$$- i_{\bar{K}_i}\omega^{(2)}\Big|_{U(M_h^r)} = \sum_{j=1}^{r} d\bar{H}_j \bar{h}_j^{(1)}(\bar{K}_i) + \sum_{s=1}^{n-r} d\tilde{f}_s\ \tilde{h}_s^{(1)}(\bar{K}_i) = d\bar{H}_i \tag{3.105}$$

for all $1 \leq i \leq r$ and

$$- i_{\tilde{F}_m}\omega^{(2)}\Big|_{U(M_h^r)} = \sum_{j=1}^{r} d\bar{H}_j \bar{h}_j^{(1)}(\tilde{F}_m) + \sum_{s=1}^{n-r} d\tilde{f}_s \tilde{h}_s^{(1)}(\tilde{F}_m) = d\tilde{f}_m \tag{3.106}$$

for all $1 \leq m \leq n-r$. Whence, from (3.105) it follows that on $U(M_h^r)$,

$$\bar{h}_j^{(1)}(\bar{K}_i) = \delta_{i,j}, \qquad \tilde{h}_s^{(1)}(\bar{K}_i) = 0 \tag{3.107}$$

for all $1 \leq i, j \leq r$ and $1 \leq s \leq n-r$, and similarly, from (3.106) it follows that on $U(M_h^r)$,

$$\bar{h}_j^{(1)}(\ \tilde{F}_m) = 0, \qquad \tilde{h}_s^{(1)}(\tilde{F}_m) = 0 \tag{3.108}$$

for all $1 \leq j \leq r$ and $1 \leq s, m \leq n-r$. Thus, the theorem is proved. □

Using these global evolution parameters $t_j : M^{2n} \to \mathbb{R}$ for the corresponding vector fields $\bar{K}_j = d/dt_j$, $1 \leq j \leq r$, and local evolution parameters $\tilde{t}_s : M^{2n} \cap U(M_h^r) \to \mathbb{R}$, $1 \leq s = n-r$, of the corresponding vector fields $\tilde{F}_s\Big|_{U(M_h^r)} := d/d\tilde{t}_s$, $1 \leq s \leq n-r$, we see directly from (3.108) that

$$t_j|_{U(M_h^r)} = \bar{t}_j, \qquad \tilde{t}_s\big|_{U(M_h^r)} = \tilde{t}_{h,s} \tag{3.109}$$

hold for all $1 \leq j \leq r$, $1 \leq s \leq n-r$, up to constant normalizations. This can be used to develop a new method, similar to that devised above, for studying the integral submanifold embedding problem in the case of the nonabelian Liouville–Arnold integrability theorem.

Before starting this development, it is interesting to note that the system of invariants

$$\mathfrak{g}_\tau := \mathfrak{g}_h \oplus \mathrm{span}_{\mathbb{R}}\{\tilde{f}_s \in \mathfrak{g}/\mathfrak{g}_h : 1 \leq s \leq n-r\}$$

constructed above, comprises a new involutive (abelian) complete algebra $\mathfrak{g}_\tau$, to which one can apply the abelian Liouville–Arnold theorem on integrability by quadratures and the integral submanifold embedding theory devised above, in order to obtain exact algebraic-analytical solutions. This proves the following result.

Corollary 3.3. *Assume that a nonabelian Lie algebra $\mathfrak{g}$ satisfies the Mishchenko–Fomenko condition (3.91) and $M_h^r \subset M^{2n}$, its integral submanifold (compact and connected) at a regular element $h \in \mathfrak{g}^*$, is diffeomorphic to the standard torus $\mathbb{T}^r \simeq M_{h,\tau}^n$. Assume also that the dual complete abelian algebra $\mathfrak{g}_\tau$ $(\dim \mathfrak{g}_\tau = n = (1/2)\dim M^{2n})$ of independent invariants constructed above is globally defined. Then its integral submanifold $M_{h,\tau}^n \subset M^{2n}$ is diffeomorphic to the standard torus $\mathbb{T}^n \simeq M_{h,\tau}^n$, and contains the torus $\mathbb{T}^r \simeq M_h^r$ as a direct product with a completely degenerate torus $\mathbb{T}^{n-r}$, that is $M_{h,\tau}^n \simeq M_h^r \times \mathbb{T}^{n-r}$.*

Thus, having successfully applied the algorithm developed above to the algebraic-analytical characterization of integral submanifolds of a nonabelian Liouville–Arnold integrable Lie algebra $\mathfrak{g}$ of invariants on the canonically symplectic manifold $M^{2n} \simeq T^*(\mathbb{R}^n)$, one can produce a wide class of exact solutions represented by quadratures - which is just what we set out to find. At this point it is necessary to note that up to now the (dual to $\mathfrak{g}$) abelian complete algebra $\mathfrak{g}_\tau$ of invariants at a regular $h \in \mathfrak{g}^*$ was constructed only on some open neighborhood $U(M_h^r)$ of the integral submanifold $M_h^r \subset M^{2n}$. As we noted above, the global existence of the algebra $\mathfrak{g}_\tau$ strongly depends on the possibility of extending these invariants to the entire manifold M^{2n}. This possibility is in one-to-one correspondence with the existence of a global complex structure [396] on the reduced integral submanifold $\tilde{M}_{h,\tau}^{2(n-r)} := M_h^k/G_h$, induced by the reduced symplectic structure $\pi_\tau^*\, \omega^{(2)} \in \Lambda^2(M_h^k/G_h)$, where $\pi_\tau : M_h^k \to M^{2n}$ is the embedding for the integrable differential system $K(\mathfrak{g}) \subset \Gamma(M^{2n})$ introduced above. If this is the case , the resulting complexified manifold ${}^{\mathbb{C}}\tilde{M}_{h,\tau}^{n-r} \simeq \tilde{M}_{h,\tau}^{2(n-r)}$ is endowed with a Kählerian structure, which makes it possible to produce the dual abelian algebra $\mathfrak{g}_\tau$ as a globally defined set of invariants on M^{2n}. This problem will be analyzed in more detail in subsequent sections.

3.5 Examples

We now consider some examples of nonabelian Liouville–Arnold integrability by quadratures covered by Theorem 3.7.

Example 3.1. Point vortices in the plane.

Consider $n \in \mathbb{Z}_+$ point vortices in the plane $\mathbb{R}^2$, described by the Hamiltonian function

$$H = -\frac{1}{2\pi} \sum_{i \neq j = 1}^{n} \xi_i \xi_j \ln |q_i - p_j| \tag{3.110}$$

with respect to the following partially canonical symplectic structure on $M^{2n} \simeq T^*(\mathbb{R}^n)$:

$$\omega^{(2)} = \sum_{j=1}^{n} \xi_j dp_j \wedge dq_j, \tag{3.111}$$

where $(p_j, q_j) \in \mathbb{R}^2$, $1 \leq j \leq n$, are coordinates of the vortices in $\mathbb{R}^2$. There exist three additional invariants

$$P_1 = \sum_{j=1}^{n} \xi_j q_j, \qquad P_2 = \sum_{j=1}^{n} \xi_j p_j, \tag{3.112}$$

$$P = \frac{1}{2} \sum_{j=1}^{n} \xi_j (q_j^2 + p_j^2),$$

satisfying the Poisson bracket conditions

$$\{P_1, P_2\} = -\sum_{j=1}^{n} \xi_j \ , \quad \{P_1, P\} = -P_2 \ , \{P_2, P\} = P_1 \ , \tag{3.113}$$

$$\{P, H\} = \ 0 \ = \ \{P_j, H\}.$$

It is evident that invariants (3.110) and (3.112) comprise on $\sum_{j=1}^{n} \xi_j = 0$ a four-dimensional Lie algebra $\mathfrak{g}$ with rank $\mathfrak{g} = 2$. Indeed, assume a regular vector $h \in \mathfrak{g}^*$ is chosen, and parametrized by real values $h_j \in \mathbb{R}$, $1 \leq j \leq 4$, where

$$h(P_i) = h_i, \ \ h(P) = h_3, \ \ h(H) = h_4, \tag{3.114}$$

and $i = 1, 2$. Then, one can easily verify that the element

$$Q_h = (\sum_{j=1}^{n} \xi_j) P - \sum_{i=1}^{n} h_i P_i \tag{3.115}$$

belongs to the Cartan Lie subalgebra $\mathfrak{g}_h \subset \mathfrak{g}$, that is

$$h(\{Q_h, P_i\}) = 0 = h(\{Q_h, P\}). \tag{3.116}$$

Since $\{Q_h, H\} = 0$ for all values $h \in \mathfrak{g}^*$, we claim that $\mathfrak{g}_h = \mathrm{span}_{\mathbb{R}}\{H, Q_h\}$ - the Cartan subalgebra of $\mathfrak{g}$. Thus, rank$\mathfrak{g}$= $\dim \mathfrak{g}_h = 2$, and it follows easily that the condition (3.91)

$$\dim M^{2n} = 2n = rank\mathfrak{g} + \dim \mathfrak{g} = 6 \tag{3.117}$$

holds only if $n = 3$. Consequently, the following result is proved.

Theorem 3.15. *The three-vortex problem (3.110) in $\mathbb{R}^2$ is nonabelian Liouville–Arnold integrable by quadratures on the phase space $M^6 \simeq T^*(\mathbb{R}^3)$ with the symplectic structure (3.111).*

As a result, the corresponding integral submanifold $M_h^2 \subset M^6$ is two-dimensional and diffeomorphic (when compact and connected) to the torus $\mathbb{T}^2 \simeq M_h^2$, on which the motions are quasiperiodic functions of the evolution parameter.

In the matter of Corollary 3.3, the dynamical system (3.110) is also abelian Liouville–Arnold integrable with an extended integral submanifold $M_{h,\tau}^3 \subset M^6$, which can be found via the scheme described in this chapter. Using simple calculations, one obtains an additional invariant $Q = (\sum_{j=1}^3 \xi_j)P - \sum_{i=1}^3 P_i^2 \notin \mathfrak{g}$, which commutes with H and P of $\mathfrak{g}_h$. Therefore, there exists a new complete dual abelian algebra $\mathfrak{g}_\tau = \mathrm{span}_{\mathbb{R}}$ $\{Q, P, H\}$ of independent invariants on M^6 with $\dim \mathfrak{g}_\tau = 3 = 1/2 \dim M^6$, whose integral submanifold $M_{h,\tau}^3 \subset M^6$ (when compact and connected) is diffeomorphic to the torus $\mathbb{T}^3 \simeq M_h^2 \times \mathbb{S}^1$.

Note also that the invariant $Q \in \mathfrak{g}_\tau$ can be naturally extended to the case of an arbitrary number $n \in \mathbb{Z}_+$ of vortices as follows: $Q = (\sum_{j=1}^n \xi_j)P - \sum_{i=1}^n P_i^2 \in \mathfrak{g}_\tau$, which obviously also commutes with invariants (3.110) and (3.112) on the entire phase space M^{2n}.

Example 3.2. A material point motion in a central potential field.

Consider the motion of a material point in the Euclidean space $\mathbb{E}^3 := (\mathbb{R}^3, <\cdot,\cdot>)$ under a central potential field whose Hamiltonian

$$H = \frac{1}{2}\sum_{j=1}^{3} p_j^2 + Q(\|q\|), \tag{3.118}$$

contains a central field $Q : \mathbb{R}_+ \to \mathbb{R}$. The motion takes place in the canonical phase space $M^6 = T^*(\mathbb{R}^3)$, and possesses three additional invariants:

$$P_1 = p_2q_3 - pq, \qquad P_2 = p_3q_1 - p_1q_3, \quad P_3 = p_1q_2 - p_2q_1, \tag{3.119}$$

satisfying the following Poisson bracket relations:

$$\{P_1, P_2\} = P_3 \ , \quad \{P_3, P_1\} = P_2 \ , \ \{P_2, P_3\} = P_1. \tag{3.120}$$

Since $\{H, P_j\} = 0$ for all $j = 1, 2, 3$, the problem under consideration has a four-dimensional Lie algebra $\mathfrak{g}$ of invariants, isomorphic to the classical rotation Lie algebra $\mathrm{so}(3) \times \mathbb{R} \simeq \mathfrak{g}$. Let us show that at a regular element $h \in \mathfrak{g}^*$, the Cartan subalgebra $\mathfrak{g}_h \subset \mathfrak{g}$ has the dimension $\dim \mathfrak{g}_h = 2 = \mathrm{rank}\mathfrak{g}$. Indeed, one easily verifies that the invariant

$$P_h = \sum_{j=1}^{3} h_j P_j \tag{3.121}$$

belongs to the Cartan subalgebra $\mathfrak{g}_h$, that is

$$\{H, P_h\} = 0, \qquad h(\{P_h, P_j\}) = 0 \tag{3.122}$$

for all $j = 1, 2, 3$. Thus, as the Cartan subalgebra $\mathfrak{g}_h = \mathrm{span}_{\mathbb{R}}\{H$ and $P_h \subset \mathfrak{g}\}$, one computes that $\dim \mathfrak{g}_h = 2 = \mathrm{rank}\mathfrak{g}_h$, and the Mishchenko–Fomenko condition (3.6)

$$\dim M^6 = 6 = \mathrm{rank}\mathfrak{g} + \dim \mathfrak{g} = 4 + 2 \tag{3.123}$$

holds. Hence, one can prove its integrability by quadratures via the nonabelian Liouville–Arnold Theorem 3.7 and obtain the following result.

Theorem 3.16. *It follows from Theorem 3.7 that the free material point motion in $\mathbb{R}^3$ is a completely integrable by quadratures dynamical system on the canonical symplectic phase space $M^6 = T^*(\mathbb{R}^3)$. The corresponding integral submanifold $M_h^2 \subset M^6$ at a regular element $h \in \mathfrak{g}^*$ (if compact and connected) is two-dimensional and diffeomorphic to the standard torus $\mathbb{T}^2 \simeq M_h^2$.*

Making use of the integration algorithm devised above, one can readily obtain the corresponding integral submanifold embedding $\pi_h : M_h^2 \to M^6$ by means of algebraic-analytical expressions and transformations.

There are clearly many other interesting nonabelian Liouville–Arnold integrable Hamiltonian systems on canonically symplectic phase spaces that arise in applications that can similarly be integrated using algebraic-analytical methods of the type describes in this chapter.

3.6 Existence problem for a global set of invariants

We proved above that in an open neighborhood $U(M_h^r) \subset M^{2n}$ of the integral submanifold $M_h^r \subset M^{2n}$ one can find by algebraic-analytical means just $n-r$ independent vector fields $\tilde{F},\ _j \in K(\mathfrak{g})/K(\mathfrak{g}_h) \cap \Gamma(U(M_h^r)$ satisfying the condition (3.93). Since each vector field $\tilde{F}_j \in K(\mathfrak{g})/K(\mathfrak{g}_h)$ is generated by an invariant $\tilde{H}_j \in \mathcal{D}(U(M_h^r)$, $1 \leq j \leq n-r$, it follows readily from (3.93) that

$$\{\tilde{H}_i, \tilde{H}_j\} = 0 \tag{3.124}$$

for all $1 \leq i, j \leq n-r$. Thus, on an open neighborhood $U(M_h^r)$ there exist just $n-r$ invariants in addition to $\tilde{H}_j \in \mathcal{D}(U(M_h^r))$, $1 \leq j \leq n-r$, all of which are in involution. We denote, as above, this new set of invariants as $\mathfrak{g}_\tau$, keeping in mind that $\dim \mathfrak{g}_\tau = r + (n-r) = n \in \mathbb{Z}_+$. Whence, on $U(M_h^r) \subset M^{2n}$ we have now the set $\mathfrak{g}_\tau$ of $n = (1/2)\dim M^{2n}$ invariants commuting with each other, thereby guaranteeing via the abelian Liouville–Arnold theorem its local complete integrability by quadratures. Consequently, there exists locally a mapping $\pi_\tau : M_{h,\tau}^k \to M^{2n}$, where $M_{h,\tau}^k := U(M_h^r) \cap M_\tau^k$ is the integral submanifold of the differential system $K(\mathfrak{g})$, and one can therefore describe the behavior of integrable vector fields on the reduced manifold $\bar{M}_{h,\tau}^{2(n-r)} := M_{h,\tau}^{k-r}/G_h$.

For global integrability properties of a given set $\mathfrak{g}$ of invariants on $(M^{2n}, \omega^{(2)})$, satisfying the Mishchenko–Fomenko condition (3.6), it is necessary to have the additional set of invariants $\tilde{H}_j \in \mathcal{D}(\ U(M_h^r))$, $1 \leq j \leq n-r$, extended from $U(M_h^r)$ to the entire phase space M^{2n}. This problem depends on the existence of extensions of vector fields $\tilde{F}_j \in \Gamma(U(M_h^r))$, $1 \leq j \leq n-r$, from the neighborhood $U(M_h^r) \subset M^{2n}$ to M^{2n}. On the other hand, as stated above, the existence of such extensions depends intimately on the properties of the complexified differential system $K^{\mathbb{C}}(\mathfrak{g})/K^{\mathbb{C}}(\mathfrak{g}_h)$, which has a nondegenerate complex metric $\omega(\tilde{K}^{\mathbb{C}}) : T(\bar{M}_{h,\tau}^{2(n-r)})^{\mathbb{C}} \times T(\bar{M}_{h,\tau}^{2(n-r)})^{\mathbb{C}} \to \mathbb{C}$, induced by the symplectic structure $\omega^{(2)} \in \Lambda^2(M^{2n})$. This point can be clarified by using the notion [3, 393, 396] of a Kähler manifold and some of its associated constructions. Namely, consider the local isomorphism $T(\bar{M}_{h,\tau}^{2(n-r)})^{\mathbb{C}} \simeq T({}^{\mathbb{C}}\bar{M}_{h,\tau}^{n-r})$, where ${}^{\mathbb{C}}\bar{M}_{h,\tau}^{n-r}$ is the complex $(n-r)$-dimensional local integral submanifold of the complexified differential system $K^{\mathbb{C}}(\mathfrak{g})/K^{\mathbb{C}}(\mathfrak{g}_h)$. This space $T(\bar{M}_{h,\tau}^{2(n-r)})$ is endowed with the standard almost complex structure

$$J : T(\bar{M}_{h,\tau}^{2(n-r)}) \to T(\bar{M}_{h,\tau}^{2(n-r)}), \quad J^2 = -1, \tag{3.125}$$

and it follows from our analysis that to extend the fields requires that the 2-form $\omega(\tilde{K}) := \operatorname{Im}\omega(\tilde{K}^{\mathbb{C}}) \in \Lambda^2(\bar{M}_{h,\tau}^{2(n-r)})$ induced from the above metric on $T(^{\mathbb{C}}\bar{M}_{h,\tau}^{n-r})$ be closed, that is $d\omega(\tilde{K}) = 0$. If this is the case, the almost complex structure on the manifold $T(\bar{M}_{h,\tau}^{2(n-r)})$ is said to be *integrable.* Define the proper complex manifold $^{\mathbb{C}}\bar{M}_{h,\tau}^{n-r}$, on which one can then define global vector fields $\tilde{F}_j \in K(\mathfrak{g})/K(\mathfrak{g}_h),\ 1 \leq j \leq n-r$, which we need for the involutive algebra $\mathfrak{g}_\tau$ of invariants on M^{2n} to be integrable by quadratures via the abelian Liouville–Arnold theorem. In summary, we have proved the following result.

Theorem 3.17. *A nonabelian set $\mathfrak{g}$ of invariants on the symplectic space $M^{2n} \simeq T^*(R^n)$, satisfying the Mishchenko–Fomenko condition (3.91), admits algebraic-analytical integration by quadratures for the integral submanifold embedding $\pi_h : M_h^r \to M^{2n}$ if the corresponding complexified reduced manifold $^{\mathbb{C}}\bar{M}_{h,\tau}^{n-r} \simeq \bar{M}_{h,\tau}^{2(n-r)} = M_{h,\tau}^{k-r}/G_h$ of the differential system $K^{\mathbb{C}}(\mathfrak{g})/K^{\mathbb{C}}(\mathfrak{g}_h)$ is Kählerian with respect to the standard almost complex structure (3.125) and the nondegenerate complex metric $\omega(\tilde{K}^{\mathbb{C}}) : T(\bar{M}_{h,\tau}^{2(n-r)})^{\mathbb{C}} \times T(\bar{M}_{h,\tau}^{2(n-r)})^{\mathbb{C}} \to \mathbb{C}$ induced by the symplectic structure $\omega^{(2)} \in \Lambda^2(M^{2n})$ is integrable, that is $d\operatorname{Im}\omega(\tilde{K}^{\mathbb{C}}) = 0$.*

Theorem 3.17 shows, in particular, that nonabelian Liouville–Arnold integrability by quadratures does not in general imply integrability via the abelian Liouville–Arnold theorem; it actually depends on certain topological obstructions associated with the Lie algebra structure of invariants $\mathfrak{g}$ on the phase space M^{2n}. Sorting this out is an intriguing problem that definitely warrants further study.

3.7 Additional examples

Finally we consider further applications of the integral submanifold embedding for abelian Liouville–Arnold integrable Hamiltonian systems on $T^*(\mathbb{R}^2)$.

3.7.1 *The Henon–Heiles system*

This flow is induced by the Hamiltonian

$$H_1 = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + q_1 q_2^2 + \frac{1}{3}q_1^3 \tag{3.126}$$

on the phase space $M^4 = T^*(\mathbb{R}^2)$ with the symplectic structure

$$\omega^{(2)} = \sum_{j=1}^{2} dp_j \wedge dq_j. \tag{3.127}$$

There exists the following additional invariant that commutes with that given by (3.126):

$$H_2 = p_1 p_2 + 1/3 q_2^3 + q_1^2 q_2, \tag{3.128}$$

that is $\{H_1, H_2\} = 0$ on the entire space M^4.

Take a regular element $h \in \mathfrak{g} := \{H_j : \ M^4 \to \mathbb{R}: \ \ j = 1, 2\}$ with fixed values $h(H_j) = h_j \in \mathbb{R}$, $j = 1, 2$. Then the integral submanifold

$$M_h^2 := \{(q, p) \in M^4 : h(H_j) = h_j \in \mathbb{R}, \ j = 1, 2\}, \tag{3.129}$$

if compact and connected, is diffeomorphic to the standard torus $\mathbb{T}^2 \simeq \mathbb{S}^1 \times \mathbb{S}^1$ owing to the Liouville–Arnold theorem, and one can find cyclic (separable) coordinates $\mu_j \in \mathbb{S}^1, \ \ j = 1, 2$, on the torus such that the symplectic structure (3.127) takes the form

$$\omega^{(2)} = \sum_{j=1}^{2} dw_j \wedge d\mu_j \ , \tag{3.130}$$

where the conjugate variables $w_j \in T^*(\mathbb{S}^1)$ on M_h^2 depend only on the corresponding variables $\mu_j \in \mathbb{S}_j^1, \ \ j = 1, 2$. In this case it is evident that the evolution along M_h^2 is separable and representable by means of quasiperiodic functions of the evolution parameters.

To show this, recall that the fundamental determining equations (3.26), based on the 1-forms $\bar{h}_j^{(1)} \in \Lambda(M_h^2)$, $j = 1, 2$, satisfy

$$\sum_{j=1}^{2} dH_j \wedge_j \bar{h}_j^{(1)} = \sum_{j=1}^{2} dp_j \wedge dq_j \ . \tag{3.131}$$

Here

$$\bar{h}_j^{(1)} = \sum_{k=1}^{2} \bar{h}_{jk}(q, p) dq_k, \tag{3.132}$$

where $j = 1, 2$. Substituting (3.132) into (3.131), one computes that

$$\bar{h}_1^{(1)} = \frac{p_1 dq_1}{p_1^2 - p_2^2} + \frac{p_2 dq_2}{p_1^2 - p_2^2}, \ \ \bar{h}_2^{(1)} = \frac{p_2 dq_1}{p_{21}^2 - p_1^2} + \frac{p_1 dq_2}{p_1^2 - p_2^2} \ . \tag{3.133}$$

In addition, the following implication holds on $M_h^2 \subset M^4$:

$$\alpha_h^{(1)} = \sum_{j=1}^{2} w_j(\mu_j;h)d\mu_j \Rightarrow \sum_{j=1}^{2} p_j dq_j \; := \alpha^{(1)}, \tag{3.134}$$

where we have assumed that the integral submanifold M_h^2 admits the local coordinates in the base manifold $\mathbb{R}^2$ endowed with the canonical 1-form $\alpha_h^{(1)} \in \Lambda(M_h^2)$ as given in (3.131). Thus, making use of the embedding $\pi_h : M_h^2 \to T^*(\mathbb{R}^2)$ in the form

$$q_j = q_j(\mu;h) \ , \quad p_j = p_j(\mu;h) \ , \tag{3.135}$$

$j = 1, 2$, one readily finds that

$$p_j = \sum_{k=1}^{2} w_k(\mu_k;h) \;\; \partial\mu_k/\partial q_j \tag{3.136}$$

hold for $j = 1, 2$ on the entire integral submanifold M_h^2.

Substituting (3.133) into (3.130) and using the characteristic relationships (3.77), one obtains after simple but lengthy calculations the following differential-algebraic expressions:

$$\partial q_1/\partial\mu_1 - \partial q_2/\partial\mu_1 = 0, \;\; \partial q_1/\partial\mu_2 + \partial q_2/\partial\mu_2 = 0, \tag{3.137}$$

with simplest solutions

$$q_1 = (\mu_1 + \mu_2)/2 \ , \quad q_2 = (\mu_1 - \mu_2)/2 \ . \tag{3.138}$$

Using expressions (3.133), one finds that

$$p_1 = w_1 + w_2 \ , \quad p_2 = w_1 - w_2 \quad , \tag{3.139}$$

where

$$w_1 = \sqrt{h_1 + h_2 - 4/3\mu_1^3}, \qquad w_2 = \sqrt{h_1 - h_2 - 4/3\mu_2^3}. \tag{3.140}$$

Consequently, it is easy to obtain the separable [14, 54, 341] Hamiltonian functions (3.126) and (3.128) in a neighborhood of the cotangent space $T^*(M_h^2)$:

$$h_1 = \frac{1}{2}w_1^2 + \frac{1}{2}w_2^2 + \frac{2}{3}(\mu_1^3 + \mu_2^3), \;\; h_2 = \frac{1}{2}w_1^2 - \frac{1}{2}w_2^2 + \frac{2}{3}(\mu_1^3 - \mu_2^3), \tag{3.141}$$

which generate the separable motions on $M_h^2 \subset T^*(\mathbb{R}^2)$

$$d\mu_1/dt := \partial h_1/\partial w_1 = \sqrt{h_1 + h_2 - 4/3\mu_1^3}, \tag{3.142}$$

$$d\mu_2/dt := \partial h_1/\partial w_2 = \sqrt{h_1 - h_2 - 4/3\mu_2^3}$$

for the Hamiltonian (3.126), and

$$d\mu_1/dx := \partial h_2/\partial w_1 = \sqrt{h_1 + h_2 - 4/3\mu_1^3}, \tag{3.143}$$

$$d\mu_2/dt := \partial h_1/\partial w_2 = -\sqrt{h_1 - h_2 - 4/3\mu_2^3}$$

for the Hamiltonian (3.128), where $x, t \in \mathbb{R}$ are the corresponding evolution parameters.

Analogously, one can show that there exists [310, 341] an integral submanifold embedding similar to (3.135) and (3.136) for the following integrable modified Henon–Heiles involutive system:

$$H_1 = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + q_1 q_2^2 + \frac{16}{3}q_1^3, \tag{3.144}$$

$$H_2 = 9p_2^4 + 36q_1 p_2^2 q_2^2 - 12p_1 p_2 q_2^3 - 2q_2^4(q_2^2 + 6q_1^2) \quad ,$$

where $\{H_1, H_1\} = 0$ on the entire phase space $M^4 = T^*(\mathbb{R}^2)$.

Using reasoning similar to the above, it is easy to deduce that

$$q_1 = -\frac{1}{4}(\mu_1 + \mu_2) - \frac{3}{8}(\frac{w_1 + w_2}{\mu_1 - \mu_2})^2, \tag{3.145}$$

$$q_2^2 = -2\sqrt{h_2}/(\mu_1 - \mu_2), \quad w_1 = \sqrt{2/3\mu_1^3 - 4/3\sqrt{h_2} - 8h_1} \ ,$$

$$p_1 = \frac{1}{2\sqrt{-6(\mu_1 + \mu_2 + 4q_{1)}}}[\frac{-2\sqrt{h_2}}{\mu_1 - \mu_2} - \mu_1\mu_2 + 4(\mu_1 + \mu_2)q_1 + 32q_1^2],$$

$$p_2 = \sqrt{h_2}(\mu_{1+}\mu_2 + 4q_1)/(3(\mu_{1-}\mu_2)) \ , \ \ w_2 = \sqrt{2/3\mu_2^3 + 4/3\sqrt{h_2} - 8h_1} \ ,$$

thereby solving the problem of finding the corresponding integral submanifold embedding $\pi_h : M_h^2 \ \rightarrow T^*(\mathbb{R}^2)$ that generates separable flows in the variables $(\mu \ , w) \in T^*(M_h^2)$.

3.7.2 *A truncated four-dimensional Fokker–Planck Hamiltonian system*

Consider the following dynamical system, called a truncated four-dimensional Fokker–Planck flow on the phase space $T^*(\mathbb{R}^2)$:

$$\frac{d}{dt}\begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} = K_1\,(q,p) := \begin{pmatrix} p_1 + \alpha(q_1 + p_2)(q_2 + p_1) \\ p_2 \\ -(q_1 + p_2) - \alpha\left[q_2 p_1 + (1/2)(p_1^2 + p_2^2 + q_2^2)\right] \\ -(q_2 + p_1) \end{pmatrix}, \tag{3.146}$$

where $K_1 : T^*(\mathbb{R}^2) \rightarrow T(T^*(\mathbb{R}^2))$ is the corresponding vector field on $T^*(\mathbb{R}^2) \ni (q,p)$, and $t \in \mathbb{R}$ is an evolution parameter. It is easy to verify that the functions $H_j : T^*(\mathbb{R}^2) \rightarrow \mathbb{R}$, $j = 1,2$, where

$$H_1 = 1/2(p_1^2+p_2^2+q_1^2)+q_1p_2+\alpha(q_1+p_2)[q_2p_1+1/2(p_1^2+p_2^2+q_2^2)] \quad (3.147)$$

and

$$H_2 = 1/2(p_1^2 + p_2^2 + q_2^2) + q_2p_1 \quad (3.148)$$

are functionally independent invariants with respect to the flow (3.91). Moreover, the invariant (3.147) is the Hamiltonian function for (3.146); that is

$$i_{K_1}\omega^{(2)} = -dH_1 \quad (3.149)$$

holds on $T^*(\mathbb{R}^2)$, where the symplectic structure $\omega^{(2)} \in \Lambda^2(T^*(\mathbb{R}^2))$ is given as follows:

$$\omega^{(2)} := d(pr^*\alpha^{(1)}) = \sum_{j=1}^{2} dp_j \wedge dq_j, \quad (3.150)$$

with $\alpha^{(1)} \in \Lambda^1(\mathbb{R}^2)$ the canonical Liouville form on $\mathbb{R}^2$:

$$\alpha^{(1)}(q;p) = \sum_{j=1}^{2} p_j dq_j \quad (3.151)$$

for any $(q,p) \in T^*(\mathbb{R}^2) \simeq \Lambda^1(\mathbb{R}^2)$.

The invariants (3.147) and (3.148) are obviously mutually commutative with respect to the Poisson bracket on $T^*(\mathbb{R}^2)$:

$$\{H_1, H_2\} = 0. \quad (3.152)$$

Therefore, it follows from the abelian Liouville–Arnold theorem [14, 3, 173] that the dynamical system (3.146) is completely integrable by quadratures on $T^*(\mathbb{R}^2)$, and we can apply the scheme above to the commuting invariants (3.147) and (3.148) for the symplectic structure (3.150). It is easy to verify that

$$\omega^{(2)} = \sum_{i=1}^{2} dH_i \wedge h_i^{(1)}, \quad (3.153)$$

where the corresponding 1-forms $\pi_h^* h_i^{(1)} := \bar{h}_i^{(1)} \in \Lambda^1(M_h^2)$, $i = 1,2$, are

$$\begin{aligned} \bar{h}_1^{(1)} &= \frac{p_2dq_1-(p_1+q_2)dq_2}{p_1p_2-(p_1+q_2)(q_1+p_2)-\alpha h_2(p_1+q_2)}, \\ \bar{h}_2^{(1)} &= \frac{-[(q_1+p_2)(1+\alpha p_2)+\alpha h_2]dq_1+(p_1+\alpha[h_2+(q_2+p_1)(q_1+p_2)])dq_2}{p_1p_2-(q_2+p_1)(\alpha h_2+\ q_1+p_2)}, \end{aligned} \quad (3.154)$$

and an invariant submanifold $M_h^2 \subset T^*(\mathbb{R}^2)$ is defined as

$$M_h^2 := \{(q,p) \in T^*(\mathbb{R}^2)\ :\ H_i(q,p) = h_i \in \mathbb{R}, i = 1,2\} \tag{3.155}$$

for parameters $h \in \mathbb{R}^2$. Using expressions (3.155) and (3.60), one can readily construct functions $\bar{P}_{ij}(w;h)$, $i,j = 1,2$, in (3.29), defined on $T^*(M_h^2) \simeq T^*(\otimes_{j=1}^2 \mathbb{S}_j^1)$ for the integral submanifold embedding map $\pi_h : M_h^2 \to T^*(\mathbb{R}^2)$ in coordinates $\mu \in \mathbb{S}_1^1 \otimes \mathbb{S}_2^1 \subset \Gamma_h^{(1)} \otimes \Gamma_h^{(2)}$, which we shall not express in detail owing to their length. Applying the criterion (3.31), we obtain the following compatibility relationships for the maps $q : (\mathbb{S}_1^1 \otimes \mathbb{S}_2^1) \times \mathbb{R}^2 \to \mathbb{R}^2$ and $p : (\mathbb{S}_1^1 \otimes \mathbb{S}_2^1) \times \mathbb{R}^2 \to T_q^*(\mathbb{R}^2)$:

$$\begin{aligned}
&\partial q_1/\partial \mu_1 - \partial q_2/\partial \mu_2 = 0,\ w_1 \partial \mathcal{L}_\mu/\partial w_1 - w_2 \partial \mathcal{L}_\mu/\partial w_2 = 0,\\
&\partial^2 q_1/\partial \mu_2 \partial h_2 + \partial^2 w_2/\partial \mu_2 \partial h_2 = 0,\\
&\partial w_1/\partial h_1 (\partial q_1/\partial h_1) = \partial w_2/\partial h_1 (\partial q_2/\partial h_1),\\
&w_1 \partial w_1/\partial h_1 - w_2 \partial w_2/\partial h_2 = 0,\\
&\partial(w_1 \partial w_1/\partial h_2)/\partial h_2 - \alpha^2 \partial q_1/\partial \mu_1 = 0, ...
\end{aligned} \tag{3.156}$$

and so on, in the variables $\mu \in \mathbb{S}_1^1 \otimes \mathbb{S}_2^1$ and $h \in \mathbb{R}^2$. Solving equations (3.156), we immediately obtain

$$\begin{aligned}
&p_1 = w_1, \quad p_2 = w_2,\\
&q_1 = c_1 + \mu_1 - w_2(\mu_2;h),\\
&q_2 = c_2 + \mu_2 - w_1(\mu_1;h),\\
&\mathcal{L}_\mu(h) = -w_1 w_2,
\end{aligned} \tag{3.157}$$

where $c_j(h_1,h_2) \in \mathbb{R}^1$, $j = 1,2$, are constant, hold on $T^*(M_h^2)$, giving rise to the Picard–Fuchs equations in the form

$$\begin{aligned}
&\partial w_1(\mu_1;h)/\partial h_1 = 1/w_1,\\
&\partial w_1(\mu_1;h)/\partial h_2 = \alpha^2 h_2/w_1,\\
&\partial w_2(\mu_2;h)/\partial h_1 = 0\\
&\partial w_2(\mu_2;h)/\partial h_2 = 1/w_2.
\end{aligned} \tag{3.158}$$

The Picard–Fuchs equations (3.158) can be easily integrated by quadratures as

$$\begin{aligned}
&w_1^2 + k_1(\mu_1) - \alpha^2 h_2 - 2h_1 = 0,\\
&w_2^2 + k_2(\mu_2) - 2h_2 = 0,
\end{aligned} \tag{3.159}$$

where $k_j : \mathbb{S}^1_j \to \mathbb{C}$, $j = 1, 2$, are still unknown functions. For them to be determined, it is necessarily to substitute (3.157) into expressions (3.147) and (3.148), making use of (3.159), which amounts to the following result:

$$k_1 = \mu_1^2, \quad k_2 = \mu_2^2 \tag{3.160}$$

under the condition that $c_1 = -\alpha h_2$, $c_2 = 0$. Thus, we have constructed owing to (3.159) the corresponding algebraic curves $\Gamma_h^{(j)}$, $j = 1, 2$, (3.63) in the explicit form:

$$\begin{aligned} \Gamma_h^{(1)} &:= \{(\lambda, w_1) : w_1^2 + \lambda^2 - \alpha^2 h_2^2 - 2h_1 = 0\}, \\ \Gamma_h^{(2)} &:= \{(\lambda, w_2) : w_2^2 + \lambda^2 - 2h_2 = 0\}, \end{aligned} \tag{3.161}$$

where $(\lambda, w_j) \in \mathbb{C} \times \mathbb{C}$, $j = 1, 2$, and $h \in \mathbb{R}^2$ are arbitrary parameters. Using expressions (3.162) and (3.157), one can construct in explicit form the integral submanifold embedding map $\pi_h : M_h^2 \to T^*(\mathbb{R}^2)$ for the flow (3.146):

$$\begin{aligned} q_1 &= \mu_1 - \sqrt{2h_2 - \mu_2^2} - \alpha h_2^2, \quad p_1 = w_1(\mu_1; h), \\ q_2 &= \mu_2 - \sqrt{2h_1 - \alpha^2 h_2^2 - \mu_1^2}, \quad p_2 = w_2(\mu_2; h), \end{aligned} \tag{3.162}$$

where $(\mu, w) \in \Gamma_h^{(1)} \otimes \Gamma_h^{(2)}$. Now, as mentioned above, formulas (3.162) together with the expressions (3.158) imply directly that the truncated Fokker–Planck flow (3.146) is integrable by quadratures.

Chapter 4

Infinite-dimensional Dynamical Systems

4.1 Preliminary remarks

Let M be an infinite-dimensional manifold, which in many cases is a subspace of the smooth vector-function space $C^{(\infty)}(S;\mathbb{R}^m)$ on a closed set $S \subset \mathbb{R}^n$ where $n, m \in \mathbb{Z}_+$ are positive integers.

As in the case of finite-dimensional manifolds a one-parameter group $\{\varphi^t\}$, $t \in \mathbb{R}$ of the phase space M of automorphisms of the infinite-dimensional manifold M is called the *dynamical system* on M.

Some of the general results concerning the dynamical systems on the finite-dimensional phase spaces mentioned in Section 1.1 carry over readily to the infinite-dimensional case. However, the theories of ergodicity and complete integrability of infinite-dimensional dynamical systems are far more complicated than in the finite-dimensional case and so are not nearly satisfactorily completed.

An infinite-dimensional dynamical system on a function space can be written locally in the form (usually a partial differential equation)

$$du/dt = K[u], \tag{4.1}$$

where $u \in M$, and $K : M \to T(M)$ is locally Fréchet smooth on M [14].

In general, $K : M \to T(M)$ can be represented as $K : J^{(p)}(S;\mathbb{R}^m) \to T(J^{(p)}(S;\mathbb{R}^m))$, where K is a smooth map of the jet space $J^{(p)}(S;\mathbb{R}^m)$ [132, 114] into $T(J^{(p)}(S;\mathbb{R}^m))$.

4.2 Implectic operators and dynamical systems

The construction of a Hamiltonian formalism for infinite-dimensional dynamical systems is intrinsically more difficult than for finite-dimensional systems. The difference is most pronounced when determining the symplectic structure $\omega^{(2)}$ on the cotangent bundle $T^*(M)$ for M as well as when treating vector fields that determine the flow (4.1). The problem of constructing an invariant measure μ for the flow (4.1) is also [121] very complicated. Later in the sequel, in fact, we shall find the Poisson formalism to be more convenient and tractable for analyzing infinite-dimensional Hamiltonian dynamical systems.

Consider the bilinear form

$$(a, b) = \int\limits_S \langle a, b\rangle dx, \tag{4.2}$$

where $a \in T^*(M),\ b \in T(M)$, which defines an inner (scalar) product structure on $T^*(M) \times T(M)$.

An operator $\vartheta : T^*(M) \to T(M)$ is called *skew-symmetric* with respect to the bilinear form (4.2) if the following holds for all $(a, b) \in T^*(M) \times T^*(M)$:

$$(a, \vartheta b) = -(\vartheta a, b). \tag{4.3}$$

Let $\mathcal{D}(M)$ be the Fréchet smooth functional space on M. We define for any $F \in \mathcal{D}(M)$ the operator grad : $\mathcal{D}(M) \to T^*(M)$ by formula $\operatorname{grad} F = F'^* \cdot 1$. In particular,

$$\operatorname{grad} F = \delta F/\delta u,$$

where $F \in \mathcal{D}(J^{(p)}(S; \mathbb{R}^n))$ and $\delta/\delta u$ is the Euler variational derivative.

For any $F, G \in \mathcal{D}(M)$ their *Poisson bracket* is the functional $\{F, G\}_\vartheta \in \mathcal{D}(M)$ determined by the formula

$$\{F, G\}_\vartheta = (\operatorname{grad} F, \vartheta \operatorname{grad} G). \tag{4.4}$$

It obviously follows from (4.3) that for all $F, G \in \mathcal{D}(M)$ we have

$$\{F, G\}_\vartheta = -\{G, F\}_\vartheta. \tag{4.5}$$

If, in addition, the operator ϑ introduced above satisfies

$$(a, (\vartheta\,'\vartheta b)c) + (b, (\vartheta\,'\vartheta c)a) + (c, (\vartheta\,'\vartheta a)b) = 0 \tag{4.6}$$

for any a, b and $c \in T(M)$, and a fortiori the Jacobi identity

$$\{F, \{G, H\}_\vartheta\}_\vartheta + \{G, \{H, F\}_\vartheta\}_\vartheta + \{H, \{F, G\}_\vartheta\}_\vartheta = 0 \tag{4.7}$$

for all $F, G, H \in \mathcal{D}(M)$, it is called *implectic* or *inversely symplectic.*

Furthermore, if the inverse operator ϑ^{-1} exists, the operator ϑ is called *co-symplectic* and ϑ^{-1} is *symplectic.* Many important algebraic properties of the Poisson bracket and their applications are considered in [176–178, 218–221, 373]. Note that the condition (4.6) in each particular case must be separately checked because the operator ϑ is in general an operator-valued functional on the manifold M. Obviously, if ϑ as a functional on M is constant, it implectic.

Now we construct evolution equations for a functional $F \in \mathcal{D}(M)$ making use of the given dynamical system (4.1). We have:

$$dF/dt = \{H, F\}_{\vartheta}, \tag{4.8}$$

where $H \in \mathcal{D}(M)$ is the Hamiltonian function which determines the system (4.1) by the formula:

$$du/dt = -\vartheta \ \mathrm{grad} H, \tag{4.9}$$

i.e. $K = -\vartheta \ \mathrm{grad} H$.

Let an implectic operator ϑ generate the dynamical system (4.1) by means of formula (4.9). We call such an operator ϑ *Nötherian* if it is skew-symmetric and satisfies

$$L_K \vartheta = \vartheta\,' K - \vartheta K'^{*} - K' \vartheta = 0, \tag{4.10}$$

where the asterisk means the ordinary conjugation and L_K is the Lie derivative with respect to a vector field $K : M \to T(M)$.

Here for any local functional Φ on M taking values in a linear space E, the expression Φ' denotes the Fréchet derivative [19, 190], which is

$$\Phi' v = \frac{d}{d\varepsilon} \Phi[u_\varepsilon]|_{\varepsilon=0}, \tag{4.11}$$

where $u_\varepsilon \in M,\ \varepsilon \in \mathbb{R}$, and for $v \in T_u(M)$

$$\left.\frac{du_\varepsilon}{d\varepsilon}\right|_{\varepsilon=0} = v.$$

In case of a linear functional space M, from (4.11) we obtain

$$\Phi'[u] v = \frac{d}{d\varepsilon} \Phi[u + v\varepsilon]|_{\varepsilon=0}, \tag{4.12}$$

where $v \in T_u(M),\ u \in M$.

By virtue of (4.11), the Fréchet derivative Φ' is a linear operator in the tangent space $T(M)$ taking values in a linear space E. Making use of the notation introduced above, we formulate the following result [137].

Lemma 4.1. *Let $\vartheta : T^*(M) \to T(M)$ be a skew-symmetric operator. Then the following statements are equivalent:*

(1) ϑ *is implectic;*
(2) ϑ *is Nötherian for the system* (4.9)
(3) for any $a, b \in T^*(M)$ *the identity*

$$(\vartheta a)'(\vartheta b) - (\vartheta b)'(\vartheta a) = \vartheta (a, \vartheta b)'^* \cdot 1 \tag{4.13}$$

holds.

Proof. Let $a, b, c \in T^*(M)$. Then, by straightforward calculations we find that equation (4.9) implies

$$\begin{aligned}&(b, (\vartheta\,'(\vartheta c) - \vartheta(\vartheta c)'^* - (\vartheta c)'\vartheta)a) \\ &\qquad = (b, (\vartheta\,'\vartheta c)a) + (c, (\vartheta\,'\vartheta a)b) + (a, (\vartheta\,'\vartheta b)c) - (b, \vartheta c - c)\vartheta a),\end{aligned} \tag{4.14}$$

owing to the definition of the conjugate operator $c'^* : T^*(M) \to T(M)$ for a mapping $c : M \to T^*(M)$ according to the formula:

$$(a, c'b) = (c'^*a, b), \tag{4.15}$$

for any $a \in T^*(M)$ and $b \in T(M)$.

Setting $c = \text{grad } H$, it follows from relations $(\text{grad } H)' = (\text{grad } H)'^*$ and (4.14) that operator ϑ is Nötherian, because the condition $c' = c'^*$ is necessary and sufficient by virtue of the Volterra criterion for the existence of a conservation law $H \in \mathcal{D}(M)$ generating equation (4.9).

As $H \in \mathcal{D}(M)$ is a conservation law for system (4.9), $\text{grad}H \in T^*(M)$ satisfies the Lax equation

$$L_K \text{grad } H = \frac{d}{dt}\text{grad } H + K'^* \text{grad } H = 0, \tag{4.16}$$

or by virtue of (4.1), the equation

$$(\text{grad } H)'K + K'^* \text{grad } H = 0. \tag{4.17}$$

In view of relation (4.17), we note that formula (4.10) turns into identity, i.e. the operator ϑ is Nötherian, so (1) $\Rightarrow$ (2).

The converse, (2) $\Rightarrow$ (1), is obtained from the conditions (4.10), (4.6) and (4.14) assuming $a = \text{grad } F$, $b = \text{grad } G$ for $F; G \in \mathcal{D}(M)$.

The equivalence of conditions (1) and (3) is obtained by (Fréchet) differentiating formula (4.13), so the proof is complete. □

From identity (4.14) we also obtain as a simple corollary that a local functional $c : M \to T^*(M)$ has the representation $c = -\text{grad}F$, where $F \in \mathcal{D}(M)$, is a conservation law of dynamical system (4.1) if and only if the operator ϑ is Nötherian and $c : M \to T^*(M)$ satisfies

$$L_K c = c'\,K + K'^*\,c = 0, \quad c' = c'^*.$$

Let an operator $\eta : T^*(M) \to T(M)$ be implectic and Nötherian. The dynamical system (4.1) is *bi-Hamiltonian* if there exists a function $\bar{H} \in \mathcal{D}(M)$ such that the following two representations hold:

$$du/dt = K[u] = -\vartheta \text{ grad } H = -\eta \text{ grad } \bar{H}. \tag{4.18}$$

An operator (ϑ, η)-pair is called *compatible* if the sum $\vartheta + \lambda\eta$ is also an implectic operator.

Let $a, b, c \in T^*(M)$. Introducing the bracket

$$|a, b, c| = (b, (\vartheta\,'\eta a)c) + (b, (\eta'\vartheta a)c), \tag{4.19}$$

we have the following simple statement (see [137, 326]).

Lemma 4.2. *The implectic operator (ϑ, η)-pair is compatible if and only if the bracket* (4.19) *satisfies the modified Jacobi identity.*

It follows directly from Lemma 4.2 that the operator $\vartheta + \lambda\eta$ is implectic for all $\lambda \in \mathbb{R}$.

The operator $\Lambda = \vartheta^{-1}\eta$ exists if ϑ is co-symplectic, and it generates the following hierarchy of dynamical systems on M:

$$du/dt = K_n[u] = -\vartheta\Lambda^n \text{grad } H, \tag{4.20}$$

where $n \in \mathbb{Z}_+$.

According to (4.18), the functional $\bar{H} \in \mathcal{D}(M)$ is a conservation law for the dynamical system (4.1), and from (4.16) and (4.20) it follows that operator $\Lambda : T^*(M) \to T^*(M)$ satisfies

$$L_K\Lambda = \Lambda' K - [\Lambda, K'^*] = 0, \tag{4.21}$$

where $[\cdot, \cdot]$ is the usual operator commutator.

The map $\Lambda : T^*(M) \to T^*(M)$ satisfying condition (4.21) for the given dynamical system (4.18) is called a *recursion operator*. If an operator Λ is also a recursion for the hierarchy of dynamical systems (4.20), then Λ is called a *hereditary recursion operator*. Next we have a result that establishes the hereditary recursion conditions for an operator Λ (see [137, 173, 326]).

Theorem 4.1. *Let $\Lambda = \vartheta^{-1}\eta$ be a hereditary recursion operator. Then the following two conditions are equivalent:*

(i) the operator (ϑ, η)-pair is compatible and the operator ϑ^{-1} exists;
(ii) the bilinear operator $[\Lambda^{*\prime}, \Lambda^*]$, where $\Lambda^* = \eta\vartheta^{-1}$ acts via the formula

$$[\Lambda^{*\prime}, \Lambda^*](a, c) = (\Lambda^{*\prime}\Lambda^* a)c - \Lambda^*(\Lambda^{*\prime} a)c, \tag{4.22}$$

for $a, c \in T(M)$, is symmetric.

Proof. Consider

$$\begin{aligned} A =&(b,(\eta'\Lambda^* a)\vartheta^{-1}c) - (b,(\eta'\Lambda^* c)\vartheta^{-1}a) \\ &- (b,\Lambda^*(\eta' a)\vartheta^{-1}c) + (b,\Lambda^*(\eta' c)\vartheta^{-1}a) \\ &- b,\Lambda^*(\vartheta\,'\Lambda^* a)\vartheta^{-1}c) + (b,\Lambda^*(\vartheta\,'\Lambda^* c)\vartheta^{-1}a) \\ &+ (b,\Lambda^*\Lambda^*(\vartheta\,' a)\vartheta^{-1}c) - (b,\Lambda^*\Lambda^*(\vartheta\,' c)\vartheta^{-1}a), \end{aligned} \tag{4.23}$$

where $a, c \in T(M), b \in T^*(M)$. It is easy to verify that for any $a, c \in T^*(M)$ and $b \in T^*(M)$

$$A = (b, [\Lambda^{*\prime}, \Lambda^*](a, c)) - (b, [\Lambda^{*\prime}, \Lambda^*](c, a)). \tag{4.24}$$

Comparing the expressions (4.23) and (4.19) one can find that

$$A = |\bar{a}, \bar{b}, \bar{c}| + |\bar{b}, \bar{c}, \bar{a}| + |\bar{c}, \bar{a}, \bar{b}|, \tag{4.25}$$

where $\bar{a} = \vartheta^{-1}a,\ \bar{c} = \vartheta^{-1}c,\ \bar{b} = \Lambda^* b$.

Taking into account (4.24), (4.25) and that the operator (ϑ, η)-pair is compatible, from Lemma 4.2 we conclude that conditions (i) and (ii) are equivalent. Thus, the proof is complete. □

Theorem 4.1 leads directly to the following result.

Theorem 4.2. *Let a pair (ϑ, η) of co-symplectic operators be compatible. Then all operators $\vartheta(\vartheta^{-1}\eta)^n, n \in \mathbb{Z},$ are co-symplectic.*

Proof. As the operator $\lambda\vartheta+\eta$ is also implectic, the operator $(\lambda\vartheta+\eta)^{-1}$ as $\lambda \to \{0, \infty\}$ is certainly symplectic. Expanding $(\lambda\vartheta+\eta)^{-1}$ in a Taylor series in $\lambda \in \mathbb{C}$ as $\lambda \to \{0, \infty\}$, we conclude that all operators $\eta^{-1}(\vartheta\eta^{-1})^n,\ n \in \mathbb{Z}$, are symplectic and their inverses are co-symplectic, which is the desired result. □

So, the dynamical bi-Hamiltonian system (4.1) with a hereditary recursion operator Λ, factorized by two implectic operators ϑ and η, obviously possesses an infinite hierarchy of nontrivial conservation laws $\gamma_j \in \mathcal{D}(M),\ j \in \mathbb{Z}_+$, which can be found using the recursion

$$\text{grad}\ \gamma_{j+1} = \Lambda\ \text{grad}\ \gamma_j, \tag{4.26}$$

starting at $\gamma_0 \in \mathcal{D}(M)$ according to (4.18).

4.3 Symmetry properties and recursion operators

We now consider symmetry properties of dynamical systems (4.1), which are in a sense dual to conservation laws. A local functional $\alpha : M \to T(M)$ is a *symmetry* of the dynamical system (4.1) if it satisfies

$$L_\alpha K = \alpha' K - K' \alpha = 0. \tag{4.27}$$

If system (4.1) is bi-Hamiltonian, it follows from (4.18) that the operator Λ^* introduced in Theorem 4.1 has the symmetry recursion property:

$$\Lambda^* \alpha = \beta, \tag{4.28}$$

where the local functional $\beta : M \to T(M)$ is also a symmetry of the dynamical system (4.1).

From (4.27), we readily find that the operator Λ^* satisfies

$$L_K \Lambda^* = \Lambda^{*\prime} K - [K', \Lambda^*] = 0. \tag{4.29}$$

Moreover, it follows directly from (4.26) that when the recursion operator Λ exists, the conservation laws $\gamma_j \in \mathcal{D}$, $j \in \mathbb{Z}_+$, for which

$$\alpha_j = -\vartheta \text{ grad } \gamma_j,$$

correspond to all the symmetries of the dynamical system (4.1). Thus, according to (4.20) all these symmetries $\alpha_j \in T(M)$ have the canonical representation:

$$\alpha_j = -\vartheta \Lambda^j \text{ grad} \gamma_0 \,. \tag{4.30}$$

The conservation laws $\gamma_j \in \mathcal{D}(M)$ are naturally related via

$$\gamma_j = \int_0^1 (\text{grad } \gamma_j[u; \lambda], u) \; d\lambda \tag{4.31}$$

to the corresponding grad $\gamma_j \in T^*(M), \; j \in \mathbb{Z}_+$. Furthermore, the conservation laws (4.31) are in involution, i.e.

$$\{\gamma_j, \gamma_k\}_\vartheta = 0 = \{\gamma_j, \gamma_k\}_\eta \tag{4.32}$$

for all $j, k \in \mathbb{Z}_+$. Consequently, it is natural to suspect that the dynamical system (4.1) is completely integrable in the sense of Liouville. This in fact is often true for such systems by virtue of the existence of a Lax representation [128, 326], which will be considered in the sequel.

4.4 Bäcklund transformations

Now we consider an important equivalence property for the two given dynamical systems related to their symmetries. Let two infinite-dimensional dynamical systems on functional manifolds M_1 and M_2, respectively, be

$$du/dt = K[u], \quad dv/dt = G[v], \tag{4.33}$$

where $u \in M_1, \ v \in M_2$.

A smooth mapping $B : M_1 \times M_2 \to Q$, where Q is an infinite-dimensional linear space, is called *admissible* if for any pair $(u, v) \in M_1 \times M_2$ satisfying the equation $B[u, v] = 0$, the linear maps $B'_u : T_u(M_1) \to Q$ and $B'_v : T_v(M_2) \to Q$ are invertible. In this case, the admissible mapping $B : M_1 \times M_2 \to Q$ is called a *Bäcklund transformation* for two flows (4.33) if the relation $B[u(t), v(t)] = 0$ holds for almost all $t \in \mathbb{R}$ when $B[u(0), v(0)] = 0$.

To study the properties of Bäcklund transformation of two systems (4.33), let us introduce the linear operator $\mathcal{R} : T_u(M_1) \to T_v(M_2)$ defined as

$$\mathcal{R} = {B'}_v^{-1} B'_u, \quad B[u, v] = 0. \tag{4.34}$$

Suppose the operator $\vartheta : T^*(M_1) \to T(M_1)$ is implectic. Identifying the tangent spaces $T_u(M_1)$ and $T_v(M_2)$, $(u, v) \in M_1 \times M_2$, with the corresponding functional spaces, we write

$$\hat{\vartheta} = \mathcal{R}\vartheta\mathcal{R}^*, \tag{4.35}$$

making use of operator (4.34). The following theorem [137, 326] is a fundamental result.

Theorem 4.3. *The operator $\hat{\vartheta}$ defined by formula* (4.35) *is implectic if and only if operator ϑ is implectic.*

Proof. The skew-symmetry of $\hat{\vartheta}$ follows from the skew-symmetry of ϑ because the map (4.35) obviously preserves this property. We now verify that the Jacobi identity holds for the operator $\hat{\vartheta}$. Indeed,

$$\begin{aligned}
&(a, \hat{\vartheta} \text{ grad } (b, \hat{\vartheta}c)) + (\text{circle}) = (a, (\hat{\vartheta}\,'_v \hat{\vartheta}c)b) + (\text{circle}) \\
&= (a, \mathcal{R}'_v \vartheta' c)\vartheta\mathcal{R}b) + (a, \mathcal{R}\vartheta({\mathcal{R}'}^*_v \hat{\vartheta}c)b) \\
&\quad + (a, \mathcal{R}(\vartheta'_v \hat{\vartheta}c)\mathcal{R}^* b) + (\text{circle}) = (\bar{a}, (\vartheta'_u \vartheta\bar{c})\bar{b}) + (\text{circle}) \\
&= (\bar{a}, \vartheta \text{ grad } (\bar{b}, \vartheta\bar{c})) + (\text{circle}) = 0,
\end{aligned} \tag{4.36}$$

where $a, b, c \in T^*(M_2)$, $\bar{a} = \mathcal{R}^* a, \bar{b} = \mathcal{R}^* b, \bar{c} = \mathcal{R}^* c \in T^*(M_1)$.

In obtaining the relations (4.36), we made use of the following easily proved identities:

$$
\begin{aligned}
&(\mathcal{R}'_v a)b = (\mathcal{R}'_c \mathcal{R} b)\mathcal{R}^{-1} a,\\
&((\mathcal{R}'^*_v a)b, c) = ((\mathcal{R}'^*_v \mathcal{R} c)b, \mathcal{R}^{-1} a) = (b(\mathcal{R}'_v \mathcal{R} c)\mathcal{R}^{-1} a) = (b, (\mathcal{R}'_v a)c), \quad (4.37)\\
&\vartheta\,'_v a = -\vartheta\,'_v \mathcal{R}^{-1} a,
\end{aligned}
$$

for all $a, b, c \in T^*(M_2)$. Thus, the proof is complete. □

The following result [137, 262, 326] is also basic.

Theorem 4.4. *If dynamical systems* (4.33) *admit Bäcklund transformation* $B[u, v]$, $(u, v) \in M_1 \times M_2$, *then formula* (4.35) *defines the Nötherian operator* $\hat{\vartheta}$ *if and only if operator* ϑ *is Nötherian.*

Proof. First, we show that operator $\hat{\vartheta}$ is Nötherian. To this end, we calculate the following values:

$$G[u] = -\mathcal{R}K[u], \quad K'_v = -K'_u \mathcal{R}^{-1}. \tag{4.38}$$

It follows from (4.38) that

$$(G\,'_v a) = -(\mathcal{R}'_v a)K + \mathcal{R}(K'_u \mathcal{R}^{-1} a) \tag{4.39}$$

for all $a \in T^*(M_2)$.

Using (4.39) and (4.37), it is easy to verify the identity

$$
\begin{aligned}
&(\hat{\vartheta}\,'_v G)a - \hat{\vartheta} G\,'^*_v a - G\,'_v \hat{\vartheta} a\\
&= (\mathcal{R}'_v G)\vartheta \mathcal{R}^* a + \mathcal{R}(\vartheta\,'_v G)\mathcal{R}^* a + \mathcal{R}\vartheta(\mathcal{R}'^*_v G)a\\
&\quad + \hat{\vartheta}[(\mathcal{R}'_v a)K]^* - \hat{\vartheta}(\mathcal{R}K'_u \mathcal{R}^{-1})^* a\\
&\quad + (\mathcal{R}'_c \hat{\vartheta} a)Ka - \mathcal{R}K'_u \mathcal{R}^{-1} \hat{\vartheta} a\\
&= \mathcal{R}[(\vartheta\,'_u K) - \vartheta K'^*_u - K'_u \vartheta]\mathcal{R} a
\end{aligned} \tag{4.40}
$$

for all $a \in T^*(M_2)$.

As the operator $\vartheta : T^*(M_1) \to T(M_1)$ is Nötherian, from (4.15) we find that expression (4.40) is identically equal to zero. Hence, the operator $\hat{\vartheta}$ of the form (4.35) is also Nötherian, which is the desired result. □

Theorems 4.3 and 4.4 are of great importance for solving the difficult problem of describing all equivalent infinite-dimensional dynamical systems. If the dynamical system (4.33) has a Bäcklund transformation $B : M_1 \times M_2 \to Q$, the equation

$$\hat{\Lambda} = \hat{\vartheta}^{-1} \hat{\eta} = \mathcal{R}^{*-1} \Lambda \mathcal{R}^*, \tag{4.41}$$

is true if the recursion operator $\hat{\Lambda}$ exists.

Assuming the formula (4.41) is the transformation rule of the recursion operators for the pair of dynamical systems (4.33), which are not necessarily Hamiltonian, one can then define a hierarchy of conservation laws $\hat{\gamma}_j \in \mathcal{D}(M_2)$ for the transformed dynamical system using the operator $\hat{\vartheta}$.

Similarly, by introducing the symmetry recursion operator

$$\hat{\Lambda}^* = \mathcal{R}\Lambda^*\mathcal{R}^{-1}, \tag{4.42}$$

we obtain an operator $\hat{\Lambda}^*$ generating a hierarchy of symmetries for the second (Bäcklund transformed) dynamical system.

4.5 Properties of solutions of some infinite sequences of dynamical systems

Let us begin an investigation of integrability of a Hamiltonian infinite-dimensional dynamical system under condition that it has an infinite hierarchy of nontrivial functionally independent conservation laws. As the Liouville integrability criteria for finite-dimensional dynamical systems are not quite applicable in the infinite-dimensional case, the problem of Liouville integrability for a general infinite-dimensional Hamiltonian system remains largely unresolved, even though substantial progress has been made over the last several decades.

One can show that the conditions of the finite-dimensional Liouville integrability theorem are necessary for the infinite-dimensional case. However, the sufficiency of the integrability conditions has only been proved for a number of concrete examples.

Consider the dynamical system (4.1) having the Hamiltonian form (4.9) with implectic operator $\vartheta : T^*(M) \to T(M)$ and Hamiltonian function $H \in \mathcal{D}(M)$. Suppose that dynamical system (4.9) also has an infinite hierarchy of smooth conservation laws $\gamma_j \in \mathcal{D}(M)$, $j \in \mathbb{Z}_+$, which are functionally independent and in involution, i.e.

$$\{\gamma_j, \gamma_k\}_\vartheta = 0 \tag{4.43}$$

for all $j, k \in \mathbb{Z}_+$ and $\gamma_2 = H$.

Taking into account formula (4.43), we obtain the following infinite sequence of dynamical systems:

$$du/dt_j = -\vartheta \text{ grad } \gamma_j, \tag{4.44}$$

$t_j \in \mathbb{R}, \;\; j \in \mathbb{Z}_+$, for which all functionals $\gamma_j \in \mathcal{D}(M)$ are obviously conservation laws.

Now, let $T_j(t): M \to M,\ j \in \mathbb{Z}_+,\ t \in \mathbb{R}$, be the operators solving the Cauchy problem for the dynamical systems (4.44) on M, i.e. for all $u_0 \in M$

$$T_j(t_j)u_0 = u(t_j) \tag{4.45}$$

is the solution of systems (4.44) with the initial value $u(0) = u_0 \in M$ for all $j \in \mathbb{Z}_+$.

Let us suppose that Cauchy problem for all equations (4.45) on M is well-posed, i.e. the operators $T_j(t_j)$ are continuous on M for all values of parameters $t_j \in \mathbb{R},\ \ j \in \mathbb{Z}_+$.

By virtue of (4.43), the identity

$$T_j(t_j)T_k(t_k) = T_k(t_k)T_j(t_j) \tag{4.46}$$

holds for all $j, k \in \mathbb{Z}_+$, i.e. $T_j(t_j),\ j \in \mathbb{Z}_+$, are mutually commutative.

Let the functional $\gamma_{N+1} \in \mathcal{D}(M)$ take the value $\gamma_{N+1} = h_{N+1} \in \mathbb{R}$ such that this value is minimal under the condition that the values of functionals $\gamma_j = h_j \in \mathbb{R},\ 0 \le j \le N$, are fixed. Then, obviously, by virtue of the minimality,

$$\text{grad } \gamma_{N+1} = \sum_{k=0}^{N} c_k \text{grad } \gamma_k \tag{4.47}$$

is satisfied, where $c_k \in \mathbb{R}$, $0 \le k \le N$, are constants.

Let $u_0 \in M$ be a solution of equation (4.47). Then we have the following lemma [227, 262, 326].

Lemma 4.3. *The expression*

$$\prod_{j=0}^{N-1} T_j(t_j)u_0 = u(t_0, t_1, \ldots, t_{N-1}) \tag{4.48}$$

for all parameters $t_j \in \mathbb{R},\ 0 \le j \le N-1$, is also a solution of the equation (4.47).

Proof. Consider the functional

$$F = \sum_{k=0}^{N} c_k \gamma_k - \gamma_{N+1},$$

which is obviously a conservation law for all flows (4.44). As the critical points of the conservation laws form [227, 262] an invariant set, the operators $T_j(t_j)$, $0 \le j \le N-1$, map the solution of (4.47) into itself, and this completes the proof. $\square$

Let us consider all functionals $\gamma_j \in \mathcal{D}(M)$ having the following canonical representation:

$$\gamma_j = \int\limits_S \gamma_j[u] d\nu, \tag{4.49}$$

where $\gamma_j : M \to \mathbb{R},\ j \in \mathbb{Z}_+$, are smooth mappings, and ν is the Lebesgue measure on the set $S, u \in M$.

Thus, equation (4.47) is an ordinary differential equation of the $2N$-th order whose solution is uniquely determined by the Cauchy (initial) data $\{u(x_0),\ u^{(1)}(x_0), \ldots\ u^{(2N-1)}(x_0)\} \in J^{(2N-1)}(S; \mathbb{R}^m)$, where $u^{(k)}(x_0)$ is the k-th derivative of function $u \in M$ at point $x_0 \in U$.

For convenience we denote the N-parameter set of functions (4.48) by M_N. The set M_N can be considered as embedded (via the Cauchy data) in the jet space $J^{(2N-1)}(S; \mathbb{R}^m)$. In addition, we consider the functionals $\gamma_j \in \mathcal{D}(M)$ to be such that the set M_N is bounded in $J^{(2N-1)}(S; \mathbb{R}^m)$, i.e. the operators $T_j(t_0),\ 0 \leq j \leq N-1$, are bounded on M. If the values $\gamma_j = h_j \in \mathbb{R},\ 0 \leq j \leq N$, are such that the grad$\gamma_j$ are linearly independent, then the following theorem [227, 262] holds.

Theorem 4.5. *The set M_N is an N-dimensional immersed manifold in $J^{(2N-1)}(S; \mathbb{R}^m)$.*

Proof. Let $u = u(t_0, t_1, \ldots, t_{N-1})$ be an arbitrary point of M_N. It was shown in Lemma 4.3 that $\gamma_j[u] = \gamma_j[u_0] = h_j$ for all $0 \leq j \leq N$. Hence, the vector fields $\alpha_j = -\vartheta$ gradγ_j are linearly independent on $M_N \subset M$. By virtue of (4.46), we have

$$u(t_0 + \varepsilon_0, t_1 + \varepsilon_1, \ldots, t_{N-1} + \varepsilon_{N-1}) = \prod_{j=0}^{N-1} T_j(\varepsilon_j) u(t_0,\ \ldots, t_{n-1}), \tag{4.50}$$

where $\varepsilon_j \in \mathbb{R},\ 0 \leq j \leq N-1$, are sufficiently small real numbers.

It follows from the linear independence of the vector fields $\alpha_j = -\vartheta$ grad$\gamma_j,\ 0 \leq j \leq N-1$, on $M_N \subset M$ that the set of points of type (4.50) forms an open subset of the N-dimensional manifold smoothly embedded in $J^{(2N-1)}(S; \mathbb{R}^m)$. Thus the proof is complete. $\square$

Observe that the tangent manifold $T_u(M_N)$ at $u \in M_N$ is generated by the vector fields $\alpha_j = -\vartheta$ grad$\gamma_j,\ 0 \leq j \leq N-1$. By virtue of the continuity of functionals $\gamma_j \in \mathcal{D}(M),\ j = 0, \ldots, N$, the equation $\gamma_j[u] = \gamma_j[u_0]$ obtains not only for points $u \in M_N$, but also for points $u \in \bar{M}_N$, where $\bar{M}_N$ is the closure of M_N.

Thus, owing to Theorem 4.5, for all $u \in \bar{M}_N$, the set of points of type (4.50) forms an open N-dimensional manifold which is smoothly embedded in $J^{(2N-1)}(U;\mathbb{R}^m)$.

Let us now study the dynamical system of type (4.33) when the functionals $\gamma_j \in \mathcal{D}(M)$, $j \in \mathbb{Z}_+$, are defined on a manifold M consisting of smooth periodic functions $u : \mathbb{R} \to \mathbb{R}$ of period 2π. Then it is easy to show from condition (4.43) that we have

$$\langle \text{grad } \gamma_j, \vartheta \text{ grad } \gamma_k \rangle = \frac{\partial}{\partial x}\mathcal{F}_{jk} \tag{4.51}$$

for all $j, k \in \mathbb{Z}_+$, where $\mathcal{F}_{jk}$ are smooth local periodic mappings of the space M into $\mathbb{R}$.

Suppose $u \in M$ is a solution of equation (4.17). When multiplied by ϑ grad γ_k, this equation makes it clear that the quantities

$$\mathcal{F}_k = \sum_{j=0}^{N} c_j \mathcal{F}_{jk} - \mathcal{F}_{N+1,k}, \tag{4.52}$$

$k \in \mathbb{Z}_+$, do not depend on the variable $x \in \mathbb{R}$.

If function $u \in M$ depends on parameters $t_m \in \mathbb{R}$, $m \in \mathbb{Z}_+$, it follows from the form of he dynamical systems (4.44) that

$$\frac{\partial \mathcal{F}_k}{\partial t_m} = \sum_{j=0}^{N} c_j \frac{\partial \mathcal{F}_{jk}}{\partial t_m} - \frac{\partial \mathcal{F}_{N+1,k}}{\partial t_m} = -(\alpha'^{*}_{m}\psi)\alpha_k - \psi(\alpha'_m \alpha_k), \tag{4.53}$$

where $\alpha_k = -\vartheta$ grad γ_k, $k \in \mathbb{Z}_+$, and

$$\psi = -\psi_{N+1} + \sum_{j=0}^{N} c_j \ \psi_j, \quad \psi_j = \text{grad } \gamma_j.$$

By virtue of Lemma 4.3, the value of ψ is equal to zero for all $t_m \in \mathbb{R}$, $m \in \mathbb{Z}_+$, so from equations (4.53) we find that the $\mathcal{F}_k$, $0 \le k \le N$, are invariants of dynamical systems (4.44) on the submanifold $M_N \subset M$. Whence, owing to the functional independence and formulas (4.49) for the conservation laws $\gamma_j \in \mathcal{D}(M)$, $j \in \mathbb{Z}_+$, we conclude that conservation laws $\mathcal{F}_k \in \mathcal{D}(J^{(2N-1)}(\mathbb{R};\mathbb{R}))$, $0 \le k \le N$, are functionally independent on $J^{(2N-1)}(\mathbb{R};\mathbb{R})$.

Let $f_k \in \mathbb{R}$ be the values of the conservation laws $\mathcal{F}_k[u] \in \mathcal{D}(J^{(2N-1)}(\mathbb{R};\mathbb{R}))$, $0 \le k \le N$, at point $u_0 \in M_N$. Denote by $V \subset J^{(2N-1)}(\mathbb{R};\mathbb{R})$ the set satisfying the conditions: $u \in V$ if $\mathcal{F}_k[u(x)] = f_k$, $0 \le k \le N$. In view of the functional independence of the $\mathcal{F}_k \in$

$\mathcal{D}(J^{(2N-1)}(\mathbb{R};\mathbb{R}))$, $0 \le k \le N$, the set V is an N-dimensional smooth submanifold of $J^{(2N-1)}(\mathbb{R};\mathbb{R})$, and we have nearly proved the following result [227, 262].

Lemma 4.4. *The closure $\bar{M}_N \subset V$ is invariant with respect to flows* (4.44) *and is compact.*

Proof. The continuity of the operators (4.45) $T_j(t_j)$, $t_j \in \mathbb{R}$, $j \in \mathbb{Z}_+$ implies that the set $\bar{M}_N$ is invariant with respect to the flows (4.44).

From the continuity of the conservation laws $\mathcal{F}_k \in \mathcal{D}(J^{(2N-1)}(\mathbb{R};\mathbb{R}))$, $0 \le k \le N$, it follows directly that $\overline{M}_N \subset V$. Hence, the assumed boundedness of M_N implies that $\bar{M}_N$ is compact, which finishes the proof. $\square$

Next, we consider the point set $M_N(u;\varepsilon)$ defined by

$$M_N(u;\varepsilon) = \prod_{j=0}^{N-1} T_j(t_j)u, \tag{4.54}$$

where $u \in \bar{M}_N$; $|t_j| < \varepsilon$, $0 \le j \le N-1$, and ε is a sufficiently small positive number. This set is an open subset of the N-dimensional manifold smoothly embedded in $J^{(2N-1)}(\mathbb{R};\mathbb{R})$, and the following lemma [72, 227, 262] holds.

Lemma 4.5. *The set $M_N(u;\varepsilon)$ defined by* (4.54) *contains a neighborhood of $u \in \bar{M}_N$.*

Proof. Let us assume the opposite. Then there is a sequence of elements $u_n \in \bar{M}_N$, $n \in \mathbb{Z}_+$, such that $u_n \to u$, $u_n \notin M_N(u;\varepsilon)$. Now, according to Lemma 4.4 the set $M_N(u_n;\varepsilon) \subset \bar{M}_N$. Hence, the sets $M_N\left(u_n;\frac{1}{2}\varepsilon\right)$ and $M_N\left(u;\frac{1}{2}\varepsilon\right)$ are disjoint for all n. Indeed, if they had an intersection point, it would follow from the representation (4.54) that

$$\prod_{j=0}^{N-1} T_j(t_j - t'_j)u = u_n \tag{4.55}$$

for some $t_j,\ t'_j,\ 0 \le j \le N-1$, with $|t_j| < \frac{1}{2}\varepsilon$, $|t'_j| < \frac{1}{2}\varepsilon$.

But equation (4.55) is equivalent to membership of $u_n, n \in \mathbb{Z}_+$, in the set $M_N(u;\varepsilon)$, which contradicts our initial assumption.

Now, we construct a subsequence $u_{n(j)}$, $n \in \mathbb{Z}_+$ such that $u_{n(j+1)}$ does not belong to each of sets $M_N(u_{n(k)};\varepsilon)$, $0 \le k \le j$. As every set $M_N(u_{n(j)};\varepsilon)$, $j \in \mathbb{Z}_+$, has a positive distance from point $u \in \bar{M}_N$, the points $u_{n(j)}$, $n \in \mathbb{Z}_+$, can be chosen arbitrarily close to $u \in \bar{M}_N$.

Again, we can select the sets $M_N\left(u_{n(j)};\frac{1}{2}\varepsilon\right)$, $j \in \mathbb{Z}_+$, to be mutually disjoint. It follows from Lemma 4.4 that the sets $M_N\left(u_{n(j)};\frac{1}{2}\varepsilon\right) \subset \bar{M}_N \subset$

V, $j \in \mathbb{Z}_+$. But this is absurd, since smooth manifold V cannot have such a complicated structure with an infinite number of fibers, so our initial assumption must be false, and the proof is complete. $\square$

Accordingly it follows from the definition of $\bar{M}_N$ that every neighborhood of an element $u \in \bar{M}_N$ contains points from M_N. Thus, some points from $M_N(u;\varepsilon)$ belong to M_N, i.e. they have the form

$$\prod_{j=0}^{N-1} T_j(t_j)u_0,$$

where $u_0 \in M_N$.

However, because of the commutativity of the operators $T_j(t_j)$, $0 \le j \le N-1$, $u \in \bar{M}_N$ has the same form as any element from M_N. This means that $\bar{M}_N = M_N$, which is the basis of a proof of the next result.

Theorem 4.6. *The set M_N is a compact, connected and open-closed subset of the manifold V, and each point of M_N is regular.*

Proof. We shall show that the manifold M_N is diffeomorphic to the N-dimensional torus $\mathbb{T}^N$, which clearly serves to establish the existence of a number $t_0 \in \mathbb{R}_+$ such that the manifold $M_N \subset M_N(u_0;t_0)$.

Indeed, as M_N is compact then there are points $u_j \in M_N$ such that the collection of sets $\{M_N(u_j;\varepsilon),\ 1 \le j \le m\}$ covers the whole manifold M_N. Now, any pair of points in the neighborhood $M_N(u_j;\varepsilon)$, $1 \le j \le m$, can be connected by the mapping

$$\prod_{j=0}^{N-1} T_j(t_j), \quad |t_j| < 2\varepsilon,$$

$j = 0,\dots,N-1$. As every point can be connected with the base point $u_0 \in M_N$ by a chain of no more than m arcs, it follows that $M_N \subset M_N(u_0;t_0)$, where $t_0 = 2m\varepsilon$.

As an immediate corollary, we deduce that there must be nonzero numbers $t_j \in \mathbb{R}$, $0 \le j \le N-1$, such that

$$\prod_{j=0}^{N-1} T_j(t_j) = 1. \tag{4.56}$$

Let $(t_0, t_1, t_2, \dots, t_{N-1}) \in \mathbb{R}^N$, $t_j \in \mathbb{R}$, $0 \le j \le N-1$, be such that formula (4.56) holds. The collection of all such sets forms the modulus $\bar{t}$ in $\mathbb{R}^N$ over the ring of integers.

As Theorem 4.5 excludes the possibility of satisfying equation (4.56) for values $|t_j| < \varepsilon$ when $\varepsilon > 0$ is sufficiently small, if $t_j \neq 0$, $0 \leq j \leq N-1$, we conclude that the modulus $\bar{t}$ is discrete, i.e. it can be represented as

$$\bar{t} = \left\{ \sum_{j=1}^{m} n_j \omega_j : n_j \in \mathbb{Z}, \quad 1 \leq j \leq m \right\},$$

where $m \leqslant N$ is its dimension and $\omega_j \in \mathbb{R}_+$, $1 \leq j \leq m$, is a basis.

Now, it follows from Lemma 4.4 that the set M_N can be represented as $M_N \simeq \mathbb{R}^N/\bar{t}$. Hence, owing to the compactness of M_N we conclude that $m = N$, i.e. $\mathbb{R}^N/\bar{t} \simeq \mathbb{T}^N$, so M_N is diffeomorphic to the N-dimensional torus. Thus, the proof is complete. □

The proof of this theorem actually leads rather directly to the following important result that is proved in the literature [227, 262].

Theorem 4.7. *The mapping*

$$\prod_{j=0}^{N-1} T_j(t_j) : (\mathbb{R}^N/\bar{t}) \leftrightarrows M_N \subset J^{(2N-1)}(\mathbb{R};\mathbb{R}) \tag{4.57}$$

is an injective map of the torus $\mathbb{T}^N \simeq \mathbb{R}^N/\bar{t}$ onto the set M_N.

In particular, $M_N \simeq \mathbb{T}^N$, i.e. M_N is diffeomorphic to the N-dimensional torus and the flows generated by dynamical systems (4.44) *in variables of $\mathbb{R}^N/\bar{t}$ are quasiperiodic.*

Theorem can be viewed as an infinite-dimensional version of the Liouville theorem on integrability; applicable to Hamiltonian dynamical systems having an infinite involutive hierarchy of functionally independent conservation laws. For many particular cases, the construction of the invariant tori $\mathbb{T}^N$ can actually be carried out explicitly by employing different mathematical techniques.

4.6 Integro-differential systems

Most of the infinite-dimensional dynamical systems arising in mathematical physics are in the form of partial differential equations. However, there are several important cases in which the dynamics is governed by integro-differential equations. We shall briefly consider two such examples here.

Example 4.1. Dynamics of a closed vortex filament in an ideal fluid

Consider a closed vortex filament moving in an ideal fluid in $\mathbb{R}^3$. A good model for the dynamics can be derived from the Biot–Savart law combined with a standard desingularization procedure [36, 38, 39]; it has the form

$$\varphi_t = \partial_t \varphi(\tau, t) = \frac{\Gamma}{4\pi} \int_{\mathbb{S}^1} \nabla \left(\sigma \left(\| \varphi(\tau, t) - \varphi(s, t) \| \right) \right) \wedge \varphi_s(s, t) ds, \tag{4.58}$$

where Γ is the vortex strength, which is assumed to be a positive constant over the filament, $\mathbb{S}^1$ is the unit circle in the complex plane $\mathbb{C}$,

$$\varphi \in M = \left\{ \psi \in C^{\infty} \left(\mathbb{S}^1 \times [0, \infty), \mathbb{R}^3 \right) : \psi(\cdot, t) \text{ is injective for every } t \geq 0 \right\},$$

∇ and $\wedge$ are the usual gradient and wedge (cross) product in $\mathbb{R}^3$, respectively, and σ is a desingularizing function, which mimics flow in a slender vortex ring in a real fluid, such that the right-hand side is smooth ($= C^{\infty}$) whenever φ is smooth.

Equation (4.58) can be shown to be a Hamiltonian dynamical system [36, 38, 81, 252], and there is a simplification called the *linear induction approximation* that was proved by Hasimoto [170] to be equivalent to the standard nonlinear Schrödinger equation - and therefore completely integrable, with soliton solutions. It is easy to show [38] that the Hamiltonian system is completely integrable if the initial filament is a perfect circle. More precisely, the circular symmetry allows reduction to a one-degree-of-freedom Hamiltonian dynamical system. In this vein, it remains an open problem to determine just how wide is the class of initial filament configurations leading to completely integrable systems in which the filament exhibits soliton like variations in the course of its motion.

Example 4.2. Continuum approximation of granular flows

There are several methods for finding continuum approximations of granular flow equations. One is the long-wave limit (used, for example, in the derivation of the KdV equation from the Fermi–Ulam–Pasta system of nonlinear oscillators). Using this approach, Nesterenko [276] was able to obtain completely integrable partial differential equation models for perfectly elastic one-dimensional particle flows, having soliton solutions that were in good agreement with experimental results.

An interesting alternative to this approach [45] produces integro-differential continuum approximations that can be written as

$$u_t + u \cdot \nabla u = e + \int_R P \left(y, u(x, t), u\left(y + x, t \right) \right) dy, \tag{4.59}$$

where u is the velocity (vector) in a region R of the appropriate one-, two- or three-dimensional space, e is the external force (such as gravity) and P is a smooth kernel, having support in a small compact neighborhood of $x \in R$, derived from the standard Mindlin–Walton–Braun particle-particle and particle-boundary (wall) interaction models [45, 276] and an assumed particle distribution [47]. If the particle-particle and particle-boundary interactions are perfectly elastic and the external force e has a smooth potential, it is not difficult to show that (4.59) is an infinite-dimensional Hamiltonian system [47], which is completely integrable in certain cases. For example, it was shown in [37] that for horizontal, perfectly elastic motion of a string of particles, (4.59) is equivalent to the completely integrable Burgers equation. The problem of finding other examples (especially in higher dimensions) of completely integrable (approximate) particle dynamics in the context of (4.59) and other continuum models is currently of considerable interest in the granular flow community.

Chapter 5

Integrability Criteria for Dynamical Systems: the Gradient-Holonomic Algorithm

5.1 The Lax representation

5.1.1 *Generalized eigenvalue problem*

Consider an infinite-dimensional dynamical system

$$u_t = K[u], \tag{5.1}$$

where $u \in M \subset C^{\infty}_{2\pi}(\mathbb{R};\mathbb{R}^n)$, belonging to a subspace of smooth 2π-periodic functions on $\mathbb{R}$, $n \in \mathbb{Z}_+$, $K : M \to T(M)$ is a local functional on M of the fixed jet-order $m \in \mathbb{Z}_+$, which is smoothly Fréchet differentiable and takes values in the tangent space $T(M)$ of the manifold M. We also consider a linear differentiable operator $L[u;\lambda]$ acting in the space $\mathcal{H} := L^{\infty}(\mathbb{R};\mathbb{C}^k)$, and also a local functional on the manifold M depending on a parameter $\lambda \in \mathbb{C}$.

Define the following generalized eigenvalue problem for the operator $L[u;\lambda] : \mathcal{H} \to \mathcal{H}$ as follows: a number $\lambda \in \mathbb{C}$ belongs to the spectrum $\sigma(L)$ of the operator $L[u;\lambda]$ when a solution $y \in \mathcal{H}$ of the equation

$$L[u;\lambda]y(x;\lambda) = 0, \tag{5.2}$$

$x \in \mathbb{R}$, is bounded, i.e.

$$\sup_{x\in\mathbb{R}} \|y(x;\lambda)\| < \infty,$$

where $\|\cdot\|$ is the standard norm in $\mathbb{C}^k$.

Let us suppose that the spectrum $\sigma(L)$ undergoes *iso-spectral deformation* for the dynamical system (5.1), i.e.,

$$\frac{d}{dt}\sigma(L) = 0$$

for all $t \in \mathbb{R}$. As, in general, the spectrum $\sigma(L)$ is a functional on M, for the case being consideration it is invariant on the manifold M.

A sufficient condition for such iso-spectrality for the dynamical system (5.1) is the existence of a parameter dependent matrix operator $\mathcal{P}(L)$: $\mathcal{H} \to \mathcal{H}$ that a local matrix functional on M, such that for all values of the parameter $\lambda \in \mathbb{C}$, the following equation holds:

$$dL/dt = [\mathcal{P}(L), L], \tag{5.3}$$

where $[\cdot, \cdot]$ is the standard operator commutator. We note that the dynamical system (5.1) is the compatibility condition for the operator equation (5.3) for all $\lambda \in \mathbb{C}$. A representation of dynamical system (5.1) in the form of (5.3) is called a *Lax representation.*

5.1.2 *Properties of the spectral problem*

We now perform a detailed analysis of the spectral properties of problem (5.2) for the dynamical system (5.1). To do this, we assume that the operator $L[u; \lambda] : \mathcal{H} \to \mathcal{H}$ can be represented in the canonical form

$$L[u; \lambda] = \frac{d}{dx} - l[u; \lambda],$$

where $l[u; \lambda]$ is a matrix functional on the manifold M. Equation (5.2) can be rewritten in the form of a linear matrix differential equation in the space $\mathcal{H}$ as

$$dy/dx = l[u; \lambda]y. \tag{5.4}$$

Let $Y(x, x_0; \lambda)$ be the fundamental matrix solution of equation (5.4) normalized at a point $x = x_0 \in \mathbb{R}$, i.e., $Y(x, x_0; \lambda) = 1$ for any $x_0 \in \mathbb{R}$, $\lambda \in \mathbb{C}$. Obviously, any solution of equation (5.4) can be represented in the form

$$y = Y(x, x_0; \lambda)y_0, \tag{5.5}$$

where $y_0 \in \mathbb{C}^k$ is an initial value at $x_0 \in \mathbb{R}$.

Consider the value $y(x, x_0; \lambda)$ at $x = x_0 \pm 2\pi N$, $N \in \mathbb{Z}_+$. By virtue of periodicity in the variable $x \in \mathbb{R}$ of the matrix $l[u; \lambda]$, we have

$$y(x_0 \pm 2\pi N, x_0; \lambda) = S^{\pm N}(x_0; \lambda)y_0, \tag{5.6}$$

where $S(x_0; \lambda) = Y(x_0 + 2\pi, x_0; \lambda)$ is called the *monodromy matrix* of the differential equation (5.4). Let $\xi(\lambda) \in \mathbb{C}$ be an eigenvalue of the matrix $S(x_0; \lambda)$, $x_0 \in \mathbb{R}$, $\lambda \in \mathbb{C}$. Then from (5.6) it directly follows [111, 128, 232, 233, 247, 262, 406] that the solution $y \in \mathcal{H}$ is bounded if and only if all eigenvalues $\xi(\lambda)$ of the matrix $S(x_0; \lambda)$ lie on the unit circle in $\mathbb{C}$, i.e., $|\xi(\lambda)| = 1$.

We now show that the eigenvalues $\xi(\lambda) \in \mathbb{C}$ do not depend on the point $x_0 \in \mathbb{R}$. With this purpose, we consider the corresponding differential equation for the matrix $S(x_0; \lambda)$:

$$dS/dx_0 = [l[u; \lambda], S]. \tag{5.7}$$

From equation (5.7), we conclude that the values $\mathrm{tr} S^n(x_0; \lambda)$, $n \in \mathbb{Z}_+$, do not depend on the point $x_0 \in \mathbb{R}$. Consequently, the eigenvalues $\xi(\lambda)$ of the matrix $S(x_0; \lambda)$, $\lambda \in \mathbb{C}$, do not depend on $x_0 \in \mathbb{R}$, i.e.,

$$\frac{d}{dx_0}\xi(\lambda) = 0.$$

Let $\bar{y} \in \mathcal{H}$ be the eigenfunction of problem (5.4) satisfying the Bloch condition [232, 247, 262, 406]

$$\bar{y}(x + 2\pi, x_0; \lambda) = \xi(\lambda)\bar{y}(x, x_0; \lambda) \tag{5.8}$$

for all $x \in \mathbb{R}$, i.e. $\bar{y} \in \mathcal{H}$ is an eigenfunction for the monodromy operator $S : \mathcal{H} \to \mathcal{H}$, $S(x; \lambda)\bar{y}(x, x_0; \lambda) = \xi(\lambda)\bar{y}(x, x_0; \lambda)$ for all $x, x_0 \in \mathbb{R}$. Under the assumption that the matrix $l[u; \lambda]$ depends analytically on parameter $\lambda \in \mathbb{C}$, we infer that the matrix $Y(x, x_0; \lambda)$ is also analytic in $\lambda \in \mathbb{C}$.

As the function $\xi : \mathbb{C} \to \mathbb{C}$ is in general algebraic on a Riemann surface Γ of infinite genus, the vector $\bar{y} \in \mathcal{H}$ is the same as the parameter $\lambda \in \Gamma \backslash \{\infty\}$, where we have denoted its essentially singular points on the surface Γ by $\{\infty\}$.

The values $\lambda \in \mathbb{C}$ for which $|\xi(\lambda)| = 1$ belong to the spectrum $\sigma(L)$, which has zone-structure [406] in the complex plane $\mathbb{C}$. For more detail on the spectrum $\sigma(L)$, it is necessary to specify the form of the operator $L[u; \lambda] : \mathcal{H} \to \mathcal{H}$ and to determine its dependence on the parameter $\lambda \in \mathbb{C}$.

5.1.3 *Analysis of a generating function for conservation laws*

Assume equation (5.3) is a Lax representation for the dynamical system (5.1). We shall show that $\xi(\lambda)$ as a functional on the manifold M, depending on the parameter $\lambda \in \mathbb{C}$, does not depend on the temporal variable $t \in \mathbb{R}$ by virtue of the dynamical system (5.1). Equation (5.3) is clearly equivalent to the system of differential equations

$$\frac{d}{dx}\bar{y} = l[u; \lambda]\bar{y}, \quad \frac{d}{dt}\bar{y} = p(l)\bar{y}, \tag{5.9}$$

where $\bar{y} \in \mathcal{H}$ is the Bloch eigenfunction of the operator $L[u; \lambda]$ defined above.

Hence, taking into consideration property (5.8), we find that

$$\frac{d\bar{y}}{dt}\xi(\lambda) + \bar{y}\frac{d\xi(\lambda)}{dt} = \xi(\lambda)p(l)\bar{y}, \tag{5.10}$$

from which, by virtue of (5.9), it follows that

$$\frac{d}{dt}\xi(\lambda) = 0, \tag{5.11}$$

i.e. the functional $\xi(\lambda) \in \mathcal{D}(M)$, $\lambda \in \mathbb{C}$, is a generating function of conservation laws for the dynamical system (5.1).

Consider equation (5.4) for the Bloch function $\bar{y} \in \mathcal{H}$,

$$d\bar{y}/dx = l\bar{y} \tag{5.12}$$

and construct the new function $\bar{z} := \bar{y}/\langle \bar{y}, \bar{c}\rangle \in \mathcal{H}$, where a vector $\bar{c} \in \mathbb{C}^k$ is chosen to be constant. Then one finds easily from (5.12) that

$$d\bar{z}/dx = l\bar{z} - \langle l\bar{z}, \bar{c}\rangle \bar{z}, \tag{5.13}$$

for which the constraint $\langle \bar{z}, \bar{c}\rangle = 1$ holds for any $x \in \mathbb{R}$. This last condition implies that the vector Riccati type equation (5.13) possesses a solution $\bar{z} \in \mathcal{H}$, which has the following asymptotic expansion as $|\lambda| \to \infty$

$$\bar{z}(x;\lambda) \sim \sum_{j\in\mathbb{Z}_+} \bar{z}_j[u]\lambda^{-j+\bar{s}}, \tag{5.14}$$

where $\bar{s} \in \mathbb{Z}_+$ and $\bar{z}_j : M \to \mathbb{C}^k,\ j \in \mathbb{Z}_+$, are local functionals.

Making use of expansion (5.14) and the condition $\langle \bar{z}, \bar{c}\rangle = 1$, one can construct a hierarchy of conservation laws naturally related to the dynamical system (5.1).

5.2 Recursive operators and conserved quantities

5.2.1 *Gradient-holonomic properties of the generating functional of conservation laws*

Consider equation (5.4) for the fundamental solution $Y(x, x_0; \lambda)$:

$$\frac{dY}{dx} = l[u;\lambda]Y, \tag{5.15}$$

where $Y(x_0, x_0; \lambda) = I$ for all $x, x_0 \in \mathbb{R}$ and $\lambda \in \mathbb{C}$. Since the fundamental solution $Y(x_0, x_0; \lambda)$, $\lambda \in \mathbb{C}$, is a functional on the linear manifold M, it follows from (5.15) one obtains an equation for the variation δY of the fundamental solution when the value $u \in M$ changes to $(u + \delta u) \in M$; namely,

$$\frac{d}{dx}\delta Y = l\delta Y + \delta l\, Y, \tag{5.16}$$

where $l[u;\lambda] := l$ and evidently $\delta Y(x_0, x_0; \lambda) = 0$ for any $x_0 \in \mathbb{R}$ and all $\lambda \in \mathbb{C}$.

The solution of equation (5.16) is

$$\delta Y = Y \int_{x_0}^{x} Y^{-1}\delta l\, Y\; dx. \tag{5.17}$$

Taking into account that $S(x_0;\lambda) = Y(x_0+2\pi, x_0;\lambda),\ \lambda \in \mathbb{C},\ x_0 \in \mathbb{R}$, from relation (5.17) we readily compute that

$$\delta S = S \int_{x_0}^{x_0+2\pi} Y^{-1}\delta l\, Y\; dx. \tag{5.18}$$

Consider now the functional

$$\Delta(\lambda) = \mathrm{tr} S(x_0;\lambda),$$

where $S(x_0;\lambda),\ x_0 \in \mathbb{R},\ \lambda \in \mathbb{C}$, is the $k \times k$-dimensional monodromy matrix considered above. By virtue of (5.11), $\Delta(\lambda)$ is a generating functional of conservation laws for the dynamical system (5.1).

From a calculation of the trace functional tr δS, we readily obtain from (5.18) the following:

$$\delta\Delta(\lambda) = \mathrm{tr}\Big(S \int_{x_0}^{x_0+2\pi} Y^{-1}\delta l\, Y\; dx\Big). \tag{5.19}$$

Making use of the formula (5.19), we find that

$$\mathrm{grad}\Delta(\lambda) = \mathrm{tr}\left(l_u'^{*} S\right), \tag{5.20}$$

or

$$\mathrm{grad}\Delta(\lambda) = \mathrm{tr}\left(S \frac{\partial}{\partial u} l\right), \tag{5.21}$$

provided that the matrix l does not explicitly depend on the derivatives of function $u \in M$.

If the matrix $l[u;\lambda]$ depends on derivatives $(u, u_x, \ldots, u_{mx}) \in J_{2\pi}^{(m)}(\mathbb{R};\mathbb{R}^k)$, then formula (5.19) will lead to an expression for $\mathrm{grad}\Delta(\lambda) \in T^*(M)$ slightly complicated than (5.21), which is necessary to calculate separately, making use of equation (5.15). For example, if $l = l(u, u_x; \lambda)$, from representation (5.19) we find that

$$\text{grad}\Delta(\lambda) = \text{tr}\left(S[l, l_{u_x}] - S\frac{d}{dx}l_{u_x} + Sl_u \right), \tag{5.22}$$

where l_{u_x} is the usual partial derivative with respect to the variable $u_x \in M$ of the matrix l and l_u is its partial derivative with respect to the variable $u \in M$.

Obviously, in general there exists a matrix-valued vector $\mathcal{A} : \mathcal{A}[l] \in (Hom\ \mathbb{C}^k)^m$, which is a local functional on M such that

$$\text{grad}\Delta(\lambda) = \text{tr}(l'^*_u S) = \text{tr}(S\mathcal{A}). \tag{5.23}$$

Suppose that the matrix-valued vector $\mathcal{A} \in Hom\ \mathbb{C}^{k\times m}$ in (5.23) is nontrivial. Then using the system of differential equations (5.7) for the functional $S : M \to Hom\ \mathbb{C}^k$ together with the relation (5.23), one can obtain the following determining expression $\text{grad}\Delta(\lambda) \in T^*(M)$:

$$\Lambda[u]\text{grad}\Delta(\lambda) = \lambda^q \text{grad}\Delta(\lambda), \tag{5.24}$$

where $\Lambda[u]$ is an integro-differential operator parametrically depending on $\lambda \in \mathbb{C}$ and $u \in M,\ q \in \mathbb{Z}$ is an integer.

Now, define

$$SpS(x;\lambda) = \Delta(\lambda) = \sum_{j=1}^{k} \exp[i\varphi_j(\lambda)], \tag{5.25}$$

where $\varphi_j(\lambda) \in \mathcal{D}(M)$ are such that $\xi_j(\lambda) = \exp[i\varphi_j(\lambda)]$, $1 \le j \le k$, are eigenvalues of the monodromy matrix $S(x_0;\lambda),\ \lambda \in \mathbb{C}$.

If the parameter $\lambda \in \mathbb{C}$ tends to the singularity $\{\infty\} \in \bar{\mathbb{C}}$ in some special way, the expression $\text{grad}\Delta(\lambda) \in T^*(M)$ in an asymptotic series in powers of the parameter $\delta(\lambda) \in \mathbb{C}$, giving rise to the relation

$$\Lambda[u]\text{grad}\gamma_j = \text{grad}\gamma_{j+q}, \tag{5.26}$$

where functionals $\gamma_j \in \mathcal{D}(M)$, $j \in \mathbb{Z}_+$, are suitable conservation laws for the dynamical system (5.1).

It is easy to check that the recursion operator $\Lambda : T^*(M) \to T^*(M)$ satisfying the relationship (5.26), also satisfies the determining equation (5.24) for the dynamical system (5.1).

5.2.2 *Involutivity of conservation laws*

We have now shown that the operator $\Lambda = \Lambda[u] : T^*(M) \to T^*(M)$, acting according to the rule (5.26), is hereditarily recursive. Assume now that there exist two implectic operators ϑ, $\eta : T^*(M) \to T(M)$ that are compatible and factorize the operator Λ as

$$\Lambda = \vartheta^{-1}\eta. \tag{5.27}$$

Thus, the operators ϑ and η are Nötherian and the dynamical system (5.1) is bi-Hamiltonian, that is

$$u_t = -\vartheta \text{grad } H = -\eta \text{grad } \bar{H}, \tag{5.28}$$

where the conservation laws H and $\bar{H} \in \mathcal{D}(M)$, in general, are linear combinations of a finite number of the conservation laws $\gamma_j \in \mathcal{D}(M)$, $j \in \mathbb{Z}_+$, of the form

$$H = \sum_{j=-q}^{\bar{m}} c_j \gamma_{j+q}, \quad \bar{H} = \sum_{j=-q}^{\bar{m}} c_j \gamma_j, \tag{5.29}$$

where $c_j \in \mathbb{R}$, $-q \le j \le m$, are constants.

Therefore, by virtue of equations (5.28), two Poisson brackets $\{\cdot,\cdot\}_\vartheta$ and $\{\cdot,\cdot\}_\eta$ are defined on the functional space $\mathcal{D}(M)$ with respect to which two generating functions $\Delta(\lambda)$ and $\Delta(\mu)$ of conservation laws with $\lambda, \mu \in \mathbb{C}$ being arbitrary parameters, are in involution. Indeed,

$$\begin{aligned} \{\Delta(\lambda), \Delta(\mu)\}_\vartheta &= (\text{grad}\Delta(\lambda), \vartheta \text{ grad}\Delta(\mu)) = (\text{grad}\Delta(\lambda), \eta \text{ grad}\Delta(\mu))\mu^{-q} \\ &= -(\eta \text{ grad}\Delta(\lambda), \text{grad}\Delta(\mu))\mu^{-q} \\ &= -(\vartheta \text{ grad}\Delta(\lambda), \text{grad}\Delta(\mu)) \ (\lambda\mu^{-1})^q \\ &= (\text{grad } \Delta(\lambda), \vartheta \text{ grad } \Delta(\mu)) \ (\lambda\mu^{-1})^q \\ &= \{\Delta(\lambda), \Delta(\mu)\}_\vartheta \ (\lambda\mu^{-1})^q. \end{aligned} \tag{5.30}$$

Whence, we find that

$$\{\Delta(\lambda), \Delta(\mu)\}_\vartheta \ [1 - (\lambda\mu^{-1})^q] = 0, \tag{5.31}$$

that is, by virtue of arbitrariness of the parameters $\lambda, \mu \in \mathbb{C}$ we obtain:

$$\{\Delta(\lambda), \Delta(\mu)\}_\vartheta = 0 = \{\Delta(\lambda), \Delta(\mu)\}_\eta. \tag{5.32}$$

From (5.32) it follows that the hierarchy of conservation laws $\gamma_j \in \mathcal{D}(M)$, $j \in \mathbb{Z}_+$, is in involution with respect to both Poisson brackets introduced above, namely for all $j, k \in \mathbb{Z}_+$ we have

$$\{\gamma_j, \gamma_k\}_\vartheta = 0 = \{\gamma_j, \gamma_k\}_\eta = 0. \tag{5.33}$$

The relationships (5.33) can be obtained in a more formal way by making use of formula (5.26). To this end, we determine the extended hierarchy of conservation laws $\{\gamma_j \in \mathcal{D}(M) : j \in \mathbb{Z}\}$ by the rule

$$\mathrm{grad}\gamma_j = \Lambda^{-1}\mathrm{grad}\gamma_{j+q}, \tag{5.34}$$

$j \in \mathbb{Z}_-$, where Λ^{-1} is the inverse operator equal to the operator $\eta^{-1}\vartheta$ by (5.27) under the assumption that operator $\eta^{-1} : T(M) \to T^*(M)$ exists.

Then from calculations analogous to (5.30), we obtain

$$\{\gamma_j, \gamma_k\}_\vartheta[1 - (\lambda\mu^{-1})^q] = 0 \tag{5.35}$$

for all $j, k \in \mathbb{Z}$ and $\lambda, \mu \in \mathbb{C}$.

By virtue of arbitrariness of the parameters $\lambda, \mu \in \mathbb{C}$, from (5.35) we find that

$$\{\gamma_j, \gamma_k\}_\vartheta = 0 = \{\gamma_j, \gamma_k\}_\eta, \tag{5.36}$$

for all $j, k \in \mathbb{Z}$. Thus, according to the Lax Theorem 4.5, the dynamical system (5.1) on the integral manifold $M_N \subset M$ determined by equation (4.50) is completely integrable in the sense of Liouville and the motion on M_N is quasiperiodic in the variable $t \in \mathbb{R}$.

To obtain the most effective formulas for the motion of the dynamical system (5.1) on a torus $\mathbb{T}^N \simeq M_N$, we shall find necessary conditions for the existence of a Lax representation (5.3), which is equivalent to equation (5.1). The existence of such a representation allows us to integrate by quadratures by means of the inverse scattering approach [406] a given nonlinear dynamical system in a regular way that is especially important for applications.

5.3 Existence criteria for a Lax representation

5.3.1 *The monodromy matrix and the Lax representation*

Again we consider the dynamical system (5.1), and assume that a Lax type representation (5.3) exists with operators $L[u; \lambda]$ and $p(L) : \mathcal{H} \to \mathcal{H}$ whose form is to be determined. We describe the gradient-holonomic technique [62, 173, 262, 308, 309, 326] for solving the direct problem of determining the Lax operator $L[u; \lambda] : \mathcal{H} \to \mathcal{H}$ starting from the form of the dynamical system (5.1).

Let $H \in \mathcal{D}(M)$ be an arbitrary conservation law of the dynamical system (5.1). Then $\mathrm{grad}H \in T^*(M)$ satisfies the Lax equation of Section 5.1.

Consider the generating function $\Delta(\lambda),\ \lambda \in \mathbb{C}$, of conservation laws which a priori exists for the dynamical system (5.1) by the assumption made above and which satisfies equation (5.24) with an unknown recursion operator Λ. Then the element $\mathrm{grad}\Delta(\lambda) \in T^*(M)$ obviously satisfies the same Lax equation (5.1) for all values of the parameter $\lambda \in \mathbb{C}$, that is

$$\frac{d}{dt}\mathrm{grad}\Delta(\lambda) + K'^*\mathrm{grad}\Delta(\lambda) = 0. \tag{5.37}$$

If $S(x_0;\lambda),\ \lambda \in \mathbb{C}$, is the corresponding monodromy matrix for the operator $L[u,\lambda] : \mathcal{H} \to \mathcal{H}$, then it follows from (5.23) that the equation (5.37) takes the form

$$\frac{d}{dt}\mathrm{tr}(S\mathcal{A}) + K'^*\mathrm{tr}(S\mathcal{A}) = 0 \tag{5.38}$$

where the matrix-valued vector $\mathcal{A} \in (Hom\ \mathbb{C}^k)^m$ is defined according to (5.29).

Some explicit formulas for the matrix-valued vector $\mathcal{A} \in (Hom\ \mathbb{C}^k)^m$ can be obtained from expressions (5.21) and (5.22). For example,

$$\mathcal{A}[l] = l_u, \tag{5.39}$$

if the matrix $l[u;\lambda]$ depends only on function $u \in M$ and not its derivatives with respect to variable $x \in \mathbb{R}$, and

$$\mathcal{A}[l] = l_u - \frac{d}{dx}l_{u_x} + [l, l_{u_x}], \tag{5.40}$$

if $l[u;\lambda] = l(u,u_x;\lambda)$, i.e., if the matrix $l[u;\lambda]$ depends only on function $u \in M$ and its derivative with respect to variable $x \in \mathbb{R}$, and so on.

In many cases, if the Fréchet derivative $K' : T(M) \to T(M)$ of a vector field $K : M \to T(M)$ is not in integro-differential form, the matrix $l[u;\lambda]$ in the operator $L[u;\lambda] : \mathcal{H} \to \mathcal{H}$ can often be chosen independent of the derivatives of the function $u \in M$.

5.3.2 *The gradient-holonomic method for constructing conservation laws*

Let $\bar{Y}(x,x_0;\lambda),\ x,x_0 \in \mathbb{R},\ \lambda \in \mathbb{C}$, be the matrix Bloch solution to equation (5.15) satisfying

$$d\bar{Y}/dx = l\bar{Y}, \quad \bar{Y}(x+2\pi,x_0;\lambda) = \bar{Y}(x,x_0;\lambda)\,Q(\lambda), \tag{5.41}$$

with diagonal matrix

$$Q(\lambda) = \{\xi_j(\lambda)\,\delta_{ij} :\ 1 \le i,j \le k\}$$

where $\xi_j(\lambda) \in \mathcal{D}(M)$, $1 \le j \le k$, are the eigenvalues of the monodromy matrix $S(x_0;\lambda)$.

Clearly, the matrix $\bar{S} = \bar{Y}C\bar{Y}^{-1}$, where $C = \{C_{ij} : 1 \le i, j \le k\}$ is an arbitrary constant matrix, satisfies the same differential equation (5.7) as does the monodromy matrix S, that is

$$d\bar{S}/dx = [l, \bar{S}], \tag{5.42}$$

where the matrix $l = l[u;\lambda],\ u \in M$, is the same as in (5.41).

Hence, by choosing the matrix $C = \{C_{ij} : 1 \le i, j \le k\}$, the monodromy matrix $S(x;\lambda)$ can be identified with the matrix $\bar{S}(x;\lambda)$, that is

$$S(x;\lambda) = \bar{Y}C\bar{Y}^{-1} \tag{5.43}$$

for all $x \in \mathbb{R},\ \lambda \in \mathbb{C}$.

Since the column $\bar{y}_n$ of the matrix $\bar{Y} = \{\bar{y}_n : 1 \le n \le k\}$ has as $\lambda \to \{\infty\}$ an asymptotic expansion of the form

$$\bar{y}_n(x, x_0;\lambda) \sim \bar{b}^{(n)}(x;\lambda) \exp\ [\int_{x_0}^{x} \sigma^{(n)}(x;\lambda)\, dx], \tag{5.44}$$

where

$$\bar{b}^{(n)}(x;\lambda) \sim \sum_{j\in\mathbb{Z}_+} \bar{b}_j^{(n)}[u]\, \delta^{-j}(\lambda), \quad \sigma^{(n)}(x;\lambda) \sim \sum_{j\in\mathbb{Z}_+} \sigma_j^{(n)}[u]\, \delta^{-j+s_n}(\lambda),$$

for some $s_n \in \mathbb{Z}_+$, then, according to (5.38) and (5.43), the Lax equation

$$d\varphi/dt + K'^{*}\varphi = 0, \tag{5.45}$$

where $\varphi = \operatorname{grad}\operatorname{tr} S(x;\lambda) \in T^*(M)$ admits the following asymptotic solution as $\delta(\lambda) \to \{\infty\}$:

$$\varphi = b(x;\lambda) \exp[\int_{x_0}^{x} \sigma(x;\lambda)\, dx + \omega(\lambda;t)]. \tag{5.46}$$

Here the vector-function $b \in T^*(M)$ is normalized by making its first component unity, $\sigma \in C^{(\infty)}(\mathbb{R};\mathbb{C})$ and we have the following asymptotic expansions

$$b(x;\lambda) \sim \sum_{j\in\mathbb{Z}_+} b_j[u]\, \delta^{-j}(\lambda), \quad \sigma(x;\lambda) \sim \sum_{j\in\mathbb{Z}_+} \sigma_j[u]\delta^{-j+s}(\lambda), \tag{5.47}$$

if $\delta(\lambda) \to \infty$.

The function $\omega : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ takes into account the part of the asymptotics (5.46) explicitly depending on the dynamical variable $t \in \mathbb{R}$, and the expansion (5.46) is fundamental for determining conservation laws $\gamma_j \in \mathcal{D}(M)$, $j \in \mathbb{Z}_+$, via the invariance of the expression

$$\gamma(\lambda) = \int\limits_{x_0}^{x_0+2\pi} \sigma(x;\lambda)\,dx \tag{5.48}$$

for any $x_0 \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ subject to dynamical system (5.1), as follows from expression (5.44). This, in particular, leads to the relationships

$$\int\limits_{x_0}^{x_0+2\pi} \sigma_n(x;\lambda)\,dx = \xi_n(\lambda), \tag{5.49}$$

where $\xi_n(\lambda) \in \mathbb{C}$, $1 \leq n \leq k$, are the eigenvalues of the monodromy matrix $S(x;\lambda)$; this follows directly from equation (5.45) owing to the periodicity of functions $b(x;\lambda)$ and $\sigma(x;\lambda)$ in $x \in \mathbb{R}$ for all $\lambda \in \mathbb{C}$.

Expanding the right-hand side of (5.48) in asymptotic series according to (5.47), we find that

$$\gamma(\lambda) \sim \sum_{j\in\mathbb{Z}_+} \gamma_j\,\delta^{-j+s(\lambda)}, \tag{5.50}$$

where owing to (5.48), the functionals

$$\gamma_j = \int_{x_0}^{x_0+2\pi} \sigma_j[u]\,dx,$$

$j \in \mathbb{Z}_+$, are conservation laws of the dynamical system (5.1). To calculate them in explicit form we have as usual to substitute solution (5.46) into equation (5.45). The recursion relations for the local functionals $\sigma_j : M \to \mathbb{R}$ and $b_j : M \to \mathbb{R}$, $j \in \mathbb{Z}_+$, appearing after the substitution, are always solvable in explicit form. Thus, we have sketched an effective algorithm for constructing conservation laws for a dynamical system (5.1).

5.3.3 *Construction of compatible implectic operators*

Suppose a recursive operator $\Lambda : T^*(M) \to T^*(M)$ for dynamical system (5.1) exists and acts on the hierarchy of conservation laws (5.50) by the rule (5.26):

$$\Lambda \text{grad}\, \gamma_j = \text{grad}\, \gamma_{j+q}, \tag{5.51}$$

where $j \in \mathbb{Z}_+$ and $q \in \mathbb{Z}$ is fixed.

To determine such an operator Λ, we represent the dynamical system (5.1) in Hamiltonian form as

$$du/dt = -\vartheta \mathrm{grad} H = K[u], \tag{5.52}$$

where

$$H = \sum_{j=-q}^{\bar{m}} c_j \gamma_{j+q}$$

and $c_j \in \mathbb{R}$, $-q \leq j \leq \bar{m}$, are constants.

To define an unknown implectic operator $\vartheta : T^*(M) \to T(M)$ by means of the known conservation laws $\gamma_j \in \mathcal{D}(M)$, $j \in \mathbb{Z}_+$, we shall use the algorithm described in what follows. Consider on a functional manifold M a Hamiltonian system

$$du/dt = -\vartheta \mathrm{grad}\ H = K[u], \tag{5.53}$$

where $H \in \mathcal{D}(M)$ is its Hamiltonian function and $\vartheta : T^*(M) \to T(M)$ is the corresponding co-symplectic structure on M. Assume now that an element $\sigma \in T^*(M)$ satisfies the invariance conditions

$$L_K \sigma = \sigma' K + K'^* \sigma = 0, \quad \sigma' \neq \sigma'^*. \tag{5.54}$$

Then, from (5.53) and (5.54) one easily computes that

$$(\sigma' - \sigma'^*) K + \sigma'^* K + K'^* \sigma = \vartheta^{-1} K + \mathrm{grad}\ (\sigma, K) = 0, \tag{5.55}$$

if the Hamiltonian function $H \in \mathcal{D}(M)$ allows the representation

$$H = (\sigma, K), \quad L_K \sigma = 0, \tag{5.56}$$

and the symplectic structure on M is given as

$$\vartheta^{-1} = \sigma' - \sigma'^*. \tag{5.57}$$

Taking into account that the $\mathbb{R} \ni x$-shift vector field

$$du/d\tau := u_x \tag{5.58}$$

always commutes with the vector field (5.53), it is easy to see that the vector field (5.58) is also Hamiltonian, namely

$$du/dx := -\vartheta \mathrm{grad} \gamma \tag{5.59}$$

for a functional $\gamma \in \mathcal{D}(M)$. Assuming that the representation

$$\gamma = (\sigma, u_x), \quad \sigma' \neq \sigma'^*, \tag{5.60}$$

holds, upon substituting $K = du/dx$ we readily obtain from (5.56) and (5.57) the symplectic structure on M if the additional condition $L_K \sigma = 0$

is satisfied. Note here that the condition $L_{\frac{d}{dx}}\sigma = 0$ is satisfied identically for any $\sigma \in T^*(M)$.

Thus, using the hierarchy $\{\gamma_j \in \mathcal{D}(M) : j \in \mathbb{Z}_+\}$ of conservation laws for the dynamical system (5.53), one can construct the corresponding co-symplectic structures on the manifold M. In particular, if $\bar{\gamma} = (\bar{\sigma}, u_x) \in \mathcal{D}(M)$, where $L_K\bar{\sigma} = 0$, is another conservation law of dynamical system (5.53), the expression

$$\eta^{-1} := \bar{\sigma}' - \bar{\sigma}'^* \tag{5.61}$$

defines on M a second co-symplectic structure $\eta : T^*(M) \to T(M)$ with respect to which the following representation

$$du/dt = -\eta \mathrm{grad}\bar{H} = K[u], \tag{5.62}$$

holds, where $\bar{H} := (\sigma, K)$ is the corresponding Hamiltonian function.

Let the obtained operator ϑ satisfy equation (5.24), so that it is Nötherian. It follows from (5.27) and (5.51) that

$$\vartheta \ \mathrm{grad}\gamma_{j+q} = \eta \ \mathrm{grad}\gamma_j \tag{5.63}$$

holds for all $j \in \mathbb{Z}_+$, so by a straightforward calculation one can find the implectic operator η.

Here, in general, we are looking for the operator η in the class of Nötherian, skew-symmetric integro-differential operators. As a result of formulae (5.62) and (5.63), we find that the dynamical system (5.1) is bi-Hamiltonian, that is

$$du/dt = \vartheta \ \mathrm{grad}\ H = -\eta \ \mathrm{grad}\bar{H}, \tag{5.64}$$

where

$$\bar{H} = \sum_{j=-q}^{\bar{m}} \bar{c}_j \gamma_j.$$

The method developed above for obtaining the implectic operators ϑ and η, by factorizing the operator Λ, is quite effective in practice. Another technique for obtaining the operator $\Lambda = \vartheta^{-1}\eta$, which is based on the small parameter expansion [84, 262, 264, 326], Fourier transforms and solving the Nötherian equations $L_K\vartheta = 0 = L_K\eta$, shall also be treated in this chapter.

5.3.4 *Reconstruction of the Lax operator algorithm*

Let operators ϑ and η obtained above be implectic and Nötherian. By virtue of Theorem 4.3, the operator $\Lambda = \vartheta^{-1}\eta$ is hereditarily recursive if the operator $[\Lambda^*, \Lambda^{*\prime}]$ is symmetric, where $\Lambda^* = \eta\vartheta^{-1}$. Then all dynamical systems $du/t_j = -\vartheta \mathrm{grad}\gamma_j,\ t_j \in \mathbb{R},\ j \in \mathbb{Z}_+$, associated with the dynamical system (5.52), possess a common system of conservation laws which are in involution, namely

$$\{\gamma_j, \gamma_k\}_\vartheta = 0 = \{\gamma_j, \gamma_k\}_\eta, \tag{5.65}$$

$j, k \in \mathbb{Z}_+$. The expressions (5.65) are valid if there exists a generating function $\gamma(\lambda), \lambda \in \mathbb{C}$, for which the general relation of the form (5.63) holds:

$$\lambda^q \vartheta \mathrm{grad}\gamma(\lambda) = \eta \mathrm{grad}\gamma(\lambda). \tag{5.66}$$

Owing to relations (5.48), (5.50) and (5.53), such a function exists and can be taken as the expression (5.48). Upon comparing equations (5.66), (5.24) and (5.25), one sees that the generating function $\gamma(\lambda) \in \mathcal{D}(M),\ \lambda \in \mathbb{C}$, can be identified with

$$\Delta(\lambda) = \mathrm{tr} S(x_0; \lambda),$$

where $S(x_0; \lambda),\ x_0 \in \mathbb{R}$, is the monodromy matrix of an operator $L[u; \lambda] : \mathcal{H} \to \mathcal{H}$ which is assumed to exist.

Thus, from (5.23) and (5.66) we obtain

$$\lambda^q \vartheta \mathrm{tr}(S\mathcal{A}) = \eta \mathrm{tr}(S\mathcal{A}) \tag{5.67}$$

for all $\lambda \in \mathbb{C}$ and $u \in M$.

The equation (5.67) is fundamental partly because it contains only differentiation and integration operations with respect to the variable $x \in \mathbb{R}$. The differentiation operation $\partial = \partial/\partial x,\ x \in \mathbb{R}$, is determined in the usual way. The integration operation (indeterminate) in the case of a periodic functional manifold M is defined here as

$$\partial^{-1}(\cdot) = \frac{1}{2}\left[\int_{x_0}^{x} (\cdot)\, dx - \int_{x_0}^{x_0+2\pi} (\cdot)\, dx\right], \tag{5.68}$$

where $x_0 \in \mathbb{R}$ is arbitrary. Obviously, in the class of periodic functions the operation (5.68) has the desired property $\partial\,\partial^{-1} = 1$, but $\partial^{-1}\,\partial = 1 + \delta_{x_0}$, where δ_{x_0} is the Dirac delta-function with support at the point $x_0 \in \mathbb{R}$.

Thus, by eliminating in (5.67) the integration and differentiation operations of the monodromy matrix S by means of equation (5.7), one can transform (5.67) to the functional trace-form:

$$\operatorname{tr}(S\mathcal{B}) = 0, \tag{5.69}$$

where the matrix-valued vector $\mathcal{B} := \mathcal{B}[u;\lambda|\, l\,] \in (Hom\ \mathbb{C}^k)^m$ is defined uniquely by the matrix $l[u;\lambda]$ for $u \in M,\ \lambda \in \mathbb{C}$.

Owing to the arbitrariness of the monodromy matrix as a solution to the differential equation (5.7), we conclude that equation (5.69) holds if and only if the matrix-valued vector $\mathcal{B}$ is identically zero, that is

$$\mathcal{B}[u;\lambda|\, l\,] = 0 \tag{5.70}$$

for all $u \in M,\ \lambda \in \mathbb{C}$.

The size of the matrix $l[u;\lambda]$ - some number $k \in \mathbb{Z}_+$ - is determined uniquely both by the orders of the integro-differential parts of operators ϑ and η and by the size of matrices representing them - the number $m \in \mathbb{Z}_+$. Namely, according to (5.67) one can show that, after transformations, the first component of the matrix-valued vector $\operatorname{tr}(S\mathcal{A})$ satisfies a linear differential equation of the minimal order ν for the operator ϑ. Then, the order $k \in \mathbb{Z}_+$ of a matrix $l[u;\lambda]$ satisfies the equation $\nu \leq k^2 - 1$, where we have taken into consideration that the quantity $\operatorname{tr}S(x_0;\lambda)$ is obviously a linear invariant of equation (5.7).

As the matrix-valued vector $\mathcal{B}[u;\lambda \mid l]$ explicitly depends on the matrix $l[u;\lambda]$, the system of equations (5.70) can be solved algebraically and the components of this matrix $l[u;\lambda]$ can be explicitly determined [69, 326] as local functionals of $u \in M$. Accordingly the problem of finding an explicit Lax representation for the given dynamical system (5.1), under assumption that such a representation exists, has been solved here by means of the gradient-holonomic technique devised in [173, 262, 308, 326, 389]. Its efficiency for applications to particular systems will be presented in the sequel of this chapter. Another approach, employing methods from differential geometry, to finding Lax representations for an integrable dynamical system is considered in [326].

5.3.5 *Asymptotic construction of recursive and implectic operators for Lax integrable dynamical systems*

Consider a smooth dynamical system

$$u_t = K[u], \tag{5.71}$$

where a function $u \in M \subset C^{\infty}_{2\pi}(\mathbb{R};\mathbb{R}^m)$ - the smooth 2π-periodic functions in the variable $x \in \mathbb{R}$, $m \in \mathbb{Z}_+$.

If the dynamical system (5.71) admits an infinite hierarchy

$$\{\gamma_j \in \mathcal{D}(M) : j \in \mathbb{Z}_+\}$$

of quasi-local conservation laws, then according to formula (5.48), the generating function

$$\gamma(\lambda) = \sum_{j\in\mathbb{Z}_+} \gamma_j \lambda^{-j+s}$$

satisfies the Lax equation

$$L_K \mathrm{grad}\gamma(\lambda) = \frac{d}{dt}\mathrm{grad}\gamma(\lambda) + K'^{*}\mathrm{grad}\gamma(\lambda) = 0. \tag{5.72}$$

Denoting $\varphi := \mathrm{grad}\gamma(\lambda)$, $\lambda \in \mathbb{C}$, we rewrite equation (5.72) as

$$L_K\varphi = d\varphi/dt + K'^{*}\varphi = 0. \tag{5.73}$$

Equation (5.73) has as $\lambda \to \{\infty\}$ an asymptotic solution in the form

$$\varphi(x,t;\lambda) = \begin{pmatrix} 1 \\ b_1(x,t;\lambda) \\ \cdots \\ b_{m-1}(x,t;\lambda) \end{pmatrix} \exp\,[\omega(t;\lambda) + \int_{x_0}^{x} \sigma(x,t;\lambda)\,dx], \tag{5.74}$$

where $x_0 \in \mathbb{R}$ is fixed, $\omega(t;\lambda)$ is the temporal dispersion function of the linearized dynamical system (5.72) in a neighborhood of $\bar{u} = 0 \in M$, and the quantities σ and b_k, $1 \le k \le m$, are local functionals on M having the asymptotic representation

$$\sigma(x,t;\lambda) \sim \sum_{j\in\mathbb{Z}_+} \sigma_j[u]\lambda^{-j+s}, \quad b_k(x,t;\lambda) \sim \sum_{j\in\mathbb{Z}_+} b_j^{(k)}[u]\lambda^{-j+s}, \tag{5.75}$$

for $u \in M$, as $|\lambda| \to \infty$ for some $s \in \mathbb{Z}_+$.

From formula (5.74) and equation (5.73), we readily find that the functional

$$\gamma(\lambda) = \int_{x_0}^{x_0+2\pi} \sigma(x,t;\lambda)\,dx,$$

as $|\lambda| \to \infty$ is a generating function of conservation laws

$$\gamma_j = \int_{x_0}^{x_0+2\pi} \sigma_j[u]\,dx,$$

$j \in \mathbb{Z}_+$, for the dynamical system (5.71).

5.3.6 *A small parameter method for constructing recursion and implectic operators*

Assume that for the conservation laws $\gamma_j \in \mathcal{D}(M)$, $j \in \mathbb{Z}_+$, the equations (5.51)

$$\Lambda \mathrm{grad}\gamma_j = \mathrm{grad}\gamma_{j+q}, \tag{5.76}$$

obtain, where $q \in \mathbb{Z}$ is a fixed integer uniquely defined by the condition $\mathrm{grad}\gamma_{j_0} = 0 = \mathrm{grad}\gamma_{j_0+q}$, $j_0 \in \mathbb{Z}_+$.

From (5.76), in particular, it is easy to see that a recursion operator $\Lambda : T^*(M) \to T^*(M)$ satisfies equation (5.35),

$$L_K\Lambda = d\Lambda/dt - [\Lambda, K'^*] = 0. \tag{5.77}$$

To find the operator Λ in explicit form, as a solution of equation (5.77), we apply a small parameter expansion technique [173, 264, 326]. Let a function $u = \varepsilon u^{(1)} \in M$, where $\varepsilon \in \mathbb{R}$ is a small parameter. Then [173, 326] the dynamical system (5.71) can be written with accuracy up to first order in $\varepsilon \in \mathbb{R}$ in the form of the linear dynamical system

$$du^{(1)}/dt_0 = K_0'[0]u^{(1)}, \tag{5.78}$$

where we took into account that

$$d/dt := d/dt_0 + \varepsilon d/dt_1 + \varepsilon^2 d/dt_2 + \dots, \tag{5.79}$$

$$K[\varepsilon u^{(1)}] = \varepsilon K^{(1)}[u^{(1)}] + \varepsilon^{(2)} K^{(2)}[u^{(1)}] + \varepsilon^3 K^{(3)}[u^{(1)}] + \dots,$$

$$K'[\varepsilon u^{(1)}] = K_0'[0] + \varepsilon K_1'[u^{(1)}] + \varepsilon^2 K_2'[u^{(1)}] + \dots.$$

A solution to (5.78) can obviously be represented by the Fourier series:

$$u_k^{(1)} = \sum_{j\in\mathbb{Z}} \bar{u}_{kj}^{(1)} \exp[\omega_{kj}(x, t_0)], \tag{5.80}$$

where $\bar{u}_{kj}^{(1)} \in \mathbb{C}$ are constant Fourier coefficients, and ω_{kj}, $1 \le k \le m$, $j \in \mathbb{Z}$, are the corresponding dispersion functions of solutions to equation (5.78). Also, at the zero order in $\varepsilon \in \mathbb{R}$ from (5.73) we obtain the linearized dynamical system

$$d\varphi^{(0)}/dt_0 + K'^*_0\varphi^{(0)} = 0. \tag{5.81}$$

The solution to equation (5.81) has the Fourier representation

$$\varphi_k^{(0)} = \sum_{j\in\mathbb{Z}} \bar{\varphi}_{kj}^{(0)} \exp\bar{\omega}_{k,j}(x, t), \tag{5.82}$$

where $\varphi^{(0)} := (\varphi_k^{(0)} : 1 \leq k \leq m)^\top \in T^*(M)$, $\bar{\varphi}_{k,j}^{(0)} \in \mathbb{C}$ are Fourier coefficients, and $\bar{\omega}_{kj}$, $1 \leq k \leq m$, $j \in \mathbb{Z}$, are the corresponding dispersion functions of the primary Fourier solutions to equation (5.81).

Let us suppose now that

$$\Lambda[\varepsilon u^{(1)}] \sim \sum_{m \in \mathbb{Z}_+} \varepsilon^m \Lambda^{(m)}[u^{(1)}]$$

is an asymptotic series expansion of a recursive operator Λ in the small parameter $\varepsilon \in \mathbb{R}$. The operator $K'^* : T^*(M) \to T^*(M)$ has the similar expansion

$$K'^* \sim \sum_{p \in \mathbb{Z}_+} \varepsilon^p K'^*_p,$$

where $K'^*_0 := K'^*[0]$.

The form of the operator $\Lambda^{(0)}$ is easily obtained from relation (5.76). To obtain the coefficients $\Lambda^{(m)}$, $m \in \mathbb{Z}_+$, we make use of equation (5.77) in the following form:

$$(d\Lambda/dt)\varphi^{(0)} + [\Lambda, K'^*]\varphi^{(0)} = 0; \tag{5.83}$$

whence, we find a series of equivalent equations:

$$\begin{aligned}
&d(\Lambda^{(0)}\varphi^{(0)})/dt_0 + K'^*_0(\Lambda^{(0)}\varphi^0) = 0, \\
&d(\Lambda^{(1)}\varphi^{(0)})/dt_0 + K'^*_0(\Lambda^{(1)}\varphi^{(0)}) + K'^*_1(\Lambda^{(0)}\varphi^{(0)}) - \Lambda^{(0)}(K'^*_1\varphi^{(0)}) = 0, \\
&d(\Lambda^{(2)}\varphi^{(0)})/dt_0 + K'^*_0(\Lambda^{(2)}\varphi^0) + K'^*_1(\Lambda^{(1)}\varphi^{(0)}) + K'^*_2(\Lambda^{(0)}\varphi^{(0)}) \\
&- \Lambda^{(0)}(K'^*_2\varphi^{(0)}) - \Lambda^{(1)}(K'^*_1\varphi^{(0)}) = 0, \ldots,
\end{aligned} \tag{5.84}$$

these equations can be solved inductively using the Fourier transform; thus, we obtain the operators $\Lambda^{(m)}[u^{(1)}]$, $m \in \mathbb{Z}_+$, and so also the desired recursive operator Λ.

The same scheme can be applied to the problem of finding exact analytical expressions for implectic structures related to integrable dynamical system (5.71). More precisely, the determining Nötherian equation for a suitable implectic operator $\vartheta : T^*(M) \to T(M)$ has the form

$$d\vartheta/dt - \vartheta K'^* - K'\vartheta = 0. \tag{5.85}$$

Then, from (5.85) and (5.79) one readily obtains

$$\begin{aligned}
&d\vartheta_0/dt_0 - \vartheta_0 K'^*_0 - K'_0\vartheta_0 = 0, \\
&d\vartheta_1/dt_0 - \vartheta_1 K'^*_0 - \vartheta_0 K'^*_1 - K'_0\vartheta_1 - K'_1\vartheta_0 = 0, \\
&d\vartheta_2/dt_0 - \vartheta_0 K'^*_2 - \vartheta_1 K'^*_1 - \vartheta_2 K'^*_0 - K'_2\vartheta_0 - K'_1\vartheta_1 - K'_0\vartheta_2 = 0,
\end{aligned} \tag{5.86}$$

where we took into account the expansion

$$\vartheta = \vartheta_0 + \varepsilon\vartheta_1 + \varepsilon^2\vartheta_2 + \ldots \tag{5.87}$$

at a point $u := \varepsilon u^{(1)} \in M$ as $\varepsilon \to 0$.

Now, it follows directly from equations (5.81), (5.86) that

$$\begin{aligned}
&d(\vartheta_0\varphi^{(0)})/dt_0 - K_0'(\vartheta_0\varphi^{(0)}) = 0,\\
&d(\vartheta_1\varphi^{(0)})/dt_0 - K_0'(\vartheta_1\varphi^{(0)}) - K_1'(\vartheta_0\varphi^{(0)}) - \vartheta_0(K'^*_1\varphi^0) = 0,\\
&d(\vartheta_2\varphi^{(0)})/dt_0 - K_0'(\vartheta_2\varphi^{(0)}) - K_1'(\vartheta_1\varphi^{(0)}) - K_2'(\vartheta_0\varphi^{(0)}) - \vartheta_1(K'^*_1\varphi^{(0)}) = 0,
\end{aligned} \tag{5.88}$$

and so on.

Now that we have solved the Fourier transform equations (5.88) for the actions $\vartheta_j\varphi^{(0)} \in T(M)$, $j \in \mathbb{Z}_+$, and extracted the exact expressions for components $\vartheta_j : T^*(M) \to T(M)$, $j \in \mathbb{Z}_+$, we obtain from (5.87) the desired implectic operator $\vartheta : T^*(M) \to T(M)$ for the dynamical system (5.71). We shall next illustrate the effectiveness of the above approach by applying it to several well-known examples.

Example 5.1. The Korteweg–de Vries (KdV) equation

The Korteweg–de Vries equation is

$$du/dt = K[u] := -u_{xxx} - 6uu_x, \tag{5.89}$$

where $u \in M \subset C^\infty_{2\pi}(\mathbb{R};\mathbb{R})$, $t \in \mathbb{R}$. Since the operator $K'^*[u] = \partial^3 + 6u\partial$, equation (5.73) has the form

$$d\varphi/dt = -6u\varphi_x - \varphi_{xxx}, \tag{5.90}$$

which has the solution

$$\varphi(x,t;\lambda) = \exp[\lambda x - \lambda^3 t + \int_{x_0}^{x} \sigma(x,t;\lambda)\,dx], \tag{5.91}$$

where $\lambda \in \mathbb{C}$, $x_0 \in \mathbb{R}$ is fixed and

$$\sigma(x,t;\lambda) \sim \sum_{j\in\mathbb{Z}_+} \sigma_j[u]\lambda^{-j} \tag{5.92}$$

is an asymptotic expansion of the local functional σ on M as $|\lambda| \to \infty$.

From (5.90)–(5.92) we find the following recursive relationships for the local functionals $\sigma_j : M \to \mathbb{R}$, $j \in \mathbb{Z}_+$:

$$\frac{d}{dt}\int_{x_0}^{x} \sigma_j[u]\,dx = -6u\delta_{j,-1} - 6u\sigma_j - \sigma_{j,xx} - 3\sigma_{j+1,x} - 3\sum_{k\in\mathbb{Z}_+}\sigma_{j-k}\sigma_{k,x}$$
$$- 3\sigma_{j+2} - 3\sum_{k\in\mathbb{Z}_+}\sigma_{j+1-k}\sigma_k - \sum_{k,n\in\mathbb{Z}_+}\sigma_{j-k-n}\sigma_k\sigma_n. \qquad (5.93)$$

Observe that

$$\gamma_j = \int_{x_0}^{x_0+2\pi} \sigma_j[u]\,dx,$$

$j = \mathbb{Z}_+$, are conservation laws for the dynamical system (5.85), and they can all be determined recursively from (5.93). The first six of these conservation laws are

$$\gamma_0 = 0, \quad \gamma_1 = -2\int_{x_0}^{x_0+2\pi} u dx,$$
$$\gamma_2 = 0, \quad \gamma_3 = -2\int_{x_0}^{x_0+2\pi} u^2 dx,$$
$$\gamma_4 = 0, \quad \gamma_5 = -12\int_{x_0}^{x_0+2\pi} uu_{xx}dx - 10\int_{x_0}^{x_0+2\pi} u_x^2 dx - 4\int_{x_0}^{x_0+2\pi} u^3 dx, \ldots. \qquad (5.94)$$

According to (5.94), the expressions for $\mathrm{grad}\gamma_j \in T^*(M)$, $j \in \mathbb{Z}_+$, are:

$$\begin{aligned}&\mathrm{grad}\gamma_0 = \mathrm{grad}\gamma_2 = \mathrm{grad}\gamma_4 = 0,\\ &\mathrm{grad}\gamma_1 = -2, \quad \mathrm{grad}\gamma_3 = -4u, \quad \mathrm{grad}\gamma_5 = -4[u_{xx} + 3u^2], \ldots,\end{aligned} \qquad (5.95)$$

and so on.

As $\Lambda\mathrm{grad}\gamma_0 = \mathrm{grad}\gamma_2 = \mathrm{grad}\gamma_{0+q} = 0$, the number $q = 2$ and the operator Λ satisfies

$$\Lambda\mathrm{grad}\gamma_{2j+1} = \mathrm{grad}\gamma_{2j+3} \qquad (5.96)$$

for all $j \in \mathbb{Z}_+$. It follows from equations (5.95) and (5.96) that the recursion operator Λ at $u = \varepsilon u^{(1)} \in M$ has the finite expansion

$$\Lambda = \Lambda^{(0)} + \Lambda^{(1)}\varepsilon, \qquad (5.97)$$

where $\Lambda^{(0)} = \partial^2$. A similar expansion holds for the operator $K'^* : T^*(M) \to T(M)$, namely

$$K'^* = K'_0 + K'_1\varepsilon, \qquad (5.98)$$

where $K_0' = \partial^3$, $K_1' = 6u^{(1)}\partial$. Here the functions $u^{(1)} \in T(M)$ and $\varphi^{(0)} \in T^*(M)$ can be represented in Fourier series as

$$u^{(1)} = \sum_{j\in 2\pi i\mathbb{Z}} \bar{u}_j^{(1)} \exp(jx - j^3 t), \quad \varphi^{(0)} = \sum_{j\in 2\pi i\mathbb{Z}} \bar{\varphi}_j^{(0)} \exp(jx - j^3 t), \quad (5.99)$$

where $i := \sqrt{-1}$.

Substituting expressions (5.97)–(5.99) into (5.84), we find that the result of action of the operator $\Lambda^{(1)}$ on the function $\varphi^{(0)}$ is

$$\Lambda^{(1)}\varphi^{(0)} = \sum_{j,k\in 2\pi i\mathbb{Z}} \left(2 + \frac{2j}{j+k}\right) \varphi_j^{(0)} \bar{u}_k^{(1)} \exp[(j+k)x - (j^3+k^3)t], \quad (5.100)$$

or equivalently

$$\Lambda^{(1)}\varphi^{(0)} = 2u^{(1)}\varphi^{(0)} + 2\partial^{-1}u^{(1)}\varphi_x^{(0)}. \quad (5.101)$$

Thus, owing to (5.101), the recursive operator (5.97) has the form

$$\Lambda = \partial^2 + 2u + 2\partial^{-1}u\partial, \quad (5.102)$$

where it was taken into consideration that $u = \varepsilon u^{(1)} \in M$. By straightforward calculations, one can readily verify that the operator Λ of form (5.102) satisfies the characteristic equation (5.77). As the dynamical system (5.85) possesses the Hamiltonian form

$$du/dt = -\vartheta \text{grad } H, \quad (5.103)$$

where $H = \frac{1}{4}\gamma_5$ and $\vartheta = \partial : T^*(M) \to T(M)$ is an implectic and Nötherian operator, it follows directly from the representation (5.102) that the operator Λ factorizes as $\Lambda = \vartheta^{-1}\eta$, where

$$\eta = \vartheta\Lambda = \partial^3 + 2(u\partial + \partial u). \quad (5.104)$$

The operator η is obviously implectic and Nötherian, and the pair (ϑ, η) is compatible because the operator $[\Lambda^*, \Lambda^{*\prime}]$ is symmetric, where $\Lambda^* = \eta\vartheta^{-1}$. Hence, Λ is also hereditarily recursive. Analogously, one can determine suitable implectic operators ϑ and $\eta : T^*(M) \to T(M)$ factorizing the recursive operator $\Lambda : T^*(M) \to T^*(M)$.

Example 5.2. The modified Korteweg–de Vries (mKdV) equation

This equation is

$$du/dt = K[u] := -6u^2 u_x - u_{xxx}, \quad (5.105)$$

where $u \in M \subset C_{2\pi}^\infty(\mathbb{R};\mathbb{R})$, $t \in \mathbb{R}$.

Taking into consideration the form of the operator $K'^{*} = \partial^3 + 6u^2\partial$, we obtain from (5.73) the following functional equation

$$d\varphi/dt = -\varphi_{xxx} - 6u^2\varphi_x, \tag{5.106}$$

whose solution $\varphi \in T^*(M)$ is

$$\varphi(x,t;\lambda) = \exp[\lambda x - \lambda^3 t + \int_{x_0}^{x} \sigma(x,t;\lambda)\,dx], \tag{5.107}$$

where $\lambda \in \mathbb{C}$, $x, x_0 \in \mathbb{R}$. As $|\lambda| \to \infty$, the local functional $\sigma : M \to \mathbb{R}$ has the asymptotic expansion

$$\sigma(x,t;\lambda) \sim \sum_{j\in\mathbb{Z}_+} \sigma_j[u]\lambda^{-j}, \tag{5.108}$$

where the local functionals $\sigma_j : M \to \mathbb{R}$, $j \in \mathbb{Z}_+$, satisfy the recursion relations

$$\begin{aligned}\frac{d}{dt}\int_{x_0}^{x}\sigma_,[u]dx &= -6u^2\left(\delta_{j,-1}+\sigma_j\right) - \sigma_{j,xx} - 3\sigma_{j+1,x} - 3\sum_{k=1}^{\infty}\sigma_{j-k}\sigma_{k,x} \\ &- 3\sigma_{j+2} - 3\sum_{k,n=1}^{\infty}\left(\sigma_{j+1-k} + \sigma_{j-k-n}\sigma_n\right)\sigma_k.\end{aligned} \tag{5.109}$$

The relationships (5.109) enable us to determine the whole hierarchy of local functionals σ_j, $j \in \mathbb{Z}_+$, in explicit form, the first five of which have the following form:

$$\begin{aligned}&\sigma_0 = 0, \quad \sigma_1 = -2u^2, \quad \sigma_2 = 4uu_x, \quad \sigma_3 = -2u_x^2 - 4uu_x - 2u^4, \\ &\sigma_4 = 4uu_{xxx} + 4u_xu_{xx} + 16u^3u_x.\end{aligned} \tag{5.110}$$

These local functionals produce the conservation laws

$$\gamma_j = \int_{x_0}^{x_0+2\pi} \sigma_j[u]\,dx,$$

for all $j \in \mathbb{Z}_+$, the first five of which are

$$\begin{aligned}&\gamma_0 = 0, \quad \gamma_1 = -2\int_{x_0}^{x_0+2\pi} u^2dx, \quad \gamma_2 = 0, \\ &\gamma_3 = -2\int_{x_0}^{x_0+2\pi} u_x^2dx - 4\int_{x_0}^{x_0+2\pi} uu_xdx - 2\int_{x_0}^{x_0+2\pi} u^4dx, \\ &\gamma_4 = \int_{x_0}^{x_0+2\pi}[4uu_{xxx} + 4u_xu_{xx} + 16u^3u_x]\,dx.\end{aligned} \tag{5.111}$$

Making use of (5.111) we find the corresponding expressions for $\text{grad}\gamma_j \in T^*(M)$, $j \in \mathbb{Z}_+$:

$$\begin{aligned} &\text{grad } \gamma_0 = \text{grad } \gamma_2 = \text{grad } \gamma_4 = 0, \\ &\text{grad } \gamma_1 = -4u, \quad \text{grad } \gamma_3 = -4u_{xx} - 8u^3, \ \dots , \end{aligned} \tag{5.112}$$

and so on.

By virtue of formula (5.112) for the operators Λ and $K'^* : T^*(M) \to T^*(M)$ when $u = \varepsilon u^{(1)} \in M$, we compute the asymptotic formulae

$$\Lambda = \Lambda^{(0)} + \varepsilon^2 \Lambda^{(2)}, \quad K'^* = K'_0 + \varepsilon^2 K'_2, \tag{5.113}$$

where $\Lambda^{(0)} = \partial^2$, $K'_0 = \partial^3$, $K'_1 = 0$, $K'_2 = 6[u^{(1)}]^2\partial$. In obtaining formulae (5.113), we took into account that by virtue of (5.112) the recursive operator Λ satisfies the equations

$$\Lambda \text{grad } \gamma_{2j+1} = \text{grad } \gamma_{2j+3} \tag{5.114}$$

for all $j \in \mathbb{Z}_+$, that is one immediately determines that $q = 2$.

Using the formulae (5.84) and Fourier series expansions for $u^{(1)} \in T(M)$ and $\varphi^{(0)} \in T^*(M)$ in the form

$$u^{(1)} = \sum_{j \in 2\pi i \mathbb{Z}} \bar{u}_j^{(1)} \exp(jx - j^3 t), \quad \varphi^{(0)} = \sum_{j \in 2\pi i \mathbb{Z}} \varphi_j^{(0)} \exp(jx - j^3 t), \tag{5.115}$$

for the result of the action of the operator $\Lambda^{(2)}$ on the function $\varphi^{(0)} \in T^*(M)$, we readily compute that

$$\begin{aligned} \Lambda^{(1)} \varphi^{(0)} = \sum_{j,k,m \in 2\pi i \mathbb{Z}} \left(\frac{k}{k+m} + \frac{k}{k+j} \right) \bar{u}_j^{(1)} \bar{u}_m^{(1)} \varphi_k^{(0)} \exp[(k+j+m)x \\ - (k^3 + j^3 + m^3)t] = 4u^{(1)} \partial^{-1} u^{(1)} \partial \varphi^{(0)}. \end{aligned} \tag{5.116}$$

Hence, the operator Λ is given as

$$\Lambda = \partial^2 + 4u\partial^{-1}u\partial, \tag{5.117}$$

where we took into account that $u = \varepsilon u^{(1)} \in M$ for all $\varepsilon \to 0$. Straightforward calculations show that the recursive operator Λ of (5.117) satisfies equation (5.77). Owing to the explicit formulae for the conservation laws (5.111), it is easy to verify that the dynamical system (5.105) admits a Hamiltonian representation in the form

$$du/dt = -\vartheta \text{grad} H, \tag{5.118}$$

where $H = \frac{1}{4}\gamma_3$ and $\vartheta = 4\partial$ is an implectic operator for (5.105).

Considering the factorization of the operator Λ in the form $\Lambda = \vartheta^{-1}\eta$, we immediately find that

$$\eta = \vartheta\Lambda = \partial^3 + \frac{1}{2}(u^2\partial + \partial u^2) - u_x\partial^{-1}u_x. \tag{5.119}$$

It is a simple matter to verify that the operator η is implectic and is Nötherian along with the operator ϑ. Moreover, the pair of operators (ϑ, η) is compatible, so the operator $\Lambda : T^*(M) \to T(M)$ is hereditarily recursive. The same procedure can also be applied for finding the corresponding implectic operators ϑ and η factorizing the recursion operator $\Lambda = \vartheta^{-1}\eta$.

Example 5.3. A nonlinear Schrödinger model (NLS equation)

This dynamical system is defined as

$$\begin{aligned} d\psi/dt &= i\psi_{xx} - 2\psi\psi^*\psi_x, \\ d\psi^*/dt &= -i\psi^*_{xx} - 2\psi\psi^*\psi^*_x, \end{aligned} \tag{5.120}$$

where $\psi \in M \subset C^\infty_{2\pi}(\mathbb{R};\mathbb{C})$ and the operator K'^* for dynamical system (5.120) is represented as

$$K'^* = \begin{pmatrix} i\partial^2 + 2\psi\partial\psi^* & -2\psi^*\psi^*_x \\ -2\psi\psi_x & -i\partial^2 + 2\psi^*\partial\psi^* \end{pmatrix}. \tag{5.121}$$

The corresponding solution $\varphi \in T^*(M)$ to equation (5.73) according to the representation (5.74) is

$$\varphi(x,t;\lambda) = \begin{pmatrix} 1 \\ b(x,t;\lambda) \end{pmatrix} \exp[\lambda x - i\lambda^2 t + \int\limits_{x_0}^{x} \sigma(x,t;\lambda)\, dx], \tag{5.122}$$

where as $|\lambda| \to \infty$ the local functionals $\sigma, b : M \to \mathbb{C}$ have the asymptotic expansions

$$\begin{aligned} \sigma(x,t;\lambda) &\sim \sum_{j\in\mathbb{Z}_+} \sigma_j[\psi,\psi^*]\lambda^{-j}, \\ b(x,t;\lambda) &\sim \sum_{j\in\mathbb{Z}_+} b_j[\psi,\psi^*]\lambda^{-j} \end{aligned} \tag{5.123}$$

for any $x \in \mathbb{R}$.

Substituting formulae (5.77) into equations (5.73) with the operator K'^* in the form of (5.121), we find for the local functionals $\sigma_j, b_j : M \to \mathbb{C}$,

$j \in \mathbb{Z}_+$, the following recursion relationships:

$$
\begin{aligned}
\frac{d}{dt}\int_{x_0}^{x} \sigma_j[\psi,\psi^*]\,dx = -i&\left[2\sigma_{j+1} + \sum_{k\in\mathbb{Z}_+}\sigma_k\sigma_{j-k} + \sigma_{j,x}\right] \\
&- 2\psi\psi_x^*\delta_{j,0} - 2\psi^*\psi_x^* b_j - 2\psi\psi^*(\delta_{j+1,0}+\sigma_j) \\
&- 2ib_{j+2} + \sum_{k\in\mathbb{Z}_+}\left(\frac{\partial}{\partial t}\int_{x_0}^{x}\sigma_k\,dx\right) b_{j-k} + b_{j,t} \\
= -2\psi\psi_x\delta_{j,0} + i&\left[2\sum_{k\in\mathbb{Z}_+}\delta_k b_{j+1-k} + \sum_{k,n\in\mathbb{Z}_+}\sigma_k\sigma_n b_{j-k-n}\right. \\
&\left. + \sum_{k\in\mathbb{Z}_+}\sigma_{k,x}b_{j-k} + 2b_{j+1,x} + 2\sum_{k\in\mathbb{Z}_+}\sigma_k b_{j-k,x} + b_{j,xx}\right] \\
&- 2\psi^*\psi_x b_j - 2\psi^*\psi\left(b_{j+1} + \sum_{k\in\mathbb{Z}_+}\sigma_k b_{j-k} + b_{j,x}\right).
\end{aligned}
\tag{5.124}
$$

These (5.124) allow us to determine local functionals $\sigma_j : M \to \mathbb{C}$, $j \in \mathbb{Z}_+$, in the explicit form

$$
\begin{aligned}
&\sigma_0 = i\psi\psi^*, \quad \sigma_1 = i(\psi_x^*\psi - \psi^*\psi_x), \\
&\sigma_2 = i(\psi\psi_{xx}^* - \psi_x\psi_x^*) - \psi\psi^*(\psi\psi_x^* - \psi_x\psi^*), \dots,
\end{aligned}
\tag{5.125}
$$

and so on. Taking into consideration that the functionals

$$
\gamma_j = \int_{x_0}^{x_0+2\pi}\sigma_j[\psi,\psi^*]\,dx,
$$

$j \in \mathbb{Z}_+$, are conservation laws for dynamical system (5.120), from (5.125) we obtain expressions for the first gradients $\mathrm{grad}\gamma_j \in T^*(M)$, $j \in \mathbb{Z}_+$, namely

$$
\begin{aligned}
&\mathrm{grad}\gamma_0 = \begin{pmatrix} i\psi^* \\ i\psi \end{pmatrix}, \quad \mathrm{grad}\gamma_1 = \begin{pmatrix} 2i\psi_x^* \\ -2i\psi_x \end{pmatrix}, \\
&\mathrm{grad}\gamma_3 = \begin{pmatrix} 2i\psi_{xx}^* + 4\psi\psi^*\psi_x^* \\ -2i\psi_{xx} + 4\psi\psi^*\psi_x^* \end{pmatrix}, \dots
\end{aligned}
\tag{5.126}
$$

and so on.

Making use of formulae (5.126) for the gradients of conservation laws, we conclude that the recursive operator Λ, if it exists, must satisfy

$$\Lambda \mathrm{grad} \gamma_j = \mathrm{grad} \gamma_{j+1} \tag{5.127}$$

for all $j \in \mathbb{Z}_+$, which shows that $q = 1$. Now, upon setting $\psi = \varepsilon q^{(1)}$, $\psi^* = \varepsilon \bar{q}^{(1)}$, from formulae (5.126) and (5.127) we obtain asymptotic expansions for the operators Λ and K'^*:

$$\Lambda = \Lambda^{(0)} + \varepsilon^2 \Lambda^{(2)}, \quad K'^* = K'^*_0 + \varepsilon^2 K'^*_2, \tag{5.128}$$

where $\Lambda^{(j)}$, K'^*_j, $j = 0, 2$, are

$$\begin{aligned} &\Lambda^{(0)} = \begin{pmatrix} \partial & 0 \\ 0 & -\partial \end{pmatrix}, \quad K'^*_0 = \begin{pmatrix} i\partial^2 & 0 \\ 0 & -i\partial^2 \end{pmatrix}, \\ &\Lambda^{(0)} = 0, \quad K'^*_1 = 0, \\ &K'^*_2 = \begin{pmatrix} 2q^{(1)}\partial\bar{q}^{(1)} & -2\bar{q}^{(1)}\bar{q}^{(1)}_x \\ -2q^{(1)}q^{(1)}_x & 2\bar{q}^{(1)}\partial\bar{q}^{(1)} \end{pmatrix}. \end{aligned} \tag{5.129}$$

Writing the Fourier series expansions for functions $(q^{(1)}, \bar{q}^{(1)})^\intercal \in T(M)$ and $\varphi^{(0)}_k \in T^*(M)$, as

$$\begin{aligned} q^{(1)} &= \sum_{j \in 2\pi i\mathbb{Z}} q^{(1)}_j \exp(jx + ij^2 t), \quad \bar{q}^{(1)} = \sum_{j \in 2\pi i\mathbb{Z}_+} \bar{q}^{(1)}_j \exp(jx + ij^2 t), \\ \varphi^{(0)}_1 &= \sum_{j \in 2\pi i\mathbb{Z}} \varphi^{(0)}_{1,j} \exp(jx + ij^2 t), \quad \varphi^{(0)}_2 = \sum_{j \in 2\pi i\mathbb{Z}} \varphi^{(0)}_{2,j} \exp(jx + ij^2 t), \end{aligned} \tag{5.130}$$

from equations (5.84) and (5.128)- (5.130) we find the following expressions for action of the operator $\Lambda^{(2)}$ on the function $\varphi^{(0)} \in T^*(M)$:

$$\begin{aligned} \Lambda^{(2)}_{11}\varphi^{(0)}_1 &= -i \sum_{j,m,k \in 2\pi i\mathbb{Z}} \frac{m+k}{j+k} q^{(1)}_j \bar{q}^{(1)}_m \varphi^{(0)}_{1,k} \exp[(j+m+k)x + i(j^2 - k^2 - m^2)t], \\ \Lambda^{(2)}_{12}\varphi^{(0)}_2 &= -i \sum_{j,m,k \in 2\pi i\mathbb{Z}} \left(2 + \frac{m-k}{j+k} + \frac{j-k}{m+k}\right) \bar{q}^{(1)}_j \bar{q}^{(1)}_m \varphi^{(0)}_{2,k} \\ &\qquad \times \exp[(j+m+k)x + i(j^2 + m^2 - k^2)t], \\ \Lambda^{(2)}_{21}\varphi^{(0)}_1 &= -i \sum_{j,m,k \in 2\pi i\mathbb{Z}} \left(2 + \frac{m-k}{j+k} + \frac{j-k}{m+k}\right) q^{(1)}_j q^{(1)}_m \varphi^{(0)}_{1,k} \\ &\qquad \times \exp[(j+m+k)x + i(j^2 + m^2 - k^2)t], \\ \Lambda^{(2)}_{22}\varphi^{(0)}_2 &= -i \sum_{j,m,k \in 2\pi i\mathbb{Z}} \frac{j+k}{m+k} \bar{q}^{(1)}_j \bar{q}^{(1)}_m \varphi^{(0)}_{2,k} \exp[(j+m+k)x + i(j^2 - m^2 + k^2)t]. \end{aligned} \tag{5.131}$$

Consequently, owing to the representations (5.130), the expressions (5.131) can be represented as

$$\begin{aligned}
\Lambda_{11}^{(2)}\varphi_1^{(0)} &= -i[\bar{q}_x^{(1)}\partial^{-1}(q^{(1)}\varphi_1^{(0)}) + \bar{q}^{(1)}\partial^{-1}(q^{(1)}\varphi_{1,x}^{(0)})],\\
\Lambda_{12}^{(2)}\varphi_2^{(0)} &= i[\bar{q}^{(1)}\bar{q}^{(1)}\varphi_2^{(0)} + \bar{q}_x^{(1)}\partial^{-1}(\bar{q}^{(1)}\varphi_2^{(0)}) - \bar{q}^{(1)}\partial^{-1}(\bar{q}^{(1)}\varphi_{2,x}^{(0)})],\\
\Lambda_{21}^{(2)}\varphi_1^{(0)} &= i[q^{(1)}q^{(1)}\varphi_1^{(0)} + q_x^{(1)}\partial^{-1}(q^{(1)}\varphi_1^{(0)}) - q^{(1)}\partial^{-1}(q^{(1)}\varphi_{1,x}^{(0)})],\\
\Lambda_{22}^{(2)}\varphi_2^{(0)} &= -i[q_x^{(1)}\partial^{-1}(\bar{q}^{(1)}\varphi_2^{(0)}) + q^{(1)}\partial^{-1}(\bar{q}^{(1)}\varphi_{2,x}^{(0)})].
\end{aligned} \tag{5.132}$$

Thus, recalling that $\varepsilon q^{(1)} = \psi$, $\varepsilon\bar{q}^{(1)} = \psi^*$, from (5.128) and (5.132) we obtain the explicit form of the recursive operator Λ, namely

$$\Lambda = \begin{pmatrix} \partial - i(\psi_x^*\partial^{-1}\psi + \psi^*\partial^{-1}\psi\partial) & i(\psi^{*2} + \psi_x^*\partial^{-1}\psi^* - \psi^*\partial^{-1}\psi^*\partial) \\ i(\psi^2 + \psi_x\partial^{-1}\psi - \psi\partial^{-1}\psi\partial) & -\partial - i(\psi_x\partial^{-1}\psi^* + \psi\partial^{-1}\psi^*\partial) \end{pmatrix}. \tag{5.133}$$

It is straightforward to check that the recursive operator Λ satisfies the characteristic equation (5.77) with operator K'^* of the form (5.121). We also notice here that the dynamical system (5.120) can be written in the Hamiltonian form

$$\frac{d}{dt}\begin{pmatrix} \psi \\ \psi_t^* \end{pmatrix} = -\vartheta \operatorname{grad} H, \tag{5.134}$$

where $H = \frac{1}{2}\gamma_2$ and the implectic operator ϑ is represented as

$$\vartheta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{5.135}$$

By virtue of formulae (5.127) and (5.135), the recursive operator Λ defined by (5.133) allows the factorization $\Lambda = \vartheta^{-1}\eta$, where

$$\eta = \begin{pmatrix} i(\psi^2 + \psi_x\partial^{-1}\psi - \psi\partial^{-1}\psi\partial) & -\partial - i(\psi_x\partial^{-1}\psi^* + \psi\partial^{-1}\psi^*\partial) \\ -\partial + i(\psi_x^*\partial^{-1}\psi + \psi^*\partial^{-1}\psi\partial) & -i(\psi^{*2} + \psi_x^*\partial^{-1}\psi^* - \psi^*\partial^{-1}\psi^*\partial) \end{pmatrix} \tag{5.136}$$

is an implectic operator. In addition, one can by long but simple calculations make sure that operators ϑ and η are compatible and Nötherian.

An important observation for algebraic structures of all of above dynamical systems is that they are *bi-Hamiltonian*; that is, for $u \in M$ one has

$$du/dt = -\vartheta \operatorname{grad} H = -\eta \operatorname{grad} \bar{H}, \tag{5.137}$$

where $\bar{H} \in \mathcal{D}(M)$ is a conservation law satisfying the obvious recursion condition

$$\operatorname{grad} H = \Lambda \operatorname{grad} \bar{H}.$$

Owing to the recursion property (5.76) for the conservation laws $\gamma_j \in \mathcal{D}(M)$, $j \in \mathbb{Z}_+$, it is easy to prove their involutivity with respect to both symplectic structures $\{\cdot,\cdot\}_\vartheta$ and $\{\cdot,\cdot\}_\eta$. Consequently, the dynamical systems (5.137) are completely integrable in the sense of Liouville which has been determined as a result of the above approach. This integrability problem is in greater depth in the sequel.

5.4 The current Lie algebra on a cycle: A symmetry subalgebra of compatible bi-Hamiltonian nonlinear dynamical systems

5.4.1 *Preliminaries*

Let us consider once more a dynamical system on a functional manifold $M \subset C^\infty(\mathbb{R};\mathbb{R}^m)$ in the form

$$u_t = K[u] \tag{5.138}$$

on the functional manifold $M \subset C^\infty(\mathbb{R};\mathbb{R}^m)$, where the vector field $K : M \to T(M)$ is autonomous and uniform with respect to the independent variable $x \in \mathbb{R}$ as a function $u \in M \simeq J^\infty(\mathbb{R};\mathbb{R}^m)$. We call a vector field $\alpha : M \to T(M)$ an *autonomous symmetry* of the dynamical system (5.138) if

$$L_K\alpha = [K,\alpha] = 0. \tag{5.139}$$

Correspondingly, we shall call a vector field $\tau : M \to T(M)$ the *non-autonomous symmetry* of dynamical system (5.138) if

$$\partial\tau/\partial t + L_K\tau = 0 \tag{5.140}$$

with the partial derivative $\partial\tau/\partial t \neq 0$ for $t \in \mathbb{R}$. The next result follows directly from these definitions

Corollary 5.1. *The subsets $\mathfrak{g}\{\alpha\}$ and $\mathfrak{g}\{\tau\}$ of autonomous and non-autonomous symmetries, respectively, are Lie subalgebras of vector fields on M.*

Now let the dynamical system (5.139) on M be bi-Hamiltonian with a compatible (ϑ,η)-pair of implectic and Nötherian operators. Suppose also that there exist two nontrivial symmetries $\alpha_0 \in \mathfrak{g}\{\alpha\}$ and $\tau_0 \in \mathfrak{g}\{\tau\}$ such that

$$L_{\tau_0}\alpha_0 = [\tau_0,\alpha_0] = \varepsilon\alpha_0, \qquad L_{\alpha_0}\vartheta = L_{\alpha_0}\eta = 0,$$

$$L_{\tau_0}\vartheta = (\xi - 1/2)\vartheta, L_{\tau_0}\eta = (\xi + 1/2)\eta, \tag{5.141}$$

where $\varepsilon, \xi \in \mathbb{R}$ are parameters. Further, we construct the following sets $\mathfrak{g}_0\{\alpha\}$ and $\mathfrak{g}_0\{\tau\}$ of symmetries:

$$\mathfrak{g}_0\{\alpha\} := \{\alpha_j : \quad \Lambda^{*j}\alpha_0 : \quad j \in \mathbb{Z}\} \subset \mathfrak{g}\{\alpha\}, \tag{5.142}$$

$$\mathfrak{g}_0\{\tau\} := \{\tau_j : \quad \Lambda^{*j}\tau_0 : \quad j \in \mathbb{Z}\} \subset \mathfrak{g}\{\tau\}.$$

Owing to the recursion symmetry of the operator $\Lambda^* : T(M) \to T^*(M)$, where $\Lambda^* = \eta\vartheta^{-1}$, the sets (5.142) turn out to be Lie subalgebras over the field $\mathbb{R}$ of the symmetry subalgebras $\mathfrak{g}\{\alpha\}$ and $\mathfrak{g}\{\tau\}$, respectively.

Theorem 5.1. *The semidirect sum $Q_0 := Q_0\{\tau\} \odot Q_0\{\alpha\}$ is a Lie subalgebra over $\mathbb{R}$ of symmetries of the dynamical system (5.138), and the following relations hold:*

$$[\alpha_j, \alpha_k] = 0, \qquad [\tau_k, \alpha_j] = (j + \varepsilon)\alpha_{j+k},$$

$$[\tau_j, \tau_k] = (k - j)\tau_{j+k}, \tag{5.143}$$

where $j, k \in \mathbb{Z}$.

Proof. The desired results can be obtained directly from simple calculations based on the relationships (5.141) and (5.142). □

The current group $G = \mathrm{Diff}\left(\mathbb{S}^1\right) \odot \mathfrak{g}$ on the circle $\mathbb{S}^1$ possesses vector fields that are Lie algebra isomorphic, in their Fourier representation, to symmetry Lie algebras (5.143). Consequently, we can conclude from the assumptions on the dynamical system (5.138) that the dynamical system (5.138) is invariant with respect to the infinite-dimensional symmetry group $G =\mathrm{Diff}\left(\mathbb{S}^1\right) \odot \mathfrak{g}$, with the Lie algebra $Q_0 = \mathfrak{g}_0\{\tau\} \odot \mathfrak{g}_0\{\alpha\}$ serving as its functional representation by means of vector fields on the manifold M. Notice here that it follows readily from (5.141)–(5.143) that we have the following equations for all $j \in \mathbb{Z}$:

$$L_{\alpha_j}\vartheta = L_{\tau_j}\eta = 0, \qquad L_{\alpha_j}\Lambda = 0, \quad L_{\tau_j}\Lambda = \Lambda^{j+1} \tag{5.144}$$

$$L_{\tau_j}\vartheta = (\xi - j - 1/2)\vartheta\Lambda^j, \qquad L_{\tau_j}\eta = (\xi - j + 1/2)\eta\Lambda^j.$$

Moreover, the formulae can be easily verified:

$$L_{\vartheta\varphi}\vartheta = -\vartheta(\varphi' - \varphi'^*)\vartheta, \quad L_\alpha\Omega = (\Omega\alpha)' - (\Omega\alpha)'^*, \tag{5.145}$$

where the tensors $\vartheta \in T^{(0,2)}(M)$ and $\Omega \in T^{(2,0)}(M)$ are, respectively, implectic and symplectic operators, $\alpha : M \to T(M)$ is an arbitrary vector field, and $\varphi : M \to T^*(M)$ is an arbitrary 1-form on the manifold M. The formulae (5.145) lead to the following result.

Theorem 5.2. *Let the tensors Ω and $\tilde{\Omega} \in T^{(2,0)}(M)$ be symplectic operators on the functional manifold M. (i) If Ω is an invertible operator, there exists a vector field $\tau : M \to T(M)$ such that*

$$L_\tau \Omega = \tilde{\Omega}; \tag{5.146}$$

and (ii) if the tensors $\vartheta, \tilde{\vartheta} \in T^{(0,2)}(M)$ are co-symplectic operators on M and ϑ be invertible, there exists a vector field $\tau : M \to T(M)$ that

$$L_\tau = \tilde{\vartheta}, \tag{5.147}$$

if and only if the operator $\vartheta^{-1}\tilde{\vartheta}\vartheta^{-1}$ is co-implectic on M.

Proof. To verify (i), we note that as the Lie derivative and exterior derivative commute, it is obvious that the Lie derivative of a symplectic operator is always a co-implectic operator. Therefore, from (5.145) we obtain

$$L_\tau \Omega = (\Omega\tau)' - (\Omega\tau)'^* = \tilde{\Omega}, \tag{5.148}$$

which has the solution

$$\tau[u] = \Omega^{-1}[u] \int_0^1 d\lambda \Omega[u\lambda]\lambda u. \tag{5.149}$$

The vector field (5.149), as usual, is not unique, since the vector field $\tau[u] \to \tau[u] + \Omega \text{ grad } f[u] \in T(M)$, where $f \in \mathcal{D}(M)$ is an arbitrary function that also satisfies equation (5.138). We can prove (ii) by first observing that (5.145) implies

$$L_\tau \vartheta = -\vartheta((\vartheta^{-1}\tau)'^{-1}\tau)'^*)\vartheta = \tilde{\vartheta}. \tag{5.150}$$

Now, from our proof of (i) we conclude that a vector field $\tau : M \to T(M)$ with the property (5.150) exists if and only if the operator $\vartheta^{-1}\tilde{\vartheta}\vartheta^{-1}$ is symplectic on M. Moreover,

$$\tau[u] = -\vartheta[u] \int_0^1 d\lambda (\vartheta^{-1}\tilde{\vartheta}\vartheta^{-1})[u\lambda]u\lambda \tag{5.151}$$

with the above mentioned non-uniqueness property of the vector field $\tau : M \to T(M)$ (5.152). Thus, (ii) is also verified and the proof is complete.□

5.4.2 *Hierarchies of symmetries and related Hamiltonian structures*

Let us consider some structural properties of the symmetry Lie algebra $\mathfrak{g}_0 = \mathfrak{g}_0\{\tau\} \odot \mathfrak{g}_0\{\alpha\}$ for a compatibly bi-Hamiltonian dynamical system (5.138) on a functional manifold M. It follows immediately from the relationships (5.144) and (5.145) that there exists an infinite hierarchy of autonomous functionals $\gamma_j \in D(M)$, $j \in \mathbb{Z}_+$, satisfying the relation

$$\alpha_j = -\vartheta \text{ grad } \gamma_j. \tag{5.152}$$

Let $\{\cdot,\cdot\}_\vartheta := (\langle \text{grad}(\cdot), \vartheta \text{ grad}(\cdot)\rangle)$ be the *Poisson bracket*, which obviously satisfies the Jacobi identity for an implectic operator $\vartheta : T^*(M) \to T(M)$. Since the symmetry Lie algebra $\mathfrak{g}_0\{\alpha\}$ is abelian, (5.152) implies that for all $j, k \in \mathbb{Z}$

$$\{\gamma_j, \gamma_k\}_\vartheta = 0 = \{\gamma_j, \gamma_k\}_\eta. \tag{5.153}$$

Thus, the Hamiltonian system (5.138) on M provides an infinite hierarchy of involutive conservation laws $\gamma_j \in \mathcal{D}(M), j \in \mathbb{Z}_+$, which can, under some additional restrictions, guarantee its complete Liouville integrability [14, 3, 173, 326]. We also note that if the initial dynamical system (5.138) has an infinite hierarchy of conservation laws $\{\gamma_j \in \mathcal{D}(M) : j \in \mathbb{Z}_+\}$, it can be used for the exact determination of the associated symplectic structures on the manifold M.

For convenience, let a functional manifold $M \subset \mathcal{S}(\mathbb{R}; \mathbb{R}^m)$ be of Schwartz type on $\mathbb{R}$. Suppose also that the dynamical system (5.138) is not only autonomous but uniform on M. This means that it possesses two commuting symmetries provided by the vector fields d/dx and d/dt. Considering the dynamical system (5.138) on M as that on the infinite-dimensional jet manifold $J^\infty(\mathbb{R}; \mathbb{R}^m)$, we denote by $\Omega \in \Lambda^2(J^\infty)$ a corresponding symplectic structure with respect to which the vector fields d/dx and d/dt are Hamiltonian. Thus, there exists a functional $\gamma \in \mathcal{D}(M)$ satisfying

$$i_{\frac{d}{dx}}\Omega = -d\gamma[u], \tag{5.154}$$

where $\gamma[u] \in \Lambda^0(J^\infty(\mathbb{R}; \mathbb{R}^m))$ is the density of a functional $\gamma \in \mathcal{D}(M)$. Using the standard Cartan representation for the Lie derivative

$$L_{d/dx}\Omega = i_{d/dx}\Omega + di_{d/dx}\Omega, \tag{5.155}$$

we immediately compute that

$$\gamma = (\sigma, u_x), \tag{5.156}$$

where $\Omega = d\Sigma$, $\Sigma := \langle \sigma, du\rangle + (d/dx)\omega^{(1)}$ for some uniquely defined 1-form $\omega^{(1)} \in \Lambda^1(J^\infty(\mathbb{R};\mathbb{R}^m))$. Taking into account the representation (5.156), it is easy to see that the vector field d/dx on the manifold can be expressed as

$$du/dx = -\text{grad}\ \gamma, \qquad \vartheta^{-1} := \sigma' - \sigma'^{*}, \tag{5.157}$$

where, obviously, $\vartheta^{-1} = \sigma' - \sigma'^{*} \simeq d\Sigma = \Omega$ as required. As $d^2\Sigma = 0$, the operator $\vartheta^{-1} : T^*(M) \to T(M)$ is symplectic on M when it is non-degenerate. Thus, each uniform conservation law $\gamma_j \in \mathcal{D}(M)$, $j \in \mathbb{Z}_+$, of the dynamical system (5.138) on the functional manifold $M \subset \mathcal{S}(\mathbb{R};\mathbb{R}^m)$ generates the vector field d/dx on M as a Hamiltonian system

$$du/dx = -\vartheta_j\ \text{grad}\gamma_j, \tag{5.158}$$

where $\vartheta_j : T^*(M) \to T(M)$, $j \in \mathbb{Z}_+$ is a hierarchy of associated implectic structures on M. Hence the given dynamical system (5.138) is Hamiltonian with respect to this hierarchy if one of the operators $\{\vartheta_j : j \in \mathbb{Z}\}$ is Nötherian. If the dynamical system (5.138) is bi-Hamiltonian, all of the operators $\vartheta_j : T^*(M) \to T(M)$, $j \in \mathbb{Z}_+$, must be Nötherian; that is, $L_K\vartheta_j = 0$, $j \in \mathbb{Z}_+$. In this regard, it is also necessary to take into account that these implectic operators θ_j, $j \in \mathbb{Z}_+$, are not uniquely defined due to the non-uniqueness of the representation of the conservation laws in the form (5.156).

Example 5.4. The procedure applied to the KdV equation on a Schwartz manifold $M \subset \mathcal{S}(\mathbb{R};\mathbb{R}^m)$

Here we write the KdV in the form

$$u_t = K[u] := u_{xxx} + uu_x, \tag{5.159}$$

which, as is well known [173, 262, 326, 406], has an infinite hierarchy of conservation laws $\gamma_j \in \mathcal{D}(M)$, $j \in \mathbb{Z}_+$,

$$\gamma_0 = \int_{\mathbb{R}} dxu, \qquad \gamma_1 = \int_{\mathbb{R}} dxu^2/2, \quad \gamma_2 = \int_{\mathbb{R}} dx(\frac{1}{2}u_x^2 - \frac{1}{6}u^3)\ldots. \tag{5.160}$$

In accordance with the representations (5.156) and (5.160), one obtains

$$\gamma_1 = \frac{1}{2}(u,u) = \frac{1}{2}(u,\partial^{-1}u_x) = -\frac{1}{2}(\partial^{-1}u,u_x) = (\sigma_1,u_x); \tag{5.161}$$

$$\gamma_2 = \frac{1}{2}(u_x,u_x) - \frac{1}{6}(u^2,u) = \frac{1}{2}(u_x,u_x) - \frac{1}{6}(u^2,\partial^{-1}u_x)$$

$$= \frac{1}{2}(u_x, u_x) + \frac{1}{6}(\partial^{-1}u^2, u_x) = (\frac{1}{2}u_x + \frac{1}{6}\partial^{-1}u^2, u_x) = (\sigma_2, u_x).$$

For two implectic operators, using (5.157) we find that

$$\vartheta_1^{-1} = \sigma_1' - \sigma_1'^* = -\partial^{-1}; \quad \vartheta_2^{-1} = \sigma_2' - \sigma_2'^* = \partial + \frac{1}{3}(u\partial^{-1} + \partial^{-1}u). \quad (5.162)$$

By means of straightforward calculations, one can easily see that the implectic operators (5.162) on the manifold M are also Nötherian; namely, $L_K\vartheta_1 = 0 = L_K\vartheta_2$, and also show that this pair $(\vartheta_1, \vartheta_2)$ is compatible on M. Consequently, from such a pair we can also construct the operators

$$\vartheta := -\vartheta_1 = \partial, \quad \eta := \vartheta_1\vartheta_2^{-1}\vartheta_1 = \partial^3 + \frac{1}{3}(\partial u + u\partial), \quad (5.163)$$

in terms of local differential expressions on M. Thus, the Korteweg-de Vries dynamical system (5.159) on M induces a compatibly bi-Hamiltonian flow, as one might expect based on its complete integrability. In order to construct the symmetry Lie subalgebra $\mathfrak{g}_0 = (\mathfrak{g}_0\{\tau\} \odot \mathfrak{g}_0\{\alpha\})$ of the KdV dynamical system (5.159) on M, let us take as generating elements for $\alpha_0, \tau_0 \in T(M)$ the following vector fields: $\alpha_0 := u_x$, $\tau_0 := xu_x/2 + u + 3t(u_{xxx} + uu_x)/2$. Whence, we obtain

$$\mathfrak{g}_0\{\alpha\} = \{\alpha_j : \quad \Lambda^{*j}\alpha_0 : \quad j \in \mathbb{Z}, \alpha_0 = u_x\}, \quad (5.164)$$

$$\mathfrak{g}_0\{\tau\} = \{\tau_j : \quad \Lambda^{*j}\tau_0 : \quad j \in \mathbb{Z}, \tau_0 = u + 1/2xu_x + 3/2tK\},$$

where

$$\Lambda^* := \eta\vartheta^{-1} = \partial^2 + \frac{1}{3}u + \frac{1}{3}\partial u\partial^{-1} \quad (5.165)$$

is a symmetry hereditary operator of (5.159) as follows from (5.163).

5.4.3 *A Lie-algebraic algorithm for investigating integrability*

The properties described above of the bi-Hamiltonian dynamical system (5.138) with a compatible implectic (θ, η)-pair of Nötherian operators are, in a certain sense, characteristic elements [54, 137, 173, 326] of an algebraic proof of its complete integrability. In particular, as a natural consequence one can prove the important fact that the dynamical system (5.138) with these conditions has isospectral Lax type deformations [2, 83, 263, 326, 406] allowing, by means of the inverse spectral transform and techniques from algebraic geometry, calculation of a wide class of its solutions in explicit analytical form.

With the aim of describing a recursive algorithm for studying any Fréchet smooth dynamical system (5.138) on an infinite-dimensional functional manifold $M \subset C^{\infty}(\mathbb{R};\mathbb{R}^m)$, we first take a more detailed look at the original algebraic structure of the symmetry Lie subalgebra (5.143) and the corresponding properties (5.144). We notice that a Lie subalgebra $\mathfrak{g}_0\{\tau\}$ of nonautonomous symmetries contains an entire generating subalgebra $Q_0\{\tau\}$ with a Lie subalgebra $\{\tau_{-1}, \tau_0, \tau_1\}$ that is isomorphic to the Lie algebra sl(2). Moreover, if the dynamical system (5.138) is Hamiltonian with implectic operator $\vartheta : T^*(M) \to T(M)$, it follows from (5.144) that

$$L_{\tau_{-1}}\vartheta = (\xi + 1/2)\vartheta\eta^{-1}\vartheta, L_{\tau_0}\vartheta = (\xi - 1/2)\vartheta, L_{\tau_1}\vartheta = (\xi - 3/2)\eta, \quad (5.166)$$

where $\xi \in \mathbb{R}$ is fixed. We conclude from (5.166) that in order to determine the second implectic operator $\eta : T^*(M) \to T(M)$ it is only necessary to calculate the Lie derivative $L_{\tau_{-1}}\vartheta$ at $\xi \neq 3/2$ or the Lie derivative $L_{\tau_{-1}}\vartheta$ at $\xi \neq -1/3$. In order to calculate a nonautonomous symmetry $\tau_1 : M \to T(M)$ (at $\xi \neq 3/2$), we see that from the determining equation

$$\partial\tau_1/\partial t + [K, \tau_1] = 0 \quad (5.167)$$

with $K = \sum_{j=0}^{N} c_j\alpha_j$ for some $N \in \mathbb{Z}_+$, it follows that

$$\tau_1 = \sum_{j=0}^{N} tc_j(j+\varepsilon)\alpha_{j+1} + x\sum_{j=0}^{N} s_j\alpha_j + a_1, \quad (5.168)$$

where $s_j, c_j \in \mathbb{R}$, $0 \leq j \leq N$, $c_N = 1$, $\alpha_j = -\vartheta$ grad γ_j, $j \in \mathbb{Z}_+$ are autonomous symmetries corresponding to an infinite hierarchy of autonomous conservation laws of (5.138), and $a_1 := a_1[u] \in T(M)$ is the uniform part of the symmetry (5.168).

Moreover, equations (5.167) and (5.168) determine both unknown values $c_j, s_j \in \mathbb{R}$, $0 \leq j \leq N$, and also a quantity $a_1 \in T(M)$, giving rise to an explicit form of the desired symmetry (5.168). Here we need to recall that an infinite hierarchy of functionally independent conservation laws $\{\gamma_j \in \mathcal{D}(M) : j \in \mathbb{Z}_+\}$ for (5.138) is considered to be known; to calculate it explicitly, one can, for example, make use of standard asymptotic approaches [254, 257, 258, 262, 265, 328]. As for solutions to equations (5.167) and (5.168) in explicit functional form, the small parameter approach with the associated Fourier transform method can be used. The same approaches are relevant also for explicitly determining solutions of the corresponding Nötherian equation. It is also necessary to observe that this asymptotic approach to constructing conservation laws is suitable only for dynamical systems (on manifolds $M \subset J^{\infty}(\mathbb{R};\mathbb{R}^m)$) of non-hydrodynamic type, that is for systems containing higher derivatives. As a result, when constructed using the methods developed here, a compatible implectic (ϑ, η) pair for the dynamical system (5.138) provides, via the *gradient holonomic algorithm* [173, 262, 263, 326], the means for determining the main analytical object of such a system - its Lax representation.

This page intentionally left blank

Chapter 6

Algebraic, Analytic and Differential Geometric Aspects of Integrability for Systems on Functional Manifolds

6.1 A non-isospectrally Lax integrable KdV dynamical system

Let us consider on a functional jet-manifold $M \subset J_{top}(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R})$ a nonautonomous cylindrical nonlinear Korteweg-de Vries (KdV) dynamical system

$$du/dt = K[t;u] := -6uu_x - u_{3x} - u/(2t), \tag{6.1}$$

where $K[t,\cdot] : M \to T(M)$ is the corresponding vector field on M, and $t \in \mathbb{R}$ is the evolution parameter.

To study the Lax integrability of (6.1) within the framework of the gradient-holonomic algorithm developed in the preceding chapter, let us examine first the existence for (6.1) of an infinite hierarchy of conservation laws $\gamma_j \in \mathcal{D}(M)$, $j \in \mathbb{Z}_+$, which also satisfies in addition an ordering spectral gradient law. For this purpose, consider asymptotic solutions of the Lax equation

$$\partial\varphi/\partial t + L_K\varphi = d\varphi/dt + 6u\varphi_x + \varphi_{3x} - \varphi/(2t) = 0 \tag{6.2}$$

depending on the spectral parameter $\mathbb{C} \ni z \to \infty$. First, we find asymptotic solutions of the Lax equation

$$du/d\tau = K[\tau;u], \tag{6.3}$$

$\tau \in (0;t]$, where we assume that the following conditions are satisfied: if $\varphi = \varphi(x,t;z) \in T^*(M)$, $\tilde{\varphi} = \tilde{\varphi}(x,\tau;\lambda(t;z)) \in T^*(M)$, then

$$\tilde{\varphi}\mid_{\tau=t} = \varphi \tag{6.4}$$

for the spectral evolution equations

$$d\lambda/dt = g(t;\lambda), \quad \lambda\mid_{t=0} = z, \tag{6.5}$$

which are meromorphic in $\lambda \in \mathbb{C}$, for any $z \in \mathbb{C}$.

It is easy to see that the function

$$\tilde{\varphi}(x;\lambda) = \tau^{1/2}\exp[\lambda^3 - \lambda x + \partial^{-1}\tilde{\sigma}(x;\lambda)], \tag{6.6}$$

where

$$\tilde{\sigma}(x;\lambda) \sim \sum_{j\in\mathbb{Z}_+} \tilde{\sigma}_j[\tau;u]\lambda^{-j}, \tag{6.7}$$

as $|\lambda| \to \infty$, satisfies the equation (6.1) if and only if the relationships

$$\partial^{-1}\tilde{\sigma}_{j,\tau} - 6u\delta_{j,-1} + 6u\tilde{\sigma}_j + \tilde{\sigma}_{j,xx} - 3\tilde{\sigma}_{j+1,x} + 3\tilde{\sigma}_{j-k,x}\tilde{\sigma}_k$$

$$+\tilde{\sigma}_{j+2} - 3\tilde{\sigma}_{j+1-k}\tilde{\sigma}_k + \tilde{\sigma}_{j-k}\tilde{\sigma}_{k-s}\tilde{\sigma}_s = 0 \tag{6.8}$$

hold for all $j \in \mathbb{Z}_+$. Solving the recursive hierarchy (6.6), one obtains

$$\begin{aligned}
&\tilde{\sigma}_0 = 0, \quad \tilde{\sigma}_1 = 2u, \quad \tilde{\sigma}_2 = 2u_x,\\
&\tilde{\sigma}_3 = 2u^2 + 2u_{xx} + \partial^{-1}u/(3\tau), \quad \tilde{\sigma}_4 = 2u_{3x} + 8uu_x + 2u/(3\tau),\\
&\tilde{\sigma}_5 = \partial^{-1}(2u^2 + \partial^{-1}u/(3\tau)) + 2u\partial^{-1}u/(3\tau) + u_x/\tau\\
&+ 10u_x^2 + 12uu_{xx} + 2u_{4x} + 4u^3,
\end{aligned} \tag{6.9}$$

and so on.

Next, we shall satisfy condition (6.4), while simultaneously determining the spectral evolution (6.5). To this end, we notice that owing to the linear equation (6.2) and from (6.4), (6.5), the following quantity

$$\gamma(z) := \int_{\mathbb{R}} dx : \tilde{\sigma}(x;\lambda)\mid_{\tau=t} := \int_{\mathbb{R}} dx : \sigma(x;z) \tag{6.10}$$

must be a generating function of conservation laws of the original dynamical system (6.1) as $|z| \to \infty$. From (6.6) and (6.10), we infer that for $\tau = t \in \mathbb{R}_+$, $d\lambda/dt = g(t;\lambda) = -\lambda/(2t)$, $t \in \mathbb{R}_+$, and

$$\varphi(x;z) = t^{1/2}\exp[-2z^3t^{-1/2} - zxt^{-1/2} + \partial^{-1}\sigma(x;z)], \tag{6.11}$$

where $\sigma(x;z) := \tilde{\sigma}(x;\lambda)\mid_{\tau=t}$, and

$$\sigma(x;z) \sim \sum_{j\in\mathbb{Z}_+} \sigma_j[t;u]z^{-j}, \tag{6.12}$$

as $|z| \to \infty$. It is also easy to find from (6.9) that

$$\begin{aligned}
&\sigma_0 = 0, \quad \sigma_1 = 2ut^{1/2}, \quad \sigma_2 = 2u_xt/3, \qquad (6.13)\\
&\sigma_3 = t^{3/2}(2u^2 - xu/(3t)), \quad \sigma_4 = t^2(u_{3x}/3 + 8uu_x/3 + 2u/(9t)),\\
&\sigma_5 = t^{5/2}(4u^3 + 2u_{xx}u + ux^2/(12t) - xu^2/t), \ \ldots,
\end{aligned}$$

and so on. Thus, all the functionals $\gamma_j := \int_{\mathbb{R}} dx\sigma_j[t;u]$, $j \in \mathbb{Z}_+$, are conserved quantities of the original flow (6.1). As a result, we conclude that the dynamical system (6.1) admits a Lax representation with a spectral parameter $\lambda := \lambda(t;z) = zt^{-1/2}$, depending on the evolution variable $t \in \mathbb{R}_+$ for all $z \in \mathbb{C}$. Notice now that the element $\varphi(x;z) \in T^*(M)$, $z \in \mathbb{C}$, must satisfy the following recursive spectral gradient relation:

$$\Lambda\varphi(x;z) = \lambda\varphi(x;z), \tag{6.14}$$

where $\Lambda : T^*(M) \to T^*(M)$ is a suitable recursion operator. Combining (6.14) with (6.2), one readily finds that

$$d\Lambda/dt = [\Lambda, K'^*] + g(t;\Lambda), \tag{6.15}$$

where $d\lambda/dt := g(t;\lambda) = -\lambda/(2t)$, $t \in \mathbb{R}_+$. Solving equation (6.15) using the small parameter method of Chapter 5, we obtain the following two nonautonomous recursive operators:

$$\begin{aligned} \Lambda^{(1)} &:= t(\partial^2 + 2u/3 - \partial^{-1}u_x/3), \\ \Lambda^{(2)} &:= -2t\partial^{-1} + xt + 2t\partial^{-2}u\partial^{-1}/3 \\ &\quad + t\partial^{-2}(\partial^{-1}u)/3 + t^2\partial^{-1}(3\partial^3 + u\partial + \partial u) \end{aligned} \tag{6.16}$$

for any $t \in \mathbb{R}_+$ and $u \in M$.

Taking into account that $\varphi(z) = \mathrm{grad}\Delta(z) \in T^*(M)$, where $\Delta(z) := \mathrm{tr}S(x;z)$, $z \in \mathbb{C}$, is the trace of the monodromy matrix $S(x;z)$ of a Lax representation corresponding to (6.16), and making use of the differential algebraic approach to finding the explicit Lax representation devised in the preceding chapter, one obtains from (6.15) and (6.16) after tedious but simple calculations that

$$\ell[u;\lambda(t;z)] = \begin{pmatrix} 0 & -1 \\ u - x/(12t) - \lambda^2/4 & 0 \end{pmatrix}, \tag{6.17}$$

where $\lambda = zt^{-1/2}$ for $t \in \mathbb{R}_+$ and $z \in \mathbb{C}$.

Thus, we have demonstrated that the nonautonomous dynamical system (6.1) on the manifold M admits a standard Lax representation with the corresponding l-matrix (6.17).

6.1.1 *A non-isospectrally integrable nonlinear nonautonomous Schrödinger dynamical system*

Let us investigate the nonautonomous nonlinear Schrödinger dynamical system:

$$\begin{pmatrix} d\Psi/dt \\ d\Psi^*/dt \end{pmatrix} = K[t;\Psi,\Psi^*] := \begin{pmatrix} -\Psi/2t + i\Psi_{xx} + 2i\Psi^*\Psi\Psi \\ -\Psi^*/2t - i\Psi^*_{xx} - 2i\Psi^*\Psi^*\Psi \end{pmatrix}, \tag{6.18}$$

on a manifold $M \subset C^\infty(\mathbb{R}/2\pi\mathbb{Z};\mathbb{C}^2)$, where $K(t,.) : M \to T(M)$ is the corresponding smooth nonautonomous vector field on M.

As before, we first consider the Lax equation (with respect to a new evolution parameter $\tau \in \mathbb{R}_+$) :

$$d\tilde{\varphi}/d\tau + K'^{*}\,\tilde{\varphi} = 0, \tag{6.19}$$

adjoint to (6.18), where $\tilde{\varphi} \in T^*(M)$, and

$$K'^{*} = \begin{pmatrix} -1/(2\tau) + i\partial^2 + 4i\Psi^*\Psi & -2i(\Psi^*)^2 \\ 2i\Psi^2 & -1/(2\tau) - i\partial^2 - 4i\Psi^*\Psi \end{pmatrix}. \tag{6.20}$$

Equation (6.19) has a solution $\tilde{\varphi} := \tilde{\varphi}(x,\tau;\lambda(\tau;z)) \in T^*(M)$ such that at $\tau = t \in \mathbb{R}_+$ and for any $z \in \mathbb{C}$ the following linear Lax equation is satisfied:

$$d\varphi/dt + K'^{*}\,\varphi = 0, \tag{6.21}$$

where $\varphi(x,t;z) := \tilde{\varphi}(x,\tau;\lambda(t;z))|_{\tau=t} \in T^*(M)$ and $d\lambda/dt = g(t;\lambda)$, $\lambda|_{t=0} := z \in \mathbb{C}$, for a meromorphic map $g(t;\cdot) : \mathbb{C} \to \mathbb{C}$ for all $t \in \mathbb{R}_+$.

We have

$$\tilde{\varphi}(x,\tau;\lambda) = (1,\tilde{b}(x,\tau;\lambda))^\intercal \exp\{\log[\tau^{1/2} + \lambda x - i\lambda^2\tau + \partial^{-1}\tilde{\sigma}(x,\tau;\lambda)]\}, \tag{6.22}$$

where

$$\begin{aligned} \tilde{b}(x,\tau;\lambda) &\sim \sum_{j\in\mathbb{Z}_+} \tilde{b}_j[x,\tau;u]\lambda^{-j}, \\ \tilde{\sigma}(x,\tau;\lambda) &\sim \sum_{j\in\mathbb{Z}_+} \tilde{\sigma}_j[x,\tau;u]\lambda^{-j}, \end{aligned} \tag{6.23}$$

as $|\lambda| \to \infty$.

Substituting expansions (6.23) into (6.22), from (6.19) one readily obtains the following recursive hierarchy

$$\begin{gathered} \partial^{-1}\tilde{\sigma}_{j,\tau} + 2i\tilde{\sigma}_{j+1} + i\tilde{\sigma}_{j-k}\tilde{\sigma}_k + i\tilde{\sigma}_{j,x} \\ +4i\Psi^*\Psi\delta_{j,0} - 2i\Psi^*\Psi\tilde{b}_j = 0, \\ \tilde{b}_{j,\tau} + 2\tilde{b}_{j-k}\partial^{-1}\tilde{\sigma}_{k,\tau} - 2i\tilde{b}_{j+2} - 2i\Psi^*\Psi^*\tilde{b}_{j-k} \\ +2i\Psi\delta_{j,0} - i\tilde{b}_{j,xx} - 2i\tilde{b}_{j+1,x} - 2i\tilde{b}_{j+1-k,x}\tilde{\sigma}_k = 0, \end{gathered} \tag{6.24}$$

for all $j \in \mathbb{Z}_+$. Solving the hierarchy (6.24) inductively, we find that

$$\begin{aligned} &\tilde{b}_0 = 0 = \tilde{b}_1, \quad \tilde{\sigma} = -2\Psi^*\Psi, \quad \tilde{b}_2 = -\Psi^2, \\ &\tilde{\sigma}_2 = -i\partial^{-1}(\Psi^*\Psi)/\tau + (\Psi_x\Psi^* - \Psi^*_x\Psi) + (\Psi^*\Psi)_x, \quad \tilde{b}_3 = -\Psi\Psi_x, \\ &\tilde{\sigma}_3 = i\Psi^*\Psi/(2\tau) - 4(\Psi^*\Psi)^2 - (\Psi_x\Psi^* - \Psi^*_x\Psi)_x/2 \\ &\qquad - (\Psi_{xx}\Psi^* - \Psi^*_{xx}\Psi) + \Psi_x\Psi_x - \partial^{-1}(\partial^{-1}\Psi^*\Psi)/\tau^2, \ \ldots \end{aligned} \tag{6.25}$$

and so on. It follows directly from its construction that the functional

$$\gamma(z) := \int_{\mathbb{R}} dx : \tilde{\sigma}(x,\tau;\lambda)\mid_{\tau=t}, \tag{6.26}$$

as $|z| \to \infty$, must be a generating function of conservation laws of the dynamical system (6.18). Clearly, this means that $d\gamma(z)/dt = 0$ for any $t \in \mathbb{R}$ and all $z \in \mathbb{C}$. Whence, we arrive at the simple conclusion:

$$\text{grad} : \left(\frac{d}{dt}\tilde{\sigma}(x,\tau;\lambda)\mid_{\tau=t} \right) = 0 \tag{6.27}$$

for all $z \in \mathbb{C}$ and $u \in M$. As a result of verifying condition (6.27), one can easily see that it is satisfied if and only if

$$d\lambda/dt = g(t;\lambda) = -\lambda t^{-1}, \quad \lambda\mid_{t=0} = z \in \mathbb{C}. \tag{6.28}$$

Thus, we have shown that the dynamical system (6.18) is endowed with an infinite hierarchy of nonautonomous conservation laws satisfying a recursive relation analogous to (6.24) with the spectral parameter $\lambda(t;z) := z/t \in \mathbb{C}$, $t \in \mathbb{R}_+$, making it possible to find the Lax type l-matrix

$$l[\Psi,\Psi^*;\lambda] = \begin{pmatrix} \lambda - ix/(4t) & i\Psi \\ i\Psi^* & -\lambda + ix/(4t) \end{pmatrix}. \tag{6.29}$$

The nonautonomous recursive operator $\Lambda : T^*(M) \to T^*(M)$ corresponding to (6.29), is given as follows: $\Lambda := \vartheta^{-1}\eta$, where

$$\vartheta = \begin{pmatrix} 0 & t^{-1} \\ -t^{-1} & 0 \end{pmatrix},$$

$$\eta = \begin{pmatrix} -2\Psi\partial^{-1}\Psi & -\partial - x/(4t) + 2\Psi\partial^{-1}\Psi^* \\ -\partial + x/(4t) + 2\Psi^*\partial^{-1}\Psi & -2\Psi^*\partial^{-1}\Psi^* \end{pmatrix} \tag{6.30}$$

for all $t \in \mathbb{R}_+$, $(\Psi,\Psi^*)^\intercal \in M$, which satisfies equation (6.15) with the function $g(t;\lambda) = -\lambda/t$, $\lambda \in \mathbb{C}$.

6.1.2 *Lagrangian and Hamiltonian analysis of dynamical systems on functional manifolds: The Poisson–Dirac reduction*

Let us consider a vector field $K : M \to T(M)$ on a smooth manifold $M \subset C^\infty(\mathbb{R}^n;\mathbb{R}^m)$, $n,m \in \mathbb{Z}_+$, which is Hamiltonian with respect of a symplectic operator $\Omega : T(M) \to T^*(M)$, that is

$$-\Omega K[u] = \text{grad } H[u] = dH[u] \tag{6.31}$$

for any $u \in M$, where $H \in \mathcal{D}(M)$ is the corresponding Hamiltonian function. The corresponding 2-form on $T(M) \times T(M)$ is closed, so $\Omega = d\varphi = \varphi' - \varphi'^{*}$ for some $\varphi \in T^{*}(M)$, where for all $\alpha \in T(M)$

$$\langle \varphi, \alpha \rangle = \int_0^1 d\lambda \, \langle \Omega[\lambda u] \, \lambda u, \alpha[u] \, \rangle. \tag{6.32}$$

It is easy to show that the dynamical system $u_t = K[u]$ on M admits the following Euler-Lagrange representation

$$\text{grad } \mathcal{L}[u, u_t] = 0 \tag{6.33}$$

where $t \in \mathbb{R}$,

$$\mathcal{L}[u, u_t] = \langle \varphi[u], u_t \rangle - H[u] \tag{6.34}$$

is a Lagrangian function density defined, as usual, up to a closed 1-form on M. The above Lagrangian system (6.33) is degenerate, since a generalized momentum variable $\pi \in T^{*}(M)$ conjugated with the independent variable $u \in M$ by the classical rule

$$\pi := \delta \mathcal{L}[u; u_t]/\delta u_t = \varphi[u] \tag{6.35}$$

cannot be inverted with respect of the dynamical variable $u_t \in T(M)$, $t \in \mathbb{R}$. We shall now obtain the original Hamiltonian system

$$u_t = K[u] = -\Omega^{-1} \text{grad } H[u] \tag{6.36}$$

on the functional manifold M as a reduction of a canonical Hamiltonian system on the phase space $T^{*}(M) \ni (u, \pi)$ by introducing certain constraints via the Dirac procedure [54, 100, 104, 184, 262, 287]. The corresponding extended Hamiltonian function $H_\alpha \in \mathcal{D}(T^{*}(M))$ is obtained by the formula:

$$H_\alpha := H + (\langle \pi - \varphi, \alpha \rangle), \tag{6.37}$$

where $\alpha \in T(M) \subset T(T^{*}(M))$ is a special vector of Lagrange multipliers that can be determined from the reduction condition; that is, on the submanifold $\{(u, \pi)^{\intercal} \in T^{*}(M) : \pi - \varphi[u] = 0\}$ the Hamiltonian (6.37) must be in involution with the constraint $\pi - \varphi[u] = 0$ with respect to the canonical symplectic structure $\Omega_0 \in \Lambda^2(T^{*}(M))$:

$$\{H_\alpha, (\langle \pi - \varphi[u], \tau \rangle)\}_0 = 0 \tag{6.38}$$

for any vector field $\tau \in T(M) \subset T(T^{*}(M))$, where $\{\cdot, \cdot\}_0$ is the Poisson bracket on $T^{*}(M)$ corresponding to $\Omega_0 \in \Lambda^2(T^{*}(M))$. For the corresponding bilinear form $\Omega_0 : T(M) \times T(M) \to \mathbb{R}$ one has

$$\Omega_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{6.39}$$

where for any two functionals $F, H \in \mathcal{D}(T^*(M))$

$$\{H, F\}_0 := (\langle \text{grad } H, \Omega_0^{-1} \text{grad } F\rangle).$$

It now follows from (6.38) and (6.39) that

$$(\langle \text{grad } H + (\varphi' - \varphi'^*)\, \alpha, \tau\rangle) = 0 \tag{6.40}$$

for all functional vectors $\tau \in T(M) \subset T(T^*(M))$. As grad $H = \Omega K = (\varphi' - \varphi'^*)K$, we immediately find that the choice $\alpha := K \in T(M)$ satisfies the above reduction. In the case of degeneracy of the co-implectic operator $\Omega : T(M) \to T^*(M)$, its kernel can be added to the above vector field $\alpha \in T(M)$. Thus, we have described the canonical Dirac procedure of lifting a given Hamiltonian system (6.36) on the manifold M to an extension on the cotangent bundle $T^*(M)$ and verified the following result in the process.

Theorem 6.1. *Let $(u, \pi) \in T^*(M)$ be local coordinates on the cotangent bundle $T^*(M)$ of a functional manifold M. Then to each nonlinear dynamical system $u_t = K[u]$ on M there is an associated extended nonlinear dynamical system on $T^*(M)$ defined as*

$$\begin{pmatrix} u_t \\ \pi_t \end{pmatrix} = K_\varphi[u, \pi] := \begin{pmatrix} K[u] \\ -K'^*\, \pi + L_K \varphi \end{pmatrix}$$

$$= -\Omega_0^{-1}[\text{grad } (\langle \pi, K\rangle) - (L_K \varphi, 0)^\intercal] \tag{6.41}$$

for some $\varphi \in T^*(M)$. If the above vector field $K : M \to T(M)$ is Hamiltonian on M, that is $\varphi' - \varphi'^* = \Omega$, where $\Omega\, K = -\text{grad } H$ for some $H \in \mathcal{D}(M)$, then

$$L_K \varphi = \text{grad } (\langle \varphi, K\rangle) + \Omega\, K = \text{grad } ((\langle \varphi, K\rangle) - H), \tag{6.42}$$

so

$$(u_t, \pi_t)^\intercal = K_\varphi[u, \pi] = -\Omega_0^{-1} \text{grad } ((\langle \pi - \varphi, K\rangle) - H)[u, \pi] \tag{6.43}$$

for any $(u, \pi)^\intercal \in T^*(M)$.

Suppose that $\varphi \in T^*(M)$ is such that the Lie derivative

$$L_K\, \varphi[u] = \text{grad } \mathcal{L}[u] \tag{6.44}$$

for some $\mathcal{L} \in \mathcal{D}(M)$ and any $u \in M$, with the corresponding co-implectic structure $\Omega \in \Lambda^2(M)$ being determined as above by $\Omega = d\varphi = \varphi' - \varphi'^*$. Then dynamical system (6.43) can be written in the form

$$(u_t, \pi_t)^\intercal = K_\varphi[u, \pi] = -\Omega_0^{-1} \text{grad } ((\langle \pi, K\rangle) - \mathcal{L})[u, \pi] \tag{6.45}$$

for all $(u,\pi)^\intercal \in T^*(M)$. If the Hamiltonian dynamical system (6.45) is reduced upon the manifold M via the mapping $M \ni u :\rightarrow (u, \varphi[u]) \in T^*(M)$, it is easy to see that the canonical symplectic structure (6.39) on the cotangent vector bundle $T^*(M)$ reduces to the symplectic structure $\Omega = d\varphi = \varphi' - \varphi'^*$ on the base manifold M.

Note that it follows from (6.41) that any dynamical system $u_t = K[u]$ on a manifold M (not necessarily Hamiltonian) always admits a non-canonical Hamiltonian extension on the cotangent bundle $T^*(M)$ given as

$$\begin{pmatrix} u_t \\ \pi_t \end{pmatrix} = K_\varphi[u,\pi] = -\Omega_\varphi^{-1}\text{grad }(\langle \pi - \varphi, K\rangle), \tag{6.46}$$

where

$$\Omega_\varphi := \begin{pmatrix} \varphi'^* - \varphi' & 1 \\ -1 & 0 \end{pmatrix} \tag{6.47}$$

for any $(u,\pi)^\intercal \in T^*(M)$.

Consider the KdV as a simple example of bi-Hamiltonian dynamical system:

$$du/dt = 6uu_x + u_{xxx} \tag{6.48}$$

on a functional manifold $M \subset C^\infty(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R})$, where $t \in \mathbb{R}$ is an evolution parameter. The dynamical system (6.48) on the manifold M is Hamiltonian, that is

$$du/dt = -\vartheta\text{grad }\int_0^{2\pi} dx(u_x^2/2 - u^3), \quad \vartheta = \partial/\partial x; \tag{6.49}$$

whence, $\Omega := \vartheta^{-1} = \partial^{-1} = \varphi' - \varphi'^*$, where $\varphi := \partial^{-1}u/2 \in T^*(M)$, $u \in M$. Then, from (6.46) one can easily compute that

$$\begin{pmatrix} u_t \\ \pi_t \end{pmatrix} = \begin{pmatrix} u_{xxx} + 6uu_x \\ \pi_{xxx} + 6u\pi_x - 3u^2/2 \end{pmatrix}$$

$$= -\Omega_0^{-1}\text{grad }(\int_0^{2\pi} dx[\pi(u_{xxx} + 6uu_x) + u^3/2])$$

$$= -\Omega_\varphi^{-1}\text{grad }(\int_0^{2\pi} dx[\pi(u_{xxx} + 6uu_x) + 3u^3/2 - 4u_x^2/2]), \tag{6.50}$$

where

$$\Omega_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Omega_\varphi = \begin{pmatrix} -\partial^{-1} & 1 \\ -1 & 0 \end{pmatrix}, \tag{6.51}$$

represents its corresponding bi-Hamiltonian extension on the cotangent bundle $T^*(M)$ by means of the special nonlocal element $\varphi = \partial^{-1}u/2 \in T^*(M)$.

We now analyze the Poisson–Dirac brackets on submanifolds of a functional manifold M endowed with a symplectic structure $\Omega \in \Lambda^2(M)$. Suppose also that we have a smooth dynamical system $u_t = K[u]$ on the manifold $M \ni u$ with $K : M \to T(M)$, the corresponding vector field on M that is Hamiltonian with respect to the symplectic structure $\Omega \in \Lambda^2(M)$ with a Hamiltonian function $H \in D(M)$, such that grad $H = -\Omega K[u]$ for any $u \in M$. Let us also be given $r \in \mathbb{Z}_+$ local functionals $\varphi_j \in T^*(M)$, $1 \leq j \leq r$, on M that are functionally independent on the functional submanifold

$$M_\varphi := \{u \in M : \ \varphi_j[u] = 0, \ 1 \leq j \leq r\} \subset M, \tag{6.52}$$

which is smoothly embedded in M. The elements of a set $\{\varphi_j \in T^*(M) : 1 \leq j \leq r\}$ are usually called holonomic restrictions or constraints. Following Dirac [100, 104, 326], we formulate a general scheme introducing the Hamiltonian formalism on the submanifold $M_\varphi \subset M$.

Lemma 6.1. *The restriction $\Omega_\varphi \in \Lambda^2(M_\varphi)$ of the symplectic structure $\Omega \in \Lambda^2(M)$ is a nondegenerate symplectic 2-form on M_φ if and only if the matrix differential-integral operator*

$$\hat{\varphi} := (\{\varphi_j, \varphi_k\}_\Omega)_{j,k=1}^r \tag{6.53}$$

at each point $u \in M_\varphi$ is also nondegenerate.

Proof. This is readily proved using straightforward calculations based on the above definitions. □

Therefore, if the matrix differential-integral operator (6.53) is not degenerate as an operator $\hat{\varphi} : (T(M))^r \to (T^*(M))^r$ at $u \in M_\varphi \subset M$, the corresponding 2-form $\Omega_\varphi \in \Lambda^2(M_\varphi)$ defines a suitable symplectic structure on M_φ. Let $K_H \in T(M_\varphi)$ be generated by a Hamiltonian function $H \in \mathcal{D}(M_\varphi)$ with respect to the symplectic structure $\Omega_\varphi \in \Lambda^2(M_\varphi)$ on M_φ. It is evident that in general $K_H \neq K_H|_{M_\varphi}$, but the following lemma holds.

Lemma 6.2. *The vector field $K_H = K_H|_{M_\varphi}$ if and only if $\{H, F_j\}_\Omega = 0$ on M_φ for all $1 \leq j \leq r$, where $F_j = (\varphi_j, \alpha_j)$ and $\alpha_j \in T(M)$ are arbitrary functional elements. Moreover, if $H \in \mathcal{D}(M_\varphi)$ and there are $a_j \in T(M_\varphi)$ such that $\sum_{k=1}^r (\hat{\varphi}_{jk} a_k, \alpha_j) = \{H, (\varphi_j, \alpha_j)\}_\Omega$ for all $1 \leq j \leq r$ on the submanifold $M_\varphi \subset M$, then $K_H = K_{H_0}|_{M_\varphi}$ where $H_0 = H + \sum_{j=1}^r (\varphi_j, a_j)$.*

Proof. If $\{H, F_j\}_\Omega = 0$ on M_φ for $1 \le j \le r$, then $K_H = -\Omega^{-1}\text{grad } H \in T(M_\varphi)$ for any $u \in M_\varphi$, so that $K_H|_{M_\varphi} = K_H$ on M_φ. In contrast, if $-\Omega^{-1}\text{grad } H = K_H = K_H|_{M_\varphi} \subset T(M_\varphi)$, then $\{H, F_j\}_\Omega = = (\text{grad } H, \Omega^{-1}\text{grad } F_j) = 0$ for $1 \le j \le r$. The last assertion of the lemma follows from the equality $K_H = K_{H_0} = K_{H_0}|_{M_\varphi}$ on M_φ, so the proof is complete. □

The Dirac-Poisson brackets on the submanifold $M_\varphi \subset M$ for arbitrary functionals $f, g \in \mathcal{D}(M_\varphi)$ with respect to the reduced symplectic structure $\Omega_\varphi \in \Lambda^2(M_\varphi)$ can be calculated using the following Dirac rule.

Theorem 6.2. (Dirac [100, 104]) *Let $\{f, g\}_{\Omega_\varphi}$ be the Poisson bracket of smooth functionals $f, g \in \mathcal{D}(M_\varphi)$ with respect to the reduced symplectic structure $\Omega_\varphi \in \Lambda^2(M_\varphi)$ on the submanifold $M_\varphi \subset M$. Then for all $u \in M_\varphi$*

$$\{f, g\}_{\Omega_\varphi} = \{f, g\}_\Omega - \sum_{i,j=1}^{r} (\{f, \varphi_j\}_\Omega, (\hat{\varphi}^{-1})_{j,k}\{\varphi_k, g\}_\Omega), \tag{6.54}$$

where $\{f, \varphi_j\}_\Omega$ and $\{\varphi_k, g\}_\Omega \in T^*(M)$ for $1 \le j, k \le r$, with the right-hand side of (6.54) *being computed for arbitrary smooth extensions of functionals $f, g \in \mathcal{D}(M_\varphi)$ to functionals $f, g \in \mathcal{D}(\tilde{M}_\varphi)$, where $\tilde{M}_\varphi \subset M$ is an open neighborhood of the submanifold $M_\varphi \subset M$.*

Proof. Let us assume that $\{f, F_j\}_\Omega = \{g, F_j\}_\Omega = 0$ on $M_\varphi \subset M$ for all $1 \le j \le r$. Then, it follows from the above results that

$$\{f, g\}_{\Omega_\varphi} = \Omega_\varphi(K_f, K_g) = \{f, g\}_\Omega|_{M_\varphi}$$

for all $u \in M_\varphi \subset M$. In general, we compute that

$$\{f, g\}_{\Omega_\varphi} = \{f + \sum_{k,j=1}^{r} (\varphi_j, (\hat{\varphi}^{-1})_{jk}\{f, \varphi_k\}_\Omega), g + \sum_{i,s=1}^{r} (\varphi_i, (\hat{\varphi}^{-1})_{is}\{g, \varphi_s\}_\Omega)\}_{\Omega_\varphi}$$

$$= \{f + \sum_{j,k=1}^{r} (\varphi_j, (\hat{\varphi}^{-1})_{jk}\{f, \varphi_k\}_\Omega), g + \sum_{i,s=1}^{r} (\varphi_i, (\hat{\varphi}^{-1})_{is}\{g, \varphi_s\}_\Omega)\}_\Omega$$

$$= \{f, g\}_\Omega + \sum_{j,k=1}^{r} ((\hat{\varphi}^{-1})_{jk}\{f, \varphi_k\}_\Omega, \{\varphi_j, g\}_\Omega) + \sum_{i,s=1}^{r} ((\hat{\varphi}^{-1})_{is}\{f, \varphi_i\}_\Omega, \{g, \varphi_s\}_\Omega)$$

$$+ \sum_{i,j,k,s=1}^{r} ((\hat{\varphi}^{-1})_{jk}(\hat{\varphi}^{-1})_{is}\hat{\varphi}_{ji}\{f, \varphi_k\}_\Omega, \{g, \varphi\}_\Omega)$$

$$= \{f, g\}_\Omega - \sum_{k,j=1}^{r} (\{f, \varphi_j\}_\Omega, (\hat{\varphi}^{-1})_{jk}\{\varphi_k, g\}_\Omega) \tag{6.55}$$

for all $u \in M_\varphi \subset M$, which yields the desired result, thereby completing the proof. □

6.1.3 *Remarks*

The theory of non-isospectrally Lax integrable dynamical systems described above expands the class of nonlinear dynamical systems from many branches of science that admit exact solutions. In particular, the following has been demonstrated: almost every nonlinear dynamical system admits a non-isospectral Lax representation, but a dynamical system is Lax integrable if and only if an evolution of the spectral parameter is independent of $u \in M$ for all Cauchy data. This leads to an effective criterion for determining whether or not a given nonlinear dynamical system on a functional manifold is non-isospectrally Lax integrable. Once such a criterion is satisfied, the reduction procedure developed in this chapter can be used to obtain the restrictions on invariant submanifolds $M_N \subset M$ which inherit the canonical Hamiltonian structure and Liouville complete integrability. Thus, powerful perturbation techniques can be brought to bear in studying the dynamical systems and deeper insights into the relationships between complete Hamiltonian theory and its various partial forms can be gained.

The embedding problem for infinite-dimensional dynamical systems with additional structures such as invariants and symmetries is as old as Newton-Lagrange mechanics, and has been the subject of extensive research using both analytical and qualitative methods. The differential geometric tools described here were developed mainly by E. Cartan at the beginning of the twentieth century. The great advance in the embedding methods during the last two decades is primarily a result of the theory of isospectral deformations for linear differential structures, constructed on special vector bundles over a space carrying a nonlinear dynamical system. Among these linear structures are the moment map $l : M \to \mathfrak{g}^*$ into the adjoint space of the related Lie algebra $\mathfrak{g}$ of symmetries acting equivariantly on the symplectic phase space M, with the Cartan-Ehresmann connection structures appearing via the generalized Wahlquist-Estabrook approach [44, 392].

General analytical structures of Hamiltonian and Lagrangian formalisms have been thoroughly studied, primarily using geometric and algebraic methods [157, 187, 218, 268, 371]. Considerable attention has also been paid to the theory of differential-difference dynamical systems on infinite-dimensional manifolds [218, 219]. Several articles have been devoted to the theory of discrete dynamical systems [92, 228], and these [231, 364] appear to have a variety of important applications.

6.2 Algebraic structure of the gradient-holonomic algorithm for Lax integrable systems

6.2.1 *Introduction*

Assume that we are given a nonlinear uniform dynamical system

$$du/dt = K[u] \tag{6.56}$$

on a 2π-periodic infinite-dimensional manifold $M \subset C^\infty(\mathbb{R}/2\pi\mathbb{Z}; \mathcal{B}^m)$, $m \in \mathbb{Z}_+$, where $K : M \to T(M)$ is a Fréchet smooth vector field on M, $t \in \mathbb{R}$ is the evolution parameter, and $\mathcal{B}$ is a Banach algebra of operators on the functional Schwartz space $\mathcal{G}(\mathbb{R}; \mathbb{C}^n)$, $n \in \mathbb{Z}_+$.

The Lax type integrability of (6.56) implies the existence of an associated linear operator $\ell[u; \lambda] : \mathcal{B}^r \to \mathcal{B}^r$, $r \in \mathbb{Z}_+$, which depends on a point on an orbit $u \in M$ of (6.56) with a spectral parameter $\lambda \in \mathbb{C}$, where the generalized spectrum [262, 406] $\mathrm{Spec}(\ell)$ of the ℓ-operator is invariant with respect to the dynamical system (6.56). As a consequence of this invariance property, the system (6.56) is Hamiltonian [3, 14, 278, 326] and has an infinite hierarchy of commuting conservation laws.

As we showed in Chapter 5, the gradient-holonomic method, which was first proposed in [262, 326], is an effective tool for determining the Lax integrability of a dynamical system. This method is based on the special gradient-holonomic relations for the monodromy matrix of the Lax associated linear ℓ-operator and has its origins in the general Lie-algebraic scheme of Adler [7] and the theory of the central extension [121] augmented by Novikov's variational theory [227, 262, 326, 406] and the Fokas–Santini approach developed in [130, 131, 230].

The aim here is to provide additional concerning the algebraic structure of the gradient-holonomic algorithm for Lax integrability.

6.2.2 *The algebraic structure of the Lax integrable dynamical system*

Let $\mathfrak{g}$ be a metrizable Lie algebra over the field $\mathbb{R}$ (or $\mathbb{C}$) with a nondegenerate invariant and symmetric scalar product $(\cdot, \cdot)$, so that for arbitrary $a, b, c \in \mathfrak{g}$ we have

$$(a, [b, c]) = ([a, b], c), \qquad (a, b) = (b, a), \tag{6.57}$$

where $[\cdot, \cdot]$ is the Lie bracket on the module $\bar{\mathfrak{g}}$. This yields a natural identification of the spaces $\mathfrak{g}^*$ and $\mathfrak{g}$: $\mathfrak{g}^* \simeq \mathfrak{g}$.

Definition 6.1. An $\mathcal{R}$-structure is a pair $(\mathfrak{g}, \mathcal{R})$ where $\mathfrak{g}$ is a Lie algebra, and $\mathcal{R} : \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}$ is a linear homomorphism such that

$$[a,b]_{\mathcal{R}} := [\mathcal{R}a, b] + [a, \mathcal{R}b], \tag{6.58}$$

is also a Lie structure on the module $\bar{\mathfrak{g}}$.

It is easy to see that the $\mathcal{R}$-homomorphism $\mathcal{R} : \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}$ (also called an $\mathcal{R}$-matrix) generates *an* $\mathcal{R}$-structure (6.58), if the following Yang–Baxter [121, 286, 287, 374] relation is satisfied for all $a, b \in \mathfrak{g}$:

$$\mathcal{R}[a,b]_{\mathcal{R}} - [\mathcal{R}a, \mathcal{R}b] = \nu[a,b], \tag{6.59}$$

where $\nu \in \mathbb{C}$ is a complex parameter. The Yang–Baxter equation (6.59) (or briefly $YB(\nu)$) has a simple solution [121, 374, 373] in the case where the Lie algebra $\mathfrak{g}$ splits into two subalgebras; that is, $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$. Namely, the following relation

$$\mathcal{R} = (P_+ - P_-)/2, \tag{6.60}$$

holds, where $P_\pm \mathfrak{g} := \mathfrak{g}_\pm$ are projectors in $\mathfrak{g}$ onto its subalgebras. If we also have $\mathfrak{g}_\pm^* \simeq \mathfrak{g}_\mp$, then obviously the $\mathcal{R}$-structure (6.60) will be *unitary*, i.e., for all $a, b \in \mathfrak{g}$ we have

$$(\mathcal{R}a, b) + (a, \mathcal{R}b) = 0. \tag{6.61}$$

In the case of the Yang–Baxter equation $YB(0)$, if a homomorphism $\mathcal{R} : \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}$, is invertible then the symplectic two-cocycle $\omega_0(a,b) := (a, \mathcal{R}^{-1}b)$, where $a, b \in \mathfrak{g}$, also models the unitary $\mathcal{R}$-structure on the Lie algebra $\mathfrak{g}$, so the homomorphism $\mathcal{R}^{-1} : \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}$ is its derivative [287]. The Lie structure can be used to define the Lie-Poisson bracket

$$\{\gamma, \mu\}_{\mathrm{Lie}}(\ell) = (\ell, [\triangledown\gamma(\ell), \triangledown\mu(\ell)]) = (\triangledown\gamma(\ell), \mathcal{L} \triangledown \mu(\ell)) \tag{6.62}$$

for any smooth functionals $\gamma, \mu \in \mathcal{D}(\mathfrak{g}^*)$, where $\ell \in \mathfrak{g}^*$ is a fixed element, $\mathcal{L} : \mathfrak{g} \to \mathfrak{g}^*$ is the induced homomorphism, and $\triangledown\gamma(l), \triangledown\mu(l) \in \mathfrak{g}$ are the standard gradients of functionals on $\mathfrak{g}^*$, defined as: $(m, \triangledown\gamma(l)) = d/d\varepsilon\gamma(\ell + \varepsilon m)|_{\varepsilon=0}$ for any smooth $\gamma \in \mathcal{D}(\mathfrak{g}^*)$ and all $m \in \mathfrak{g}^*$. The $\mathcal{R}$-structure (6.60) generates on $\mathfrak{g}^*$ a new Poisson bracket via the formula

$$\{\gamma, \mu\}_\vartheta(\ell) = (\ell, [\triangledown\gamma(\ell), \triangledown\mu(\ell)]_{\mathcal{R}}) := (\triangledown\gamma(\ell), \vartheta \triangledown \mu(\ell)), \tag{6.63}$$

where $\ell \in \mathfrak{g}^*$ and $\vartheta = \mathcal{R}^*\mathcal{L} + \mathcal{L}\mathcal{R} : \mathfrak{g} \to \mathfrak{g}^*$ is the induced map. For the Jacobi identity to hold for the bracket (6.63), a homomorphism $\mathcal{R} : \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}$ must satisfy the following generalized equation [16, 18, 333]:

$$\ell([X, [\mathcal{R}Y, \mathcal{R}Z] - \mathcal{R}[Y,Z]_{\mathcal{R}}]) + \text{cycle}$$

$$+\ell([X,\{\ell(Y),\mathcal{R}Z\}_{\vartheta}-\{\ell(Z),\mathcal{R}Y\}_{\vartheta}])+\text{cycle}=0 \tag{6.64}$$

for any $\ell \in \mathfrak{g}^*$ and all X, Y, and Z in $\mathfrak{g}$, where $\ell(X) := (\ell, X)$, $\ell(Y) := (\ell, Y)$, and $\ell(Z) := (\ell, Z)$. It is clear that in the case $\nabla\mathcal{R}(\ell) = 0$ for all $\ell \in \mathfrak{g}^*$, the Yang–Baxter equation (6.59) is sufficient for (6.63) to be a Poisson bracket.

Let the metrizable Lie algebra $\mathfrak{g}$ be generated by the associative underlying algebraic structure over $\mathbb{C}$ with the standard commutator Lie structure $[X,Y] = XY - YX$ for all $X, Y \in \mathfrak{g}$. Then any $\mathfrak{g}^*$-independent $\mathcal{R}$-structure (6.58) on $\mathfrak{g}$ generates two $\mathfrak{g}^*$-dependent $\mathcal{R}_{\pm}$-homomorphisms via [18, 333]

$$\mathcal{R}_{\pm}(X) = P_{+}(\ell X) \pm P_{-}(X\ell), \tag{6.65}$$

where $\ell, X \in \mathfrak{g}$, $P_{\pm} := (\frac{1}{2} \pm \mathcal{R})$. It is important to point out that the brackets $\{\cdot,\cdot\}_{\vartheta}$ and $\{\cdot,\cdot\}_{\eta}$ are compatible, where, by definition,

$$\{\gamma,\mu\}_{\eta}(\ell) = (\ell, [\nabla\gamma(\ell), \nabla\mu(\ell)]_{\mathcal{R}_{-}}) = (\nabla\gamma(\ell), \eta\nabla\mu(\ell)), \tag{6.66}$$

$\ell \in \mathfrak{g}^*$, $\gamma, \mu \in \mathcal{D}(\mathfrak{g}^*)$ and $\eta : \mathfrak{g} \to \mathfrak{g}^*$ is the corresponding induced map. Now, assume that the functional $\gamma \in I(\mathfrak{g}^*)$ is of the Casimir type. Then for all $\ell \in \mathfrak{g}^* \simeq \mathfrak{g}$ we have

$$\mathrm{ad}^*_{\nabla\gamma(\ell)}\ell[\ell, \nabla_{\gamma}(\ell)] = 0, \tag{6.67}$$

where ad^* is the coadjoint action of the Lie algebra $\mathfrak{g}$ on the adjoint space $\mathfrak{g}^*$. The following result is very important [7, 121, 206, 326, 373, 385].

Lemma 6.3. *The set of Casimir functionals $I(\mathfrak{g}^*)$ is involutive with respect to the two Poisson brackets* (6.62) *and* (6.63)*. Moreover, the set $I(\mathfrak{g})$ is involutive with respect to the bracket* (6.66)*.*

Let us assume now that $\nabla\mathcal{R}(\ell) = 0$ for all $\ell \in \mathfrak{g}^*$ and construct the Hamiltonian vector field on the manifold $\mathfrak{g}^*$ via the Poisson bracket (6.63) and the Hamiltonian functional $\gamma \in I(\mathfrak{g}^*)$. For all $t \in \mathbb{R}$ and $\ell \in \mathfrak{g}^*$ we obtain

$$d\ell/dt = \mathrm{ad}^*_{\mathcal{R}\nabla\gamma(\ell)}\ell = [\ell, \mathcal{R}\nabla\gamma(\ell)]. \tag{6.68}$$

Equation (6.68) has the standard Lax commutator form and may be used in the theory of integrable dynamical systems. Indeed, let us suppose that an element $\ell \in \mathfrak{g}^*$ is a locally Fréchet smooth functional on an operator manifold $M \subset C^{\infty}(\mathbb{R}/2\pi\mathbb{Z}; \mathcal{B}^m)$, that is, $\ell := \ell[u] \in \mathfrak{g}^*$ for all $u \in M$. Then, it is easy to see that the Lax evolution equation (6.68) is equivalent to a nonlinear dynamical system on the operator manifold M. Owing to (6.56) and the properties of the Poisson brackets $\{\cdot,\cdot\}_{\vartheta}$ and $\{\cdot,\cdot\}_{\eta}$, the nonlinear dynamical system obtained will be bi-Hamiltonian via Magri [242, 243] and also

Liouville–Lax integrable [2, 173, 262, 278] on the manifold M. To expand the class of Liouville–Lax integrable dynamical systems on the manifold M described above, let us make use of the powerful techniques of central extensions of the basic Lie current algebra $\mathfrak{g}$, as given in [262, 263, 326]. To this end, we denote by $\tilde{\mathfrak{g}} := \mathfrak{g} \oplus \mathbb{C}$ the standard central extension [121, 278] of the metrizable Lie algebra $\mathfrak{g}$ of currents on the circle by means of the two-cocycle

$$\omega_p(a,b) = \int_0^{2\pi} dx \, \left(\lambda^p a, \frac{db}{dx}\right),$$

where $a, b \in C^\infty(\mathbb{R}/2\pi\mathbb{Z}; \mathfrak{g})$, and $p \in \mathbb{Z}$ is arbitrary. Using an obvious analog of the theory above, we find that a Casimir functional $\gamma \in I(\tilde{\mathfrak{g}}^*)$ is defined by the condition

$$\lambda^p d \bigtriangledown \gamma(\ell)/dx = [\ell, \bigtriangledown\gamma(\ell)] \tag{6.69}$$

for $p \in \mathbb{Z}$, $\ell \in \tilde{\mathfrak{g}}^* \simeq \tilde{\mathfrak{g}}$, $x \in \mathbb{R}$. The corresponding $\mathcal{R}$-structure on the Lie algebra $\tilde{\mathfrak{g}}$ is defined as

$$[(a,\alpha),(b,\beta)]_{\mathcal{R}} = ([a,b]_{\mathcal{R}}, \tilde{\omega}_p(a,b)), \tag{6.70}$$

$$\tilde{\omega}_p(a,b) := \omega_p(a,\mathcal{R}b) + \omega_p(\mathcal{R}a,b), \tag{6.71}$$

where $p \in \mathbb{Z}$, and (a,α), $(b,\beta) \in \tilde{\mathfrak{g}}$ are any fixed elements. The $\mathcal{R}$-structure (6.70) satisfies the classical Yang–Baxter relation with parameter $\nu = 1/4$ for all $p \in \mathbb{Z}$. The standard proof of bi-Hamiltonicity of Lax integrable nonlinear dynamical systems goes through for the centrally extended Lie algebra $\tilde{\mathfrak{g}} \simeq \mathfrak{g} \oplus \mathbb{C}$, after a simple modification using as instruments the infinite hierarchy of two-cocycles $\omega_p(\cdot,\cdot)$, $p \in \mathbb{Z}$, and the Dirac theory [54, 100, 104, 121, 173, 184, 326] of Poisson brackets on the constrained Poisson invariant manifolds above. Thus, it is possible to construct the Hamiltonian vector field via expression (6.68) on the extended Lie algebra $\tilde{\mathfrak{g}}$ as

$$\frac{d\ell}{dt} = [\ell, \mathcal{R} \bigtriangledown \gamma(\ell)] - \lambda^p \frac{d\mathcal{R} \bigtriangledown \gamma(\ell)}{dx}, \tag{6.72}$$

where $\ell \in \mathfrak{g}^* \simeq \mathfrak{g}$, for all $t \in \mathbb{R}$, $x \in \mathbb{R}$, $p \in \mathbb{Z}$, $c = 1$ and the Casimir functional $\gamma \in I(\tilde{\mathfrak{g}}^*)$ on $\tilde{\mathfrak{g}}^*$. It is clear that (6.72) is equivalent to the Lax equation

$$\frac{d\ell}{dt} = \left[\ell - \lambda^p \frac{d}{dx}, \mathcal{R} \bigtriangledown \gamma(\ell)\right] \tag{6.73}$$

for $p \in \mathbb{Z}$, $\ell \in \mathfrak{g}^* \simeq \mathfrak{g}$, and all $x, t \in \mathbb{R}$. Note that the commutator equation (6.73) is the compatibility condition for the following important class of linear systems:

$$df/dx = \lambda^{-p}\ell f, \quad df/dt = -\mathcal{R} \bigtriangledown \gamma(\ell) f, \tag{6.74}$$

where $p \in \mathbb{Z}$, $f \in W(\mathbb{R}; \mathcal{B}^r)$ is an element of a suitable Sobolev space, $x, t \in \mathbb{R}$, and $r = \dim(\operatorname{rep} \mathfrak{g})$ is the dimension of a matrix representation of the Lie algebra $\mathfrak{g}$. If we are given the smooth moment map [162, 164–166, 191–193, 206, 224, 358] $\ell : M \to \mathfrak{g}^*$, then the Lax representation (6.73) is equivalent to a Liouville–Lax nonlinear Hamiltonian system of type (6.56) on the manifold M. To identify these nonlinear integrable Hamiltonian equations, we must describe the structure of the basic Lie algebra $\mathfrak{g}$. More precisely, let the Lie algebra $\mathfrak{g}$ have the following affine structure $\mathfrak{g} = \mathrm{gl}(r; \mathcal{B}) \otimes \mathbb{C}(\lambda, \lambda^{-1})$, where $r \in \mathbb{Z}_+$, and $\lambda \in \mathbb{C}$ is the spectral parameter. The metric structure on $\mathfrak{g}$ is defined as follows: for all $a, b \in \mathfrak{g}$ let

$$(a, b) = \operatorname{res}_{\lambda=0} \int_0^{2\pi} \operatorname{Tr}(a\, b)(x; \lambda) dx. \tag{6.75}$$

Here $\lambda \in \mathbb{C}$ and, by definition, for any element $c \in \mathfrak{g}$

$$\operatorname{Tr} c(x; \lambda) = \operatorname{tr} \operatorname{Sp} c(x; \lambda),$$

where tr is the ordinary matrix trace, and Sp denotes an invariant trace operation on the associative operator algebra $\mathcal{B}$. Now, we construct the $\mathcal{R}$-structure on the Lie algebra $\mathfrak{g}$ as follows: Set $\mathcal{R} = (P_+ - P_-)/2$, where $P_\pm \mathfrak{g} = \mathfrak{g}_\pm$, $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ and define

$$\mathfrak{g}_+ := \left\{ \sum_{j=0}^{n<\infty} u_j(x)\lambda^j : \ u_j(x) \in \mathrm{gl}(r; \mathcal{B}),\ j \in \mathbb{Z}_+,\ x \in \mathbb{R}/2\pi\mathbb{Z} \right\},$$

$$\mathfrak{g}_- := \left\{ \sum_{j \in \mathbb{Z}_+} a_j(x)\lambda^{-(j+1)} : \ \ a_j(x) \in \mathrm{gl}(r; \mathcal{B}),\ j \in \mathbb{Z}_+,\ x \in \mathbb{R}/2\pi\mathbb{Z} \right\}. \tag{6.76}$$

The role of the associative operator algebra $\mathcal{B}$ is to be played here by one of the following: (i) $\mathcal{B}$ is the set of trace-class integral operators acting in the Schwartz space of functions; (ii) $\mathcal{B}$ is the set of symbols for micro-differential operators acting in the Schwartz space of functions. We mention here that case (i) was considered earlier in essentially different ways [7] and also in [202]. Case (ii) was investigated in [130, 131, 333], where the Lie algebra $\mathfrak{g}$ was chosen as $\mathrm{gl}(2; \mathcal{B})$. Illustrative examples of applications of the above algebraic scheme will be presented in the sequel.

6.2.3 *The periodic problem and canonical variational relationships*

Construct the monodromy transfer matrix $S(x;\lambda)$, $\lambda \in \mathbb{C}$, for the first linear equation in (6.74) at $p = 0$, where $\ell : M \to \mathfrak{g}^*$ is the suitable momentum map:

$$S(x;\lambda) := F(x + 2\pi, x; \lambda) \tag{6.77}$$

for all $x \in \mathbb{R}$, $\lambda \in \mathbb{C}$. Here $F(x, x_0; \lambda)$, $\lambda \in \mathbb{C}$, is the fundamental operator-valued solution of the first equation in (6.74) at $p = 0$, which is normalized to the identity at a point $x = x_0 \in \mathbb{R}$ for all $\lambda \in \mathbb{C}$. It is easy to verify that the monodromy matrix (6.77) satisfies the following Novikov–Lax equation

$$dS/dx = [\ell, S], \tag{6.78}$$

where $\ell := \ell(x;\lambda) \in \mathfrak{g}$ is a fixed element, 2π-periodic in $x \in \mathbb{R}$. It follows from equation (6.78) that the functional

$$\Delta(\zeta) := \operatorname{Tr} S(x;\lambda)$$

$\zeta \in \mathbb{C}$, is invariant in relation to the vector field d/dx, that is

$$d\Delta(\zeta)/dx = 0$$

for all $x \in \mathbb{R}$. As a consequence of the Casimir equation (6.69) at $p = 0$, we observe the obvious identification: $\bigtriangledown\gamma(\ell)(x;\lambda) = S(x;\lambda)k(\lambda)$ for all $x \in \mathbb{R}$, $\lambda \in \mathbb{C}$, that is $\gamma(\ell) \in I(\tilde{\mathfrak{g}})$ is a Casimir related to the monodromy matrix, and $k(\lambda) \in \mathbb{C}$ is a constant with respect to $x \in \mathbb{R}$. Let us now consider some variational properties of the monodromy matrix (6.77) via the methods of [262, 263, 333]. It follows directly from the analytic properties of (6.69) that as $|\zeta| \to \infty$

$$\begin{aligned}\delta\Delta(\zeta) &= \int_0^{2\pi} dx\ \operatorname{Tr}(S(x;\zeta)\delta\ell(x;\zeta)) \\ &= \operatorname{res}_{\lambda=\zeta} \int_0^{2\pi} dx\ \operatorname{Tr}\left(\frac{1}{\lambda-\zeta} S(x;\lambda)\delta\ell(x;\lambda)\right) \\ &= (\operatorname{grad}\Delta(\zeta), \delta\ell),\end{aligned} \tag{6.79}$$

where $(\cdot,\cdot)$ above denotes a nondegenerate invariant and symmetric scalar product on the Lie algebra $\mathfrak{g}$, $\operatorname{res}_{\lambda=\zeta}$ is the usual residue, and grad $(\cdot) := \bigtriangledown(\cdot)$. Owing to (6.79), we compute that

$$\{\Delta(\lambda), \Delta(\zeta)\}_\vartheta = 0 = \{\Delta(\lambda), \Delta(\zeta)\}_\eta, \operatorname{grad}\Delta(\zeta)(l) = S(x;\lambda)/(\lambda-\eta)$$

for all $x \in \mathbb{R}$ and $\zeta, \lambda \in \mathbb{C}$. Since $\bigtriangledown\gamma(\ell)(x;\lambda) = S(x;\lambda) \cdot k(\lambda)$ for all $x \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, we obtain an important identity for the whole hierarchy of Casimir functionals on $\mathfrak{g} : \gamma(\ell)(\zeta) = \Delta(\zeta)$, if $k(\lambda) = (\lambda - \zeta)^{-1}$ as $|\lambda| \to \infty$. The following result [173, 355, 262, 406] can be easily proved by employing Floquet theory.

Theorem 6.3. *The set of Casimir functionals $I(\tilde{\mathfrak{g}}^*)$ on the Lie algebra $\tilde{\mathfrak{g}}$ coincides with the set of trace functionals $\Delta(\lambda) = \operatorname{Tr} S(x;\lambda)$, $\lambda \in \mathbb{C}$, for the monodromy matrix of the first Lax equation in* (6.74)*.*

This theorem allows us to describe the infinite set of gradients $\{\bigtriangledown\gamma_j \in T^*(M) : j \in \mathbb{Z}_+\}$ by means of asymptotic solutions of (6.78); namely, we expand

$$\Delta(\lambda) \sim \sum_{j\in\mathbb{Z}_+} \gamma_j \lambda^{-j},$$

where $\gamma_j \in \mathcal{D}(M)$, $j \in \mathbb{Z}_+$, in relation to the expansion

$$S(x;\lambda) \sim \sum_{j\in\mathbb{Z}_+} S_j(x)\lambda^{-j} \tag{6.80}$$

as $|\lambda| \to \infty$. Substituting the expansion (6.80) into (6.78), we obtain a system of recursion formulas for the coefficients $S_j(x)$, $j \in \mathbb{Z}_+$. As a consequence of the equation $\bigtriangledown\gamma(\ell)(x;\lambda) = S(x;\lambda)\, k(\lambda)$ for all $x \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, we can determine the operator quantity $\mathcal{R} \bigtriangledown \gamma(\lambda) \in \mathfrak{g}$ in the second equation of (6.74). More precisely, using the equality $\mathcal{R} = (P_+ - P_-)/2$ we find the hierarchy of homogeneous expressions for the quantities $\mathcal{R} \bigtriangledown \gamma_n(\ell) \in \mathfrak{g}$, $n \in \mathbb{Z}_+$:

$$\begin{aligned} \mathcal{R} \bigtriangledown \gamma_n(\ell) &= \frac{1}{2}(P_+ - P_-) \sum_{j\in\mathbb{Z}_+} S_j(x)\lambda^{n-j-1} \\ &= \frac{1}{2}[(\lambda^{n-1}S(x;\lambda))_+ - (\lambda^{n-1}S(x;\lambda))_-], \end{aligned} \tag{6.81}$$

where $(\lambda^{n-1}S(x;\lambda))_\pm \in \mathfrak{g}_\pm$ is as defined in (6.76).

With the above equation in hand, we turn to the search for the induced hierarchy of evolution operator equations (6.72). For each $n \in \mathbb{Z}_+$ we have

$$\frac{dl}{dt_n} = \frac{1}{2}\left([l, (\lambda^{n-1}S)_+] - [l, (\lambda^{n-1}S)_-] - \frac{d(\lambda^{n-1}S)_+}{dx} + \frac{d(\lambda^{n-1}S)_-}{dx}\right), \tag{6.82}$$

where $t_n \in \mathbb{R}$, $n \in \mathbb{Z}_+$, is a new hierarchy of evolution parameters. Recalling the main determining equations (6.69) at $p = 0$ and (6.78), we also obtain

$$\frac{d(\lambda^{n-1}S)_+}{dx} - [\ell, (\lambda^{n-1}S)_+] = -\frac{d(\lambda^{n-1}S)_-}{dx} + [\ell, (\lambda^{n-1}S)_-] \tag{6.83}$$

for all $n \in \mathbb{Z}_+$; whence from (6.82) and (6.83) it is easy to see that for all $n \in \mathbb{Z}_+$

$$\frac{d\ell}{dt_n} = \left[\ell - \frac{d}{dx}, (\lambda^{n-1}S)_+\right], \tag{6.84}$$

giving rise to an infinite hierarchy of evolution Lax operator equations. To identify equation (6.84) with nonlinear Liouville–Lax integrable dynamical systems of type (6.56) on the operator manifold M, we assume once more that $\mathfrak{g}^* \ni \ell(x;\lambda) := \ell[u;\lambda]$, where $u \in M$ and $\lambda \in \mathbb{C}$. Hence, we deduce from equation (6.79) that

$$\begin{aligned} \delta\gamma_j(\ell) &= \mathrm{res}_{\lambda=\infty} \int_0^{2\pi} dx\ \mathrm{Tr}(S(x;\lambda)\delta\ell[u;\lambda]\lambda^{j-1}) \\ &= (\lambda^{j-1}S, \delta\ell) := (\mathrm{grad}\ \gamma_j[u], \delta u), \end{aligned} \tag{6.85}$$

where $\Delta(\lambda) \sim \sum_{j\in\mathbb{Z}_+} \gamma_j(\ell)\lambda^{-j}$ as $|\lambda| \to \infty$, grad $\gamma_j[u] \in T^*M)$, $j \in \mathbb{Z}_+$, and $(\cdot,\cdot)$ is the standard bilinear form on $T^*(M) \times T(M)$. Then it follows directly from (6.85) that for all $u \in M$ and $j \in \mathbb{Z}_+$ we have

$$\varphi_j[u] := \mathrm{grad}\ \gamma_j[u] = \mathrm{res}_{\lambda=0}\mathrm{Sp}(\lambda^{j-1}\ell'^*[u;\lambda]S(x;\lambda)), \tag{6.86}$$

where $S(x;\lambda)$, $\lambda \in \mathbb{C}$, is the canonical asymptotic matrix solution of equation (6.78), $\ell' : T(M) \to \mathfrak{g}^*$ denotes the Fréchet derivative of a local $\mathfrak{g}$-valued functional on the manifold M, and ℓ'^* is its conjugation with respect to the bracket $(\cdot,\cdot)$ on $T^*(M)\times T(M)$. As a consequence of (6.85), we have the general formula

$$\varphi(x;\lambda) := \mathrm{grad}\ \Delta(\lambda)[u] = \mathrm{Sp}(\ell'^*[u;\lambda]S(x;\lambda)) \tag{6.87}$$

for all $x \in \mathbb{R}$, $\lambda \in \mathbb{C}$, which is very useful for further applications and which was first introduced in [227, 263, 262, 308]. Note that the element $\varphi \in T^*(M)$ for all $\lambda \in \mathbb{C}$ satisfies the Lax equation [227],

$$d\varphi/dt + K'^*[u;\zeta]\,\varphi = 0, \tag{6.88}$$

where the generating dynamical system $K : M \to T(M)$ is defined as

$$du/dt = K[u;\zeta] := -\vartheta \mathrm{grad}\ \Delta(\zeta)[u], \tag{6.89}$$

where $t \in \mathbb{R}$ and $\zeta \in \mathbb{C}$ denotes the labeling index of the hierarchy (6.84).

Next, from the asymptotic expansion of the monodromy matrix, we obtain the following recurrent gradient identity:

$$q(\lambda)\vartheta\varphi(x;\lambda) = \eta\varphi(x;\lambda), \tag{6.90}$$

where $q(\lambda) \in \mathbb{C}$ is a *recursion number function.* Indeed, for all $\lambda, \zeta \in \mathbb{C}$ we have

$$\{\Delta(\lambda), \Delta(\zeta)\}_{\vartheta} = (\varphi(x;\lambda), \vartheta\varphi(x;\lambda)) = 0 = \{\Delta(\lambda), \Delta(\zeta)\}_{\eta}$$

and, by the properties of the functional $\Delta(\zeta)$, $\zeta \in \mathbb{C}$, the function $q : \mathbb{C} \to \mathbb{C}$ is Casimir on the manifold M for all $\zeta \in \mathbb{C}$. Observe that the implectic and Nötherian operators $\vartheta, \eta : T^*(M) \to T(M)$ are defined in full correspondence with the Poisson brackets (6.63) and (6.66). In summary, we can apply identity (6.90) to the initial dynamical system (6.89) to derive the involution of an infinite hierarchy of conservation laws on the manifold M, that is

$$\{\gamma_j, \gamma_k\}_{\vartheta} = 0 = \{\gamma_j, \gamma_k\}_{\eta}$$

for all $j, k \in \mathbb{Z}_+$. Since the implectic pair (ϑ, η) in (6.90) is also compatible [137, 173, 242, 262], it follows that the recursion operator $\Lambda = \vartheta^{-1}\eta : T^*(M) \to T^*(M)$ satisfies the Nijenhuis condition: for all $\alpha, \beta \in T(M)$,

$$\Lambda^*([\Lambda^*\alpha, \beta] + [\alpha, \Lambda^*\beta]) - [\Lambda^*\alpha, \Lambda^*\beta] = \Lambda^{*2}[\alpha, \beta]. \tag{6.91}$$

Accordingly the dynamical system (6.89) is bi-Hamiltonian and completely Liouville–Lax integrable on the operator manifold M. The variational properties of the monodromy matrix (6.77) also allow us to deduce the following interesting identity, which was introduced in [333, 389]

$$\delta\left(\frac{\lambda^{-\nu}S(x;\lambda)}{\lambda - \zeta}, \frac{d\ell}{d\lambda}\right) = \left(\zeta^{-\nu} \cdot \frac{d}{d\zeta} \cdot \zeta^{\nu}\right)\left(\frac{\lambda^{-\nu}S(x;\lambda)}{\lambda - \zeta}, \delta\ell\right), \tag{6.92}$$

where δ denotes the variation operation on the manifold M, and $\nu \in \mathbb{Q}$ is a uniformity number of the matrix solution to equation (6.78), $\lambda \in \mathbb{C}$, that is

$$\operatorname{ord} S(x;\lambda) := 0, \quad \operatorname{ord} \lambda^{-\nu}S(x;\lambda) := -\deg \lambda^{-\nu}.$$

The next lemma is valid owing to the results in [173, 262] and the identity (6.79).

Lemma 6.4. *Let $\delta\ell \in \mathfrak{g}^*$ be an arbitrary variation of the Lax operator $\ell \in \mathfrak{g}^* \simeq \mathfrak{g}$ in the initial linear systems of equations* (6.74). *Then the induced trace of the monodromy matrix variation $\delta\Delta(\zeta) = \operatorname{Tr}\, \delta\, S(x;\zeta)$, $\zeta \in \mathbb{C}$, is determined by the following expression:*

$$\delta\Delta(\zeta) = \left(\frac{S}{\lambda - \zeta}, \delta\ell\right). \tag{6.93}$$

Proof. Substitute the condition $\delta\ell := (d\ell/d\lambda)\,\delta\lambda,\ \lambda \in \mathbb{C}$ into (6.93). Then, it follows that for all $\zeta \in \mathbb{C}$

$$\frac{d\Delta(\zeta)}{d\zeta} = \left(\frac{S}{\lambda-\zeta}, \frac{dl}{d\lambda}\right). \tag{6.94}$$

Similarly, by applying the identity $d\delta\Delta(\zeta)/d\zeta \equiv \delta(d\Delta(\zeta)/d\zeta)$, where $\zeta \in \mathbb{C}$ is arbitrary, we deduce from (6.93) and (6.94) that:

$$\delta\left(\frac{S}{\lambda-\zeta}, \frac{d\ell}{d\lambda}\right) = \frac{d(S/(\lambda-\zeta), \delta\ell)}{d\zeta}. \tag{6.95}$$

Since the operator matrix $\lambda^{-\nu}S(x;\lambda)$ is also a solution to the determining equation (6.78) for all $\nu \in \mathbb{Q}$, $\lambda \in \mathbb{C}$, it follows from (6.94) that

$$\begin{aligned}
&\delta\left(\frac{\lambda^{-\nu}S(x;\lambda)}{\lambda-\zeta}, \frac{d\ell}{d\lambda}\right) = \delta\left(\frac{S(x;\lambda)}{\lambda-\zeta}, \frac{d\ell}{d\lambda}\right)\zeta^{-\nu} \\
&= \zeta^{-\nu}\frac{d}{d\zeta}\left(\frac{S(x;\lambda)}{\lambda-\zeta}, \delta\ell\right) = \left(\frac{d}{d\zeta} + \frac{\nu}{\zeta}\right)\left(\frac{\lambda^{-\nu}S(x;\lambda)}{\lambda-\zeta}, \delta\ell\right) \\
&= \left(\zeta^{-\nu}\frac{d}{d\zeta}\zeta^{\nu}\right)\left(\frac{\lambda^{-\nu}S(x;\lambda)}{\lambda-\zeta}, \delta\ell\right),
\end{aligned} \tag{6.96}$$

which completes the proof. □

We note that the proof of identity (6.92) presented here, is considerably simpler than the original proof in [389]. The results obtained above can be summarized as follows:

Theorem 6.4. *Consider an operator algebra $\mathcal{B}$-valued linear periodic spectral problem in the form*

$$df/dx = \ell[u;\lambda]f, \tag{6.97}$$

where $f \in W(\mathbb{R};\mathcal{B})$, $x \in \mathbb{R}$, $\ell[u;\lambda] \in \mathfrak{g}^* \subset \mathrm{gl}(\mathrm{r};\mathcal{B}) \otimes \mathbb{C}(\lambda,\lambda^{-1})$, $\lambda \in \mathbb{C}$ is the spectral parameter, and $u \in M \subset C^{(\infty)}(\mathbb{R}/2\pi\mathbb{Z};\mathcal{B}^m)$ is the dynamical vector variable. Then all operator nonlinear dynamical systems generated by the expression (6.89) *of the form*

$$du/dt = -\vartheta \mathrm{Sp}(\ell'^*[u;\lambda]S(x;\lambda)), \tag{6.98}$$

where $S(x;\lambda)$, $\lambda \in \mathbb{C}$, is the asymptotic operator solution (6.80) *of the determining equation* (6.78) *as $|\lambda| \to \infty$, are Liouville–Lax integrable and bi-Hamiltonian on the operator manifold M. Moreover, the corresponding Lax representations for* (6.98) *are given by the expression (equivalent to* (6.84)*)*

$$\frac{d\ell}{dt} = \left[\ell - \frac{d}{dx}, \left(\frac{S(x;\lambda)}{\zeta-\lambda}\right)_+\right], \tag{6.99}$$

for all $\lambda, \zeta \in \mathbb{C}$ as $|\lambda| \to \infty$ and $|\lambda/\zeta| < 1$.

Theorem 6.4 gives rise to an effective and easily computable algorithm for the construction of the Liouville–Lax integrable dynamical systems on the operator manifold M. In addition, the algorithm can also be inverted to provide the integrability criterion on the inverse question: When is a given nonlinear dynamical system on the operator manifold M integrable in the Liouville–Lax sense?

6.2.4 *An integrable nonlinear dynamical system of Ito*

Consider on a functional manifold $M \subset C^\infty(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^2)$ the following Ito dynamical system [61], which generalizes the KdV equation:

$$\begin{aligned} u_t &= u_{xxx} + 6uu_x + 2vv_x, \\ v_t &= 2u_x v + 2v_x u, \end{aligned} \tag{6.100}$$

where $(u, v)^\intercal \in M$ and $t \in \mathbb{R}$ is an evolution parameter. To simplify the calculations, we make in (6.100) the change of variables $M \ni (u, v^2)^\intercal \to (u, v)^\intercal \in M$ that yields

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = K[u, v] := \begin{pmatrix} u_{xxx} + 6uu_x + v_x \\ 4u_x v + 2uv_x \end{pmatrix}, \tag{6.101}$$

where $K : M \to T(M)$ is a smooth vector field on M.

In accordance with the asymptotic method devised above, we find an infinite hierarchy of conservation laws for this system. Toward this end, we consider once more the linear Lax equation

$$d\varphi/dt + K'^*\varphi = 0, \tag{6.102}$$

where $\varphi \in T^*(M)$ and the operator $K'^* : T^*(M) \to T^*(M)$ has the following form:

$$K'^* = \begin{pmatrix} -\partial^3 + 6u\partial & -2v\partial - 2\partial u \\ -\partial & 2\partial u - 4u\partial \end{pmatrix}. \tag{6.103}$$

Assuming, further, that equation (6.102) admits the following asymptotic solution $\varphi \in T^*(M)$:

$$\varphi(x; \lambda) = (1, a(x; \lambda))^\intercal \exp[\lambda^3 t - \lambda x + \partial^{-1}\sigma(x; \lambda)], \tag{6.104}$$

where $\lambda \in \mathbb{C}$ is a complex parameter,

$$\partial^{-1}(.) := \frac{1}{2}\left(\int_{x_0}^{x} (\cdot)\, dx - \int_{x}^{x_0} (\cdot)\, dx\right)$$

is the operator of inverse derivation, $\partial \cdot \partial^{-1} := 1$, $x_0 \in \mathbb{R}$ is an arbitrary fixed point, and

$$a(x;\lambda) \sim \sum_{j\in\mathbb{Z}_+} a_j[u,v]\lambda^{-j},$$
$$\sigma(x;\lambda) \sim \sum_{j\in\mathbb{Z}_+} \sigma_j[u,v]\lambda^{-j} \tag{6.105}$$

are the corresponding asymptotic expansions as $|\lambda| \to \infty$. Substituting (6.105) into (6.104), from (6.102) one obtains the following infinite system of recurrence formulas:

$$\frac{d}{dt}\partial^{-1}\sigma_j - \sigma_{j,xx} - 3\sigma_{j+1,x} - 3\sigma_{j-k}\sigma_k - 3\sigma_{j+2}$$
$$-3\sigma_{j-k+1}\sigma_k - \sigma_{j-k}\sigma_{k-s}\sigma_s - 6u\sigma_{j,-1} - 6u\sigma_j$$
$$-4v(a_{j,x} + a_{j+1} + a_{j-k}\sigma_k) - 2v_x a_j = 0,$$
$$\frac{d}{dt}a - j + a_{j+3} + a_{j-k}\partial^{-1}\sigma_{k,t} - \sigma_{j,-1} - 2ua_{j+1}$$
$$-\sigma_j - 2u\sigma_{j-k}a_k + 2u_x a_j - 2ua_{j,x} = 0, \tag{6.106}$$

where $j \in \mathbb{Z}_+$, and the Einstein convention is used. Solving the hierarchy (6.106) recursively yields

$$\sigma_0 = 0, \quad \sigma_1 = -2u, \quad \sigma_2 = 2u_x,$$
$$\sigma_3 = -2u_{xx} - 2u^2 - 2v,$$
$$a_2 = 1, \quad a_0 = a_1 = a_3 = 0,$$
$$\sigma_4 = 2u_{xxx} + uu_x - 2v_x/3,$$
$$\sigma_5 = -4u^3 - 2u_{4x} - 12uu_{xx} - 10u_x^2 - 4uv, \ \dots, \tag{6.107}$$

and so on. Owing to the representation (6.104), all the functionals

$$\gamma_j := \int_0^{2\pi} dx : \sigma_j[u,v], \tag{6.108}$$

$j \in \mathbb{Z}_+$, are conservation laws of the dynamical system (6.101). Next, we find an implectic (ϑ,η)-pair of operators on the manifold M that are Nötherian for (6.101) using the small parameter method devised above. For this, we need to solve the Nötherian equation

$$d\vartheta/dt - \vartheta K'^{*} - K'\,\vartheta = 0 \tag{6.109}$$

for some $\vartheta : T^*(M) \to T^*(M)$. Putting $(u = \varepsilon u^{(1)},\, v = \varepsilon v^{(1)})^T \in T(M)$, as $\varepsilon \to 0$, it is a simple matter to use (6.109) to find the corresponding equations for the components $\vartheta_n : T^*(M) \to T(M)$, $n \in \mathbb{Z}_+$, and expressions

$$K_0' = \begin{pmatrix} \partial^3 & \partial \\ 0 & 0 \end{pmatrix}, \quad K_1' = \begin{pmatrix} 6\partial u^{(1)} & 0 \\ 2\partial v^{(1)} + 2v^{(1)}\partial & 4\partial u^{(1)} - 2u^{(1)}\partial \end{pmatrix},$$

$$K_0'^* = \begin{pmatrix} -\partial^3 & 0 \\ -\partial & 0 \end{pmatrix}, \quad K_1'^* = \begin{pmatrix} -6\partial u^{(1)} & -2\partial v^{(1)} - 2v^{(1)}\partial \\ 0 & -4\partial u^{(1)} + 2u^{(1)}\partial \end{pmatrix}. \tag{6.110}$$

Multiplying equations (6.109) by an element $\varphi^{(0)} \in T^*(M)$ satisfying

$$d\varphi^{(0)}/dt_0 = -K_0'^*\,\varphi^{(0)},$$

and taking into account that $d/dt = d/dt_0 + \varepsilon d/dt_1$, $t_0, t_1 \in \mathbb{R}$, we obtain the following recursive system of vector equations:

$$d(\vartheta_0\varphi^{(0)})/dt_0 = K_0'\,(\vartheta_0\varphi^{(0)}),$$

$$d(\vartheta_1\varphi^{(0)})/dt_0 = \vartheta_0 K'^*\,\varphi^{(0)} + K_1'(\vartheta_0\varphi^{(0)}) + K_0'(\vartheta_1\varphi^{(0)}), \ldots, \tag{6.111}$$

which can be solved by Fourier expansions by virtue of the 2π-periodicity of the original manifold M:

$$u^{(1)} = \sum_{ik\in\mathbb{Z}} \overline{u}_k^{(1)} \exp(kx + k^3 t_0) + \sum_{ik\in\mathbb{Z}} \overline{\overline{u}}_k^{(1)} \exp(kx),$$

$$v^{(1)} = -\sum_{ik\in\mathbb{Z}} \overline{u}_k^{(1)} k^2 \exp(ikx), \quad \varphi_1^{(0)} = \sum_{ik\in\mathbb{Z}} \overline{\varphi}_{1,k}^{(0)} \exp(kx + k^3 t_0),$$

$$\varphi_2^{(0)} = \sum_{ik\in\mathbb{Z}} \overline{\varphi}_{1,k}^{(0)} \exp(kx + k^3 t_0) + \sum_{ik\in\mathbb{Z}} \overline{\varphi}_{2,k}^{(0)} \exp(kx), \tag{6.112}$$

where the values $\overline{u}_k^{(1)}, \overline{\overline{u}}_k^{(1)}$, $\overline{\varphi}_{1,k}^{(0)}$ and $\overline{\varphi}_{2,k}^{(0)} \in \mathbb{C}$, $k \in i\mathbb{Z}$, are arbitrary and constant. To obtain these equations, we made use of the following trivial equality:

$$\frac{d}{dt_0}(u^{(1)}, v^{(1)})^\intercal = K_0'(u^{(1)}, v^{(1)})^\intercal \tag{6.113}$$

for $(u^{(1)}, v^{(1)})^\intercal \in T(M)$. Substituting (6.112) into (6.111) and solving by Fourier transforms, we find that

$$\vartheta_0\varphi^{(0)} = (\partial_1\varphi^{(0)}, 0)^\intercal, \quad \vartheta_n\varphi^{(0)} = 0,$$

$$\vartheta_1\varphi^{(0)} = (0, 2v^{(1)}\partial\varphi_2^{(0)} + 2\partial v^{(1)}\varphi_2^{(0)})^\intercal \tag{6.114}$$

for all $n-2 \in \mathbb{Z}_+$. Thus, from (6.114) one can easily extract the first Nötherian operator $\vartheta : T^*(M) \to T^*(M)$, namely

$$\vartheta = \begin{pmatrix} \partial & 0 \\ 0 & 2v\partial + 2\partial v \end{pmatrix}. \tag{6.115}$$

It is easy to verify that the Nötherian operator (6.115) is implectic on M. To find the second Nötherian operator $\eta : T^*(M) \to T(M)$ for the dynamical system (6.101), we will use the differential geometric conservation law approach developed in the preceding chapter. Before doing this, we notice that the functional

$$\gamma = 2\gamma_5 = \int_0^{2\pi} dx : (u_x^2 - 2u^3 - 2uv) \tag{6.116}$$

is a conservation law of the dynamical system (6.101), that $d\gamma/dt = 0$ for any $t \in \mathbb{R}$. Owing to the representation

$$\gamma = (\sigma, (u_x, v_x)^\intercal),$$

$\sigma = (\sigma_1, \sigma_2)^\intercal \in T^*(M)$, where $\sigma_1 = u_x + 2\partial^{-1}u^2$, $\sigma_2 = 2\partial^{-1}u$, we find that there exists a co-implectic operator $(\vartheta^{(1)})^{-1}$ for the vector field d/dx on M such that

$$\vartheta_1^{-1} = \begin{pmatrix} \sigma'_{1,u} - \sigma'^{*}_{1,u} & \sigma'_{1,v} - \sigma'^{*}_{1,v} \\ \sigma'_{2,u} - \sigma'^{*}_{2,u} & \sigma'_{2,v} - \sigma'^{*}_{2,v} \end{pmatrix}$$

$$= \begin{pmatrix} 2\partial + 4(\partial^{-1}u + u\partial^{-1}) & 2\partial^{-1} \\ 2\partial^{-1} & 0 \end{pmatrix}. \tag{6.117}$$

Now with this co-implectic operator (6.117) together with the implectic operator (6.115), we compute that

$$\eta := \vartheta(\vartheta^{(1)})^{-1}\vartheta = \begin{pmatrix} \partial^3 + 2(u\partial + \partial u) & 2(v\partial + \partial v) \\ 2(v\partial + \partial v) & 0 \end{pmatrix}. \tag{6.118}$$

Straightforward calculations show that operator (6.118) is both Nötherian for (6.101) and implectic on M. From the construction it also follows that the implectic (ϑ, η)-pair obtained above is compatible.

We now proceed to find a Lax representation in the form (5.3), which obviously must exist for system (6.101). For this, we make use of the differential algebraic approach. The gauge condition $\operatorname{tr} l[u, v; \lambda] = 0$, where

$$df/dx = l[u, v; \lambda]f, \tag{6.119}$$

$f \in \mathcal{H}$, is the corresponding first order derivative Lax representation for (6.101) with a matrix $l[u, v; \lambda] \in \mathrm{sl}(p; \mathbb{C})$, gives rise to the following equation in the dimension $p \in \mathbb{Z}_+$: $p^2 - 1 \leq 2 + \mu$. Since the vector dimension of the corresponding space $C^\infty(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{C}^\nu)$, augmenting the cotangent vector space $T^*(M)$ when representing (6.119) in pure differential first order form, equals $\nu = 2$, one finds that $p = 2$; that is, $l[u, v; \lambda] \in \mathrm{sl}(2; \mathbb{C})$, $(u, v)^\intercal \in M$, $\lambda \in \mathbb{C}$. Thus, calculating the corresponding matrix expression (5.67) using equations (5.7), the (ϑ, η)-pairs (6.115) and (6.118) obtained above, yields the following matrix equations:

$$
\begin{aligned}
\lambda^2[l'_u, l] + \lambda^2 l'_{u,x} &= l'_{u,xxx} + [l'_{u,xx}] \\
&+\frac{d}{dx}[l'_{u,x}, l] + [[l'_u, l], l] + \frac{d}{dx}[l'_u, l] \\
&+\frac{d}{dx}[[l'_u, l], l] + [[[l'_u, l], l], l] + 4ul'_{u,x} \\
&+4u[l'_u, l] + 2u_x l'_u + 2v_x l'_v + 4vl'_{v,x} + 4v[l'_v, l], \\
4\lambda^2 l'_{v,x} + 4\lambda^2 v[l'_v, l] + 2\lambda^2 v_x l'_v & \\
&= 2v_x l'_u + 4vl'_{u,x} + 4v[l'_u, l],
\end{aligned} \tag{6.120}
$$

where we have used the fact that $l[u, v; \lambda] = l(u, v; \lambda)$ and $\delta(\lambda) = \lambda^2$, $\lambda \in \mathbb{C}$. Since the matrix equations (6.120) are valid for any $(u, v)^\intercal \in M$, we have the following two additional relationships:

$$
l'_{u,x} = l'_{v,x} = 0, \quad \lambda^2 l'_v = l'_u. \tag{6.121}
$$

Solving equations (6.120) and (6.121) employing the matrix Lie algebra $sl(2; \mathbb{C})$ representations, we obtain the matrix

$$
l[u, v; \lambda] = \begin{pmatrix} 0 & 1 \\ \lambda^2/4 - u - \lambda^{-2} v & 0 \end{pmatrix}, \tag{6.122}
$$

inducing the Lax representation (6.119) with the spectral parameter $\lambda \in \mathbb{C}$. It is also evident that operator (6.122) reduces to the well-known [2, 262, 278, 406] Sturm–Liouville equation. The first implectic operator (6.115), found above via the small parameter approach applied to the Nötherian equation (6.109), can also be obtained using the differential algebraic method devised above. Namely, it follows from (6.106) and (6.107) that the functional

$$
\gamma = \int_0^{2\pi} dx(u^2/2 + v/2) \tag{6.123}
$$

is a conservation law of the dynamical system (6.101). Therefore, representing functional (6.123) as follows:

$$\begin{aligned}\gamma &= (u/2, u) + (\sqrt{v}/2, \sqrt{v}) \\ &= (u/2, \partial^{-1}u_x) + (\sqrt{v}/2, \partial^{-1}(\sqrt{v})_x) \\ &= (-\partial^{-1}u/2, u_x) - (\partial^{-1}\sqrt{v}, \sqrt{v}v_x/4) \\ &= (\partial^{-1}u/2, u_x) - (\sqrt{v}\partial^{-1}\sqrt{v}/4),\end{aligned} \tag{6.124}$$

from (6.124) we obtain that

$$\gamma = (\sigma, (u_x, v_x)^\intercal),$$

$$\sigma = (\sigma_1, \sigma_2)^\intercal := (-\partial^{-1}u/2, -\sqrt{v}\partial^{-1}\sqrt{v}/4)^\intercal.$$

Whence

$$(\vartheta^{(1)})^{-1} := \sigma' - \sigma'^*$$

$$= \begin{pmatrix} \sigma'_{1,u} - \sigma'^*_{1,u} & \sigma'_{1,v} - \sigma'_{2,v} \\ \sigma'_{2,u} - \sigma'_{2,v} & \sigma'_{2,v} - \sigma'_{2,v} \end{pmatrix} = \begin{pmatrix} -\partial^{-1} & 0 \\ 0 & -\frac{1}{4\sqrt{v}}\partial^{-1}\frac{1}{\sqrt{v}} \end{pmatrix}. \tag{6.125}$$

Using the inverse of operator matrix (6.125), one can easily show that $-\vartheta^{(1)} = \vartheta$ is the operator determined by expression (6.115).

6.2.5 *The Benney–Kaup dynamical system*

Another interesting nonlinear dynamical system named after Benney and Kaup is

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = K[u, v] := \begin{pmatrix} uu_x + v_x - u_{xx}/2 \\ (uv)_x - v_{xx}/2 \end{pmatrix}, \tag{6.126}$$

defined on a functional manifold $M \subset C^\infty(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{R}^2)$. In order to prove that it is Lax integrable, we shall make use of the gradient-holonomic algorithm of Chapter 5. First, we prove the existence of an infinite hierarchy of nontrivial functionally independent conservation laws. The Lax equation

$$L_K\varphi = d\varphi/d\tau + K'^*\varphi = 0 \tag{6.127}$$

for an element $\varphi(\lambda) \in T^*(M)$, $\lambda \in \mathbb{C}$, is solved by the asymptotic method to obtain

$$\varphi(x; \lambda) \sim (1, a(x; \lambda))^\intercal \exp[\lambda^2 t/2 + \lambda x + \partial^{-1}\sigma(x; \lambda)],$$

$$a(x;\lambda) \sim \sum_{j\in\mathbb{Z}_+} a_j[u,v]\lambda^{-j},$$
$$\sigma(x;\lambda) \sim \sum_{j\in\mathbb{Z}_+} \sigma_j[u,v]\lambda^{-j}, \tag{6.128}$$

where $a_j, \sigma_k : M \to \mathbb{C}$, $j,k \in \mathbb{Z}_+$, are local functionals on the manifold M, which can be determined by means of substitution of (6.128) into (6.127) and equating terms of like power of the parameter. This yields

$$\partial^{-1}\sigma_{j,t} = u\sigma_j + u\sigma_{j,-1} + \sigma_{j,x}/2 + \sigma_{j-k}\sigma_k/2$$
$$+v\sigma_{j+1} + va_{j,x} + va_{j+1} + va_{j-k}\sigma_k,$$
$$a_{j,t} + a_{j-k}\partial^{-1}\sigma_{k,t} + a_{j+2}/2 = \sigma_{j,-1} + \sigma_j + ua_{j,x}$$
$$ua_{j+1} + ua_{j-k}\sigma_k - a_{j,xx}/2 - a_{j+1,x} - a_{j-k}\sigma_k$$
$$-a_{j-k}\sigma_{k-s}\sigma_s/2 - a_{j-k+1}\sigma_k, \tag{6.129}$$

where $2+j \in \mathbb{Z}_+$. Solving the equations (6.129) recursively, one finds that

$$\sigma_0 = -u, \quad \sigma_1 = -3v + u_x,$$
$$\sigma_2 = -9uv + 2u + 4v_x - u_{xx} + uu_x, \ \ldots,$$
$$a_0 = 0, \quad a_1 = 2, \quad a_2 = 6u, \ \ldots, \tag{6.130}$$

and so on. Thus, the functionals

$$\gamma := \int_0^{2\pi} dx : \sigma_j[u,v],$$

$j \in \mathbb{Z}_+$, are conservation laws of the dynamical system (6.126); in particular,

$$\gamma_0 = -\int_0^{2\pi} dx : u, \quad \gamma_1 = -3\int_0^{2\pi} dx : v,$$
$$\gamma_2 = -9\int_0^{2\pi} dx : uv - 2\gamma_0, \tag{6.131}$$

and so on. Applying to the first terms of series (6.131) the differential algebraic approach developed above, one readily shows that the dynamical system (6.126) is endowed with the following compatible implectic (ϑ,η)-pair of Nötherian operators:

$$\vartheta = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}, \quad \eta = \begin{pmatrix} 2\partial & \partial u - \partial^2 \\ u\partial + \partial^2 & v\partial + \partial v \end{pmatrix}. \tag{6.132}$$

Expressions (6.132) can also be obtained using the small parameter method. Now, constructing the Lax representation as before, we find that

$$L = \partial/\partial x - l[u, v; \lambda],$$

where

$$l[u, v, \lambda] = \begin{pmatrix} (\lambda - u)/2 & -v \end{pmatrix}, \; dL/dt = [L, p(l)],$$

$$p(l) = \begin{pmatrix} (u_x + \lambda^2 - u^2)/4 & -v(u+\lambda)/2 - v_x/2 \\ (u+\lambda)/2 & (u^2 - \lambda^2 - u_x)/4 \end{pmatrix}. \tag{6.133}$$

Consequently, the dynamical system (6.129) on M is a Lax integrable bi-Hamiltonian flow, having an infinite hierarchy of functionally independent conservation laws (6.131) in involution. It is evident that, owing to the compatibility of the implectic (ϑ, η)-pair (6.132), all operators $\vartheta^{(n)} := \eta\Lambda^n$, where $n \in \mathbb{Z}$, $\Lambda := \vartheta^{-1}\eta$, are also implectic and Nötherian for (6.126), and satisfy the following Hamiltonian equations for all $n \in \mathbb{Z}$:

$$(u_t, v_t)^\intercal = -\vartheta^{(n)}\mathrm{grad}H_n := \frac{1}{9}\Lambda^n(\mathrm{grad}\gamma_2 + 2\mathrm{grad}H_0). \tag{6.134}$$

In particular, from (6.134) one can easily show that

$$K[u, v] = -\vartheta^{(-1)}\mathrm{grad}H_{-1} = -\vartheta^{(0)}\mathrm{grad}H_0 = -\vartheta^{(1)}\mathrm{grad}H_1, \tag{6.135}$$

where $\vartheta^{(0)} = \eta$, and $\vartheta^{(1)} = \vartheta$, $\vartheta^{(-1)} = \eta\vartheta^{-1}\eta : T^*(M) \to T(M)$. From (6.132) one easily shows that

$$\vartheta^{(-1)} = \begin{pmatrix} 2(u\partial + \partial u) & 2(v\partial + \partial v) + \nu(u - \partial)^2 \\ 2(v\partial + \partial v) + (u + \partial)^2\partial & (u + \partial)(v\partial + \partial v) + (v\partial + \partial v)(u - \partial) \end{pmatrix} \tag{6.136}$$

is the third implectic and Nötherian local operator for the dynamical system (6.126) in the (u, v)-variables of the functional manifold $M \simeq J_{top}(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{R}^2)$. Thus, the dynamical system (6.126) possesses three locally defined operators that are Nötherian and compatible.

6.2.6 *Integrability analysis of the inverse Korteweg–de Vries equation (inv KdV)*

There have been numerous studies of the KdV nonlinear dynamical system [2, 54, 173, 262, 278, 406]

$$du/dt = uu_x + u_{xxx}, \tag{6.137}$$

where $t \in \mathbb{R}$ is the evolution parameter, and $u \in M \subset C^{(\infty)}(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R})$ is an element of an infinite-dimensional 2π-periodic functional manifold M. For example, the complete Liouville integrability of system (6.137) was established some time ago. Investigation of the complete integrability of the inverse Korteweg–de Vries (invKdV) equation, derived from equation (6.137) by means of the inversion map $\mathbb{R} \ni x \leftrightarrows t \in \mathbb{R}$, is of particular interest from the analytical point of view. This invKdV can be represented as

$$u_t = v, \qquad v_t = p, \qquad p_t = u_x + uv,$$

or

$$w_t := \frac{d}{dt}\begin{pmatrix} u \\ p \\ v \end{pmatrix} = K[w] = K[u,p,v] := \begin{pmatrix} v \\ u_x + uv \\ p \end{pmatrix}, \tag{6.138}$$

where $K : M \to T(M)$ is a Fréchet smooth polynomial vector field given on the infinite-dimensional functional manifold M. Applying the gradient-holonomic algorithm to the nonlinear dynamical system (6.138), we shall show that the invKdV dynamical system (6.138) has the standard Lax representation on the manifold M and is a completely integrable Hamiltonian flow. First of all, let us study the question of existence of an infinite hierarchy of conservation laws for the dynamical system (6.137). For this purpose, we consider the Lax equation

$$\varphi_t + K'^{*}\varphi = 0, \tag{6.139}$$

where $\varphi \in T^*(M)$, the prime denotes, as usual, the Fréchet derivative of the nonlinear local functional $K[u,p,v]$, * is conjugation with respect to the bilinear form on $T^*(M) \times T(M)$, and

$$K'^{*} = \begin{pmatrix} 0 & v-\partial & 0 \\ 0 & 0 & 1 \\ 1 & u & 0 \end{pmatrix}, \quad \partial = \frac{\partial}{\partial x}.$$

Equation (6.139) is assumed to have an asymptotic solution of the form

$$\varphi(x,t;\lambda) = (1, b(x,t;\lambda), c(x,t;\lambda))^{\intercal} \exp[\omega(\lambda)t + \partial^{-1}\sigma(x,t;\lambda)], \tag{6.140}$$

where $\lambda \in \mathbb{C}$ is the parameter, $\intercal$ denotes the transpose, $\omega : \mathbb{C} \to \mathbb{C}$ is the temporal dispersion functions, which takes into account the dependence of the vector $\varphi \in T^*(M)$ on the complex parameter $\lambda \in \mathbb{C}$, $\partial^{-1}(\cdot) = \frac{1}{2}\left[\int_{x_0}^{x}(\cdot)ds - \int_{x}^{x_0+2\pi}(\cdot)ds\right]$ is the operator of the inverse differentiation, $\partial \cdot$

$\partial^{-1} = 1$, and $x_0 \in \mathbb{R}$ is a fixed arbitrary point. The solution (6.140) can be found as $|\lambda| \to \infty$ by from the expansions

$$b(x,t;\lambda) \sim \sum_{j\in\mathbb{Z}_+} b_j[u,p,v]\lambda^{-j},$$

$$c(x,t;\lambda) \sim \sum_{j\in\mathbb{Z}_+} c_j[u,p,v]\lambda^{-j}, \tag{6.141}$$

$$\sigma(x,t;\lambda) \sim \sum_{j\in\mathbb{Z}_+} \sigma_j[u,p,v]\lambda^{-j}.$$

Substituting expression (6.140) into the Lax equation (6.139) and taking into consideration that $|\lambda| \to \infty$, we find

$$\varphi(x,t;\lambda) = (1, b(x,t;\lambda), c(x,t;\lambda))^{\intercal} \exp[\lambda^3 x + \lambda t + \partial^{-1}\sigma(x,t;\lambda)], \tag{6.142}$$

and an infinite system of recurrence relations:

$$\delta_{j,-1} + \partial^{-1}(\partial\sigma_j/\partial t) + vb_j - \partial b_j/\partial x - b_{j+3} - \sum_k b_{j-k}\sigma_k = 0,$$

$$\partial b_j/\partial t + \sum_k (\partial b_k/\partial x) b_{j-k} + \sum_k b_{j+3-k}b_k + \sum_{k,s} b_{j-k}b_{k-s}\sigma_s$$

$$-v\sum_k b_{j-k}b_k + c_j = 0, \quad k,s \in \mathbb{Z}_+, \tag{6.143}$$

$$\partial c_j/\partial t + \sum_k (\partial b_k/\partial x) c_{j-k} + \sum_k c_{j+3-k}b_k + \delta_{j,0}$$

$$+ub_j + \sum_{k,s} c_{j-k}b_{k-s}\sigma_s - v\sum_k c_{j-k}b_k = 0$$

for all $j \in \mathbb{Z}_+$, from which it is possible to define the local functionals $\sigma_j : M \to \mathbb{R}$, $j \in \mathbb{Z}_+$, the first few of which are

$$\sigma_1 = \frac{p}{3} - \frac{u^2}{6}, \qquad b_1 = 0, \qquad c_1 = -1,$$

$$\sigma_2 = \frac{2}{3}u_x, \qquad b_2 = 1, \qquad c_2 = 0,$$

$$\sigma_3 = \frac{v^2}{18} - \frac{up}{9} + \frac{u^3}{27}, \qquad b_3 = 0, \qquad c_3 = -\frac{2}{3}u, \tag{6.144}$$

$$\sigma_4 = \frac{1}{9}uu_x, \quad b_4 = \frac{u}{3}, \qquad c_4 = 0,$$

$$\sigma_5 = \frac{1}{18}(u^2p - p^2 - \frac{1}{3}u^4 - uu_x + u_xv), \qquad b_5 = \frac{v}{5}, \qquad c_5 = -\frac{1}{6}u^2, \ldots,$$

and so on. From the representation (6.141), we readily deduce that all of the functionals

$$\gamma_j = \partial_{2\pi}^{-1}(\sigma_j[u,p,v]), \tag{6.145}$$

$j \in \mathbb{Z}_+$, where $\partial_{2\pi}^{-1}$ is the integral over the 2π-period, are conservation laws for dynamical system (6.138), and are functionally independent by virtue of their construction. Using equalities (6.144) and (6.145), we calculate the explicit expressions for grad $\gamma_j \in T^*(M)$, $j \in \mathbb{Z}_+$:

$$\text{grad}\gamma_0 = \text{grad}\gamma_2 = \text{grad}\gamma_4 = (0,0,0)^\intercal,$$

$$\text{grad}\gamma_1 = \left(-\frac{u}{3}, \frac{1}{3}, 0\right)^\intercal, \qquad \text{grad}\gamma_3 = \frac{1}{9}(u^2 - p, -u, v)^\intercal, \tag{6.146}$$

$$\text{grad}\gamma_5 = \frac{1}{9}\left(up - u^3 - v_x, \frac{1}{2}u^2 - p, u_x\right)^\intercal,$$

and so on. We now show that the dynamical system (6.138) is bi-Hamiltonian, that is,

$$w_t = K[w] = -\vartheta\text{grad}H = -\eta\text{grad}\bar{H}, \tag{6.147}$$

where ϑ, η are implectic operators, and $H, \bar{H}$ are the corresponding Hamiltonian functionals on M for the dynamical system (6.138). To find the implectic operators ϑ, η in explicit form, we make use of the approach devised above. We assume that $w = (u,p,v)^\intercal \in M$ is a point of first order with respect to a small parameter $\varepsilon > 0$: $u = \varepsilon u_1$, $p = \varepsilon p_1$, $v = \varepsilon v_1$. Then, representing ϑ and K in the expansions

$$\vartheta = \vartheta_0 + \varepsilon\vartheta_1 + \varepsilon^2\vartheta_2 + \ldots, \quad K = \varepsilon K^{(1)} + \varepsilon^2 K^{(2)} + \ldots,$$

where $K^{(j+1)'} = K_j'$, $\frac{d}{dt} = \frac{d}{dt_0} + \varepsilon\frac{d}{dt_1} + \ldots$, and substituting these expressions into the corresponding Nötherian equation and comparing the coefficients having the same orders of the small parameter, we find that

$$\vartheta_0 K_0'^* + K_0'\vartheta_0 = 0, \quad d\vartheta_1/dt_0 = \vartheta_0 K_1'^* + \vartheta_1 K_0'^* + K'\vartheta_1 + K_1'\vartheta_0,$$

$$d\vartheta_2/dt_0 = \vartheta_0 K_2'^* + \vartheta_1 K_1'^* + \vartheta_2 K_0'^* + K_0'\vartheta_2 + K_1'^*\vartheta_1 + K_2'\vartheta_0 - \vartheta_1 K^{(2)}, \ldots, \tag{6.148}$$

where

$$K_0' = \begin{pmatrix} 0 & 0 & 1 \\ \partial & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad K'^* = \begin{pmatrix} 0 & -\partial & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$K_1' = \begin{pmatrix} 0 & 0 & 0 \\ v_1 & 0 & u_1 \end{pmatrix}, \quad K_1'^* = \begin{pmatrix} 0 & v_1 & 0 \\ 0 & 0 & 0 \\ 0 & u_1 & 0 \end{pmatrix}. \tag{6.149}$$

Multiplying equations (6.148) by the vector $\varphi^{(0)} \in T^*(M)$ satisfying the equation following from (6.139)

$$d\varphi^{(0)}/dt_0 + K_0'^* \varphi^{(0)} = 0, \tag{6.150}$$

we obtain

$$d(\vartheta_0 \varphi^{(0)})/dt_0 = K_0'(\vartheta_0 \varphi^{(0)}),$$

$$d(\vartheta_1 \varphi^{(0)})/dt_0 - K_0'(\vartheta_1 \varphi^{(0)}) = \vartheta_0(K'^* \varphi^{(0)}) + K_1'(\vartheta_0 \varphi^{(0)}), \tag{6.151}$$

and so on. Owing to the 2π-periodicity of the manifold M, we can use Fourier series to find the solution to equations (6.151). We write the functions $w_1 \in T(M)$, $\varphi^{(0)} \in T^*(M)$ in the form

$$u_1 = \sum_{k(\pm)} \overline{u}_{1,k(\pm)} \Psi_{k(\pm)} + \sum_{k(0)} \overline{u}_{1,k(0)} \Psi_{k(0)},$$

$$p_1 = \sum_{k(\pm)} \beta_\pm^2 k^2(\pm) \overline{u}_{1,k(\pm)} \Psi_{k(\pm)} + \sum_{k(0)} k^2(0) \overline{u}_{1,k(0)} \Psi_{k(0)},$$

$$v_1 = \sum_{k(\pm)} \beta_\pm k(\pm) \overline{u}_{1,k(\pm)} \Psi_{k(\pm)} + \sum_{k(0)} k(0) \overline{u}_{1,k(0)} \Psi_{k(0)}, \tag{6.152}$$

$$\varphi_1^{(0)} = -\sum_{k(\pm)} \beta_\pm k(\pm) \overline{\varphi}_{3,k(\pm)}^{(0)} \Psi_{k(\pm)} - \sum_{k(0)} k(0) \overline{\varphi}_{3,k(0)}^{(0)} \Psi_{k(0)},$$

$$\varphi_2^{(0)} = -\sum_{k(\pm)} (\beta_\pm k(\pm))^{-1} \overline{\varphi}_{3,k(\pm)}^{(0)} \Psi_{k(\pm)} - \sum_{k(0)} (k(0))^{-1} \overline{\varphi}_{3,k(0)}^{(0)} \Psi_{k(0)},$$

$$\varphi_3^{(0)} = \sum_{k(\pm)} \overline{\varphi}_{3,k(t)}^{(0)} \Psi_{k(t)} - \sum_{k(0)} k(0) \overline{\varphi}_{3,k(0)}^{(0)} \Psi_{k(0)},$$

where

$$\beta_\pm = (-1 \pm i\sqrt{3})/2, \;\; \Psi_{k(\pm)} = \exp(k^3 x + \beta_\pm k t), \;\; \Psi_{k(0)} = \exp(k^3 x + k t),$$

the numbers $\overline{u}_{1,k(\pm)}$, $\overline{\varphi}^{(0)}_{3,k(\pm)}$, $\overline{u}_{1,k(0)}$, $\overline{\varphi}_{3,k(0)} \in \mathbb{C}$ are arbitrary constants, $k \in i\mathbb{Z}$, and the equalities $d\varphi^{(0)}/dt_0 = -K_0'^{*}\varphi^{(0)}$, $dw_1/dt_0 = K_0' w_1$ are used. Substituting expansions (6.152) into equations (6.151) and using the relationship $\vartheta_0\varphi^{(0)} = a^{(0)}$, where $\varphi^{(0)} = (\varphi_1^{(0)}, \varphi_2^{(0)}, \varphi_3^{(0)})^{\intercal}$, $a^{(0)} = (a_1^{(0)}, a_2^{(0)}, a_3^{(0)})^{\intercal} := (u_1, p_1, v_1)^{\intercal}$ have form (6.151), $\vartheta_0 = \{\vartheta_0^{(ij)} : i, j = 1, 2, 3\}$, we find by straightforward calculations that the linear space of solutions for the operator $\vartheta_0 : T^*(M) \to T(M)$ is three-dimensional with respect to each differential order of its degree. At this point we must mention that in this space there are only two linearly independent basic elements which generate the implectic pair of Nötherian operators for the dynamical system (6.138) under consideration and, as a result, we obtain one of the possible seed operators in the form

$$\vartheta_0 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & \partial & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{6.153}$$

To find the operator $\vartheta_1 = \{\vartheta_1^{(ij)} : i, j = 1, 2, 3\}$, it is necessary to solve the linear differential equations (6.151). Denoting $\vartheta_1\varphi^{(0)} := \alpha = (\alpha_1, \alpha_2, \alpha_3)^{\intercal}$ and using the explicit form of the operators $\vartheta_0, K_1', K_1'^{*}$, we obtain the system of equations

$$\begin{aligned} d\alpha_1/dt_0 - \alpha_3 &= -u_1\varphi_2^{(0)} := g_1, \\ d\alpha_2/dt_0 - (\alpha_1)_x &= -v_1\varphi_3^{(0)} + u_1\varphi_1^{(0)} = g_2, \\ d\alpha_3/dt_0 - \alpha_2 &= v_1\varphi_2^{(0)} = g_3, \end{aligned} \tag{6.154}$$

from which we find the equation

$$\frac{d^3}{dt_0^3}\alpha_1 - (\alpha_1)_x = \frac{d^2}{dt_0^2}g_1 + \frac{d}{dt_0}g_3 + g_2,$$

whose solution allows us to establish that

$$\vartheta_1^{(11)}\varphi_1^{(0)} = 0, \quad \vartheta_1^{(12)}\varphi_2^{(0)} = 0, \quad \vartheta_1^{(13)}\varphi_3^{(0)} = 0. \tag{6.155}$$

Similarly, from system (6.154) we find the equations

$$\frac{d^3}{dt_0^3}\alpha_2 - (\alpha_2)_x = \frac{d^2}{dt_0^2}g_2 + \frac{d}{dt_0}g_{1,x} + g_3,$$

$$\frac{d^3}{dt_0^3}\alpha_3 - (\alpha_3)_x = \frac{d^2}{dt_0^2}g_3 + \frac{d}{dt_0}g_2 + g_{1,x},$$

from which we obtain

$$\vartheta_1^{(21)}\varphi_1^{(0)} = 0, \quad \vartheta_1^{(22)}\varphi_2^{(0)} = 0, \qquad \vartheta_1^{(23)}\varphi_2^{(0)} = -u_1\varphi_3^{(0)},$$

$$\vartheta_1^{(31)}\varphi_1^{(0)} = 0, \qquad \vartheta_1^{(32)}\varphi_2^{(0)} = u_1\varphi_2^{(0)}, \qquad \vartheta_1^{(33)}\varphi_3^{(0)} = 0. \tag{6.156}$$

From (6.155) together with (6.156) we compute that

$$\vartheta_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -u_1 \end{pmatrix}.$$

The operator ϑ_2 is defined from the equation

$$d(\vartheta_2\varphi^{(0)})/dt_0 = \vartheta_1 K_1'^{*}\varphi^{(0)} + K_1'(\vartheta_1\varphi^{(0)}) - (\vartheta_1' K^{(2)})\varphi^{(0)}, \tag{6.157}$$

where $K^{(2)} = (0, u_1v_1, 0)^{\intercal}$. Calculating the right-hand side of equation (6.157), we find that $\alpha_2 = 0$. Since $\alpha_3 = \alpha_4 = ... = 0$, we obtain the operator $\vartheta : T^*(M) \to T(M)$ in the form

$$\vartheta = \begin{pmatrix} 0 & 0 & -1 \\ 0 & \partial & -u \\ 1 & u & 0 \end{pmatrix}. \tag{6.158}$$

As the Nötherian condition is satisfied identically, the operator ϑ is Nötherian and implectic. Similarly, the second implectic operator is found to be

$$\eta = \frac{1}{3}\begin{pmatrix} 0 & 3\partial - v & -u \\ 3\partial + v & 2(\partial u + u\partial) & p - u^2 \\ u & u^2 - p & -3\partial \end{pmatrix}, \tag{6.159}$$

since the seed operator

$$\eta_0 = \begin{pmatrix} 0 & \partial & 0 \\ \partial & 0 & 0 \\ 0 & 0 & -\partial \end{pmatrix} \tag{6.160}$$

satisfies the first equation of system (6.151). If the operator

$$\tilde{\eta}_0 = \begin{pmatrix} \partial & 0 & 0 \\ 0 & 0 & -\partial^2 \\ 0 & \partial^2 & 0 \end{pmatrix} \tag{6.161}$$

is taken as a solution to equation (6.151), then after some simple calculations we find one more of the feasible implectic operators in the form

$$\tilde{\eta} = \tilde{\eta}_0 + \begin{pmatrix} 0 & 6\partial u & 3p - u^2 \\ 6u\partial & (\partial u^2 + u^2\partial) + 4u\partial u - 3(\partial p + p\partial) & 3up - u^3 - 3v\partial \\ u^2 - 3p & u^3 - 3up - 3\partial v & -3(u\partial + \partial u) \end{pmatrix}$$

$$+\begin{pmatrix} -v\partial^{-1}v & v\partial^{-1}u\partial - v\partial^{-1}uv & -v\partial^{-1}p \\ -uv\partial^{-1}v - \partial u\partial^{-1}v & (\partial u\partial^{-1}u\partial - \partial u\partial^{-1}uv + uv\partial^{-1}u\partial - uv\partial^{-1}uv) & -\partial u\partial^{-1}p \\ -p\partial^{-1}v & p\partial^{-1}u\partial - p\partial^{-1}uv & -uv\partial^{-1}p - p\partial^{-1}p \end{pmatrix}, \tag{6.162}$$

which can be represented by means of the formula $\tilde{\eta} = 9\ \eta\vartheta^{-1}\eta$, where the operator $\eta : T^*(M) \to T(M)$ is given by expression (6.159), and the operator

$$\vartheta^{-1} = \begin{pmatrix} u\partial^{-1}u & -u\partial^{-1} & 1 \\ -\partial^{-1}u & \partial^{-1} & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

is the inverse of the implectic operator $\vartheta : T^*(M) \to T(M)$ written in the form (6.158). Note that by application of the gradient-holonomic algorithm, using implectic operators (6.158), (6.159) and (6.162) for the dynamical system (6.138), it should be possible to obtain two different representations of the Lax type, which will be gauge invariant. Now, we represent the hereditary recursion operator Λ in the form $\Lambda = \vartheta^{-1}\eta$, where the operators ϑ and $\eta : T^*(M) \to T(M)$ satisfy the equalities

$$\vartheta\,\mathrm{grad}\,\gamma_{2j+3} = \eta\,\mathrm{grad}\,\gamma_{2j+1} \tag{6.163}$$

for all $j \in \mathbb{Z}$. As there exists the (ϑ, η)-pair (6.158), (6.159), all of the operators $\vartheta^{(n)} = \vartheta\Lambda^n$, $n \in \mathbb{Z}$, are implectic and Nötherian for the dynamical system (6.138), and the recursion operator $\Lambda : T^*(M) \to T^*(M)$ satisfying the Lax equation $d\Lambda/dt = [\Lambda, K'^*]$ is

$$\Lambda = \frac{1}{3}\begin{pmatrix} -2u - u\partial^{-1}v & (u\partial^{-1}u\partial - u\partial^{-1}uv - u^2 - p) & -u\partial^{-1}p - 3\partial \\ 3 + \partial^{-1}v & -\partial^{-1}u\partial + \partial^{-1}uv + 2u & -\partial^{-1}p \\ 0 & -3\partial + v & u \end{pmatrix}. \tag{6.164}$$

Therefore, the dynamical system (6.138) possesses an infinite hierarchy of functionally independent that are involutive with respect to the Poisson brackets $\{\cdot,\cdot\}_\vartheta$, $\{\cdot,\cdot\}_\eta$ integrals $\gamma_{2j+1} \in \mathcal{D}(M)$, $j \in \mathbb{Z}$, for which $\mathrm{grad}\gamma_{2j+1} = \Lambda^j\mathrm{grad}\gamma_1$, and $\{H, \gamma_j\}_\vartheta = 0 = \{\bar{H}, \gamma_j\}_\eta$, where

$$H = 9\gamma_3 = \partial_{2\pi}^{-1}\left(-up + \frac{1}{2}v^2 + \frac{1}{3}u^3\right), \bar{H} = 3\gamma_1 = \partial_{2\pi}^{-1}\left(p - \frac{u^2}{2}\right). \tag{6.165}$$

Hence, the following theorem holds.

Theorem 6.5. *The dynamical system* (6.138) *induces a hierarchy of bi-Hamiltonian completely integrable flows on* M *which are representable in the form*

$$w_t = -\vartheta \operatorname{grad} H = -\eta \operatorname{grad} \bar{H}, \tag{6.166}$$

where $H, \bar{H} \in \mathcal{D}(M)$ *are Hamiltonian functionals* (6.165) *on the manifold* M*, and* $\vartheta, \eta : T^*(M) \to T(M)$ *are implectic operators* (6.158) *and* (6.159)*, that factorize recursion operator* Λ (6.164)*.*

We now show that the dynamical system (6.138) also possesses the standard Lax representation which allows us to integrate it by the inverse scattering transform [2, 278, 406] in explicit form. If $L = L[u, p, v; \lambda] : \mathcal{H} \to \mathcal{H}$ is the L-operator in a Lax representation for system (6.138) and $S = S(x; \lambda)$ is its monodromy matrix, $x \in \mathbb{R}$, $\lambda \in \mathbb{C}$, then obviously

$$\lambda^q \vartheta^{(n)} \varphi(\lambda) = \vartheta^{(n+1)} \varphi(\lambda) \tag{6.167}$$

is valid for the gradient $\varphi(\lambda) := \operatorname{grad} \operatorname{Sp} S(x_0; \lambda) \in T^*(M)$, where $q = 2$, and $\lambda^2 \in \mathbb{C}$ is the eigenvalue of recursion operator (6.164). The monodromy matrix $S(x; \lambda)$, $\lambda \in \mathbb{C}$, satisfies the Novikov-Marchenko equation

$$dS/dx = [l, S]. \tag{6.168}$$

Here we have represented the Lax operator $L[u; \lambda]$ in the following $(m \times m)$-matrix form, $m \in \mathbb{Z}_+$:

$$L = \frac{d}{dx} - l[u, p, v; \lambda], \quad \operatorname{Sp} l[u, p, v; \lambda] = 0, \tag{6.169}$$

where $l[u, p, v; \lambda]$ is a local Fréchet smooth matrix functional on the manifold M, depending on the spectral parameter $\lambda \in \mathbb{C}$. The order $m \in \mathbb{Z}_+$ of the matrix $l[u, p, v; \lambda]$ in equation (6.168) is defined by the formula $m^2 - 1 \leq \nu + 3$, where $\nu + 3$ is the minimal vector dimension of the matrix linear differential equation of the first order for the vector $\tilde{\varphi}(\lambda) \in T^*(M) \times C^{(\infty)}(\mathbb{R}; \mathbb{C}^\nu)$, which is equivalent to equation (6.167). The realization of these properties by means of the gradient-holonomic algorithm allows the construction of the matrix $l[u, p, v; \lambda]$, $\lambda \in \mathbb{C}$, in explicit form along with the L-operator for system (6.138). It follows directly from (6.167) and (6.168) that

$$-\lambda^2 l_v = [l_p, l] + \frac{d}{dx} l_p - \frac{1}{3} v l_p - \frac{1}{3} u l_v,$$

$$\lambda^2 \left([l_p, l] + \frac{d}{dx} l_p - u l_v \right) = [l_u, l] - \frac{d}{dx} l_u + \frac{1}{3} v l_u$$

$$+\frac{2}{3}u_x l_p + \frac{4}{3}u[l_p,l] + \frac{4}{3}u\frac{d}{dx}l + \frac{1}{3}pl_v - \frac{1}{3}u^2 l_v, \tag{6.170}$$

$$\lambda^2(l_u + ul_p) = \frac{1}{3}ul_u + \frac{1}{3}u^2 l_p - \frac{1}{3}pl_p - [l_v,l] - \frac{d}{dx}l_v.$$

In obtaining equations (6.170), we took into account the fact that owing to expressions (6.158) and (6.159) the relationship $l[u,p,v;\lambda] = l(u,p,v;\lambda)$, $\lambda \in \mathbb{C}$, holds, which gives rise to

$$\lambda^2 \vartheta \ \mathrm{Sp} : (Sl'_w) = \eta \ \mathrm{Sp} : (Sl'_w),$$

where $l'_w = (l_u, l_p, l_v)^\intercal$ is a local vector functional on M. For the order of the matrix $l(u,p,v;\lambda)$, $\lambda \in \mathbb{C}$, there is the inequality $m^2 - 1 \le 3$, giving $m = 2$. Now let us represent the matrix $l(u,p,v;\lambda)$, $\lambda \in \mathbb{C}$, as follows:

$$l(u,p,v;\lambda) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = a\sigma_3 + b\sigma_+ + c\sigma_-, \tag{6.171}$$

where the matrices $\sigma_3, \sigma_+, \sigma_- \in \mathrm{sl}(2;\mathbb{C})$ have the form

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Since equations (6.170) are valid for all $(u,p,v)^\intercal \in M$, it follows that $dl_v/dx = 0$, $dl_p/dx = 0$, which leads to the matrix equations

$$\lambda^2 l_v = [l_v,l] - \frac{1}{3}vl_p - \frac{1}{3}ul_v,$$

$$\lambda^2([l_p,l] - ul_v) = [l_u,l] - \frac{d}{dx}l_u + \frac{1}{3}vl_u$$

$$+\frac{2}{3}u_x l_p + \frac{4}{3}u[l_p,l] + \frac{1}{3}pl_v - \frac{1}{3}u^2 l_v, \tag{6.172}$$

$$\lambda^2(l_u + ul_p) = \frac{1}{3}ul_u + \frac{1}{3}u^2 l_p - \frac{1}{3}pl_p - [l_v,l].$$

Substituting (6.171) into equations (6.172), we obtain some relationships for the unknown elements of the matrix $l(u,p,v;\lambda)$, $\lambda \in \mathbb{C}$, from which one finds that:

$$a = -\frac{\lambda^3}{2} + \frac{\lambda u}{6} - \frac{v}{6}, \quad c = \frac{1}{3}u - \lambda^2,$$

$$b = \frac{1}{6}\left(-\lambda^2 u - p + \lambda v + \frac{1}{3}u^2\right).$$

As a result, the L-operator can be written in the Lax form

$$L = \frac{d}{dx} - \frac{1}{6}\begin{pmatrix} -3\lambda^3 + \lambda u - v & -\lambda^2 u - p + \lambda v + \frac{1}{3}u^2 \\ 2u - 6\lambda^2 & 3\lambda^3 - \lambda u + v \end{pmatrix}. \tag{6.173}$$

The infinite hierarchy of Lax integrable involutive flows can be associated with the L-operator (6.173) as follows:

$$\alpha_j = (\Lambda^*)^{-j}\alpha_0, \quad \alpha_0 = (u_x, p_x, v_x)^\intercal,$$

where $\alpha_j : M \to T(M)$, $j \in \mathbb{Z}$, are vector fields on M called the highest inverse Korteweg–de Vries equations. In particular, from the above one obtains

$$\alpha_{-1}[w] = \alpha_{-1}[u,p,v] := K[u,p,v] = (v, u_x + uv, p)^\intercal, \tag{6.174}$$

$$\alpha_1[w] = \alpha_1[u,p,v] = \frac{1}{3}\Big(3p_x - 2uu_x - vp + \frac{1}{2}u^2 v,$$

$$\frac{1}{2}u^2 u_x - 2pu_x + up_x + 3v_{xx} + \frac{1}{2}vu^3 - upv + vv_x,$$

$$3v_{xx} + uv_x - p^2 + \frac{1}{2}u^2 p\Big)^\intercal.$$

It is worthwhile mentioning that the nonlinear inverse dynamical system $w_t = (u,p,v)_t^\intercal = \alpha_1((u,p,v)^\intercal)$ is a new completely integrable Hamiltonian flow, whose application to hydrodynamics, plasma physics and other fields is of great interest. The following result has also been established.

Theorem 6.6. *The dynamical systems $w_t = \alpha_{-1}[w] := K[w]$ and $w_t = \alpha_1[w]$, where α_{-1} and α_1 are given by formulae* (6.175)*, have a common Lax operator in the form* (6.173) *and are completely integrable bi-Hamiltonian hydrodynamic flows, whose implectic (ϑ, η)-pair is given in* (6.158) *and* (6.159)*.*

6.2.7 *Integrability analysis of the inverse nonlinear Benney–Kaup system*

In this section, we shall analyze the dynamical system [110]

$$\begin{pmatrix} u_t \\ p_t \\ q_t \\ nu_t \end{pmatrix} = K[u,p,q,\nu] := \begin{pmatrix} p_x - \nu - pu \\ u \\ v \\ -q_x + p\nu + uq \end{pmatrix}, \tag{6.175}$$

on an infinite-dimensional 2π-periodic functional manifold $M \subset C^\infty(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^4)$. Here, $K : M \to T(M)$ is a polynomial and a fortiori Fréchet smooth vector field M. This system is obtained from the nonlinear dynamical Benney–Kaup system

$$u_t = u_{xx} + \nu_x - uu_x,$$

$$\nu_t = -\nu_{xx} + (u\nu)_x,$$

via the inversion $\mathbb{R} \ni x \leftrightarrows t \in \mathbb{R}$. We shall show that system (6.175) on M is a completely integrable Hamiltonian flow and that it admits the Lax representation which can be found in explicit form. The Lax representation will be obtained by using the gradient-holonomic algorithm. First, we prove that the system (6.175) possesses an infinite hierarchy of functionally independent conservation laws. For this purpose, we consider the Lax equation

$$d\varphi/dt + K'^{*}\varphi = 0. \tag{6.176}$$

The operators K' and K'^{*} can be written in the explicit form:

$$K' = \begin{pmatrix} -u & \partial & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \nu & 0 & -\partial & u \end{pmatrix}, \quad K'^{*} = \begin{pmatrix} -u & 1 & 0 & \nu \\ -\partial & 0 & 0 & 0 \\ 0 & 0 & 0 & \partial \\ -1 & 0 & 1 & u \end{pmatrix}.$$

Equation (6.176) admits a vector solution of the form

$$\varphi(x,t;\lambda) = \begin{pmatrix} 1 \\ a(x,t;\lambda) \\ b(x,t;\lambda) \\ c(x,t;\lambda) \end{pmatrix} \exp[\omega(t;\lambda) + \partial^{-1}\sigma(x,t;\lambda)], \tag{6.177}$$

where, $x, t \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ is a complex parameter, $\omega(t;\lambda)$ is a dispersion function and

$$\partial^{-1}(\cdot) = \frac{1}{2}\left[\int_{x_0}^{x} (\cdot) ds - \int_{x}^{x_0+l} (\cdot) ds\right]$$

represents a right inverse of differentiation. As a result, we have the following asymptotic expansions as $|\lambda| \to \infty$:

$$a(x,t;\lambda) \sim \sum_{j\geq -1} a_j[u,p,q,\nu]\lambda^{-j},$$

$$b(x,t;\lambda) \sim \sum_{j\geq 0} b_j[u,p,q,\nu]\lambda^{-j}, \tag{6.178}$$

$$c(x,t;\lambda) \sim \sum_{j\geq 1} c_j[u,p,q,\nu]\lambda^{-j},$$

$$\sigma(x,t;\lambda) \sim \sum_{j\geq 0} \sigma_j[u,p,q\nu]\lambda^{-j}.$$

The dispersion function $\omega(t;\lambda) = \omega_1(\lambda)t$, $\lambda \in \mathbb{C}$, is found as a solution of the equation

$$(1,\bar{a},\bar{b},\bar{c})^\intercal \omega_1(\lambda)\exp[\omega_1(\lambda)t + \omega_2(\lambda)x]$$

$$+ \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\partial & 0 & 0 & 0 \\ 0 & 0 & 0 & \partial \\ -1 & 0 & 1 & 0 \end{pmatrix} \exp(\omega_1(\lambda)t + \omega_2(\lambda)x) = 0. \tag{6.179}$$

Equation (6.179) follows from equation (6.176) if we set in the latter $u := p = q = \nu = 0$ and take $\varphi \in T^*(M)$ in the form $(1,\bar{a},\bar{b},\bar{c})^\intercal \exp[\omega_1(\lambda)t + \omega_2(\lambda)x]$, where $\bar{a},\bar{b}$ and $\bar{c} \in \mathbb{C}$ are constants. Hence, from equation (6.179) we obtain

$$\bar{a} = -\omega_1(\lambda), \quad \bar{b} = \frac{1}{2}, \quad \bar{c} = \frac{1}{2\omega_1(\lambda)}, \quad \omega_2(\lambda) = -\omega_1^2(\lambda).$$

Now setting $\omega_1(\lambda) = \lambda \in \mathbb{C}$, we can write the solution of (6.178) in the form

$$\varphi(x,t;\lambda) \sim \begin{pmatrix} 1 \\ a(x,t;\lambda) \\ b(x,t;\lambda) \\ c(x,t;\lambda) \end{pmatrix} \exp(\lambda t - \lambda^2 x - \partial^{-1}\sigma). \tag{6.180}$$

By using (6.180), we obtain

$$\lambda + \partial^{-1}\sigma_t - u + a + \nu c = 0,$$

$$a_t + a(u - a - \nu c) + \lambda^2 - \sigma = 0,$$

$$b_t + b(u - a + \nu c) + c_x(\sigma - \lambda^2) = 0,$$

$$c_t + c(u - a - \nu c) - 1 + b + uc = 0.$$

Owing to the asymptotic expansions (6.178), we obtain from these equations the recurrence formulas

$$\delta_{-1,j} + \partial^{-1}(\partial\sigma_j/\partial t) - u\delta_{0,j} + a_j + \nu c_j = 0,$$

$$\partial a_j/\partial t + ua_j - \sum_{k=-1}^{j+1} a_k a_{j-k} - \nu \sum_{k=-1}^{j} a_k c_{j-k} - \delta_{-2,j} - \sigma_j = 0,$$

$$\partial b_j/\partial t + ub_j - \sum_{k=-1}^{j} a_k b_{j-k} + \nu \sum_{k=0}^{j} c_k b_{j-k} + \sum_{k=0}^{j} (\partial c_k/\partial x)\sigma_{j-k} - \partial c_{j+2}/\partial x = 0,$$

$$\partial c_j/\partial t + uc_j - \sum_{k=-1}^{j} a_k c_{j-k} - \nu \sum_{k=0}^{j} c_k c_{j-k} - \delta_{0,j} + b_j + uc_j = 0,$$

where $j \in \mathbb{Z}_+$. Solving this system recursively yields

$$a_{-1} = -1, \qquad a_0 = p/2, \qquad b_0 = 1/2, \qquad c_1 = 1/2,$$

$$\sigma_0 = (u + q + p^2/2)/2, \qquad a_1 = u/2, \qquad b_1 = -p/2, \qquad c_2 = -p/4,$$

$$\sigma_1 = -(p_x + pq - \nu)/2, \qquad a_2 = (\nu + pu - 2p_x)/4,$$

$$\sigma_2 = -(2u_x + q_x + pp_x + p^2 q + uq - p\nu + q^2/2)/4,$$

$$b_2 = (u + p^2 + q/2)/4, \qquad c_3 = (2u + p^2 + q)/8,$$

$$a_3 = (q_x - pp_x + p^2 u + 2u_x - qp^2 + qu - q^2/2)/8,$$

$$\sigma_3 = -(2(pq - \nu)_x + 2\sigma_{0,x} + p^2 p_x - p^2 \nu$$

$$+2p_x q - 2q\nu + 2qpu + p^3 q + 2q^2 p - 2u\nu)/8,$$

and so on. It is easy to verify that all functionals of the form

$$\gamma_j = \int_{x_0}^{x_0+2\pi} dx \sigma_j[u, p, q, \nu], \tag{6.181}$$

$j \in \mathbb{Z}_+$, are conservation laws for dynamical system (6.175). Then, by using the formula

$$\text{grad} : \gamma_j = (\delta\gamma_j/\delta u, \delta\gamma_j/\delta p, \delta\gamma_j/\delta q, \delta\gamma_j/\delta \nu)^\intercal,$$

we can write the gradients of functionals (6.179):

$$\text{grad} : \gamma_0 = \frac{1}{2}(1, p, 1, 0)^\intercal,$$

$$\text{grad} : \gamma_1 = -\frac{1}{2}(0, q, p, -1)^\intercal,$$

$$\text{grad} : \gamma_2 = \frac{1}{4}(u, pq - \nu, p^2 + u + q, -p)^\intercal,$$

and so on. We must find an implectic and Nötherian operator $\vartheta : T^*(M) \to T(M)$ that enables us to rewrite system (6.175) in the Hamiltonian form

$$dw/dt = -\vartheta \,\text{grad}\, H,$$

with $w = (u, p, q, \nu)^\intercal$, where H is a conservation law of system (6.175). If the operator $\vartheta : T^*(M) \to T(M)$ exists, it satisfies the Nötherian condition:

$$L_K \vartheta = \vartheta' K - \vartheta K'^* - K' \vartheta = 0. \tag{6.182}$$

Whence, using the small parameter method, we can write the following asymptotic expansions for the operators ϑ, K, K', and K'^*:

$$\frac{d}{dt} = \frac{d}{dt_0} + \varepsilon \frac{d}{dt_1}, \qquad K' = K'_0 + \varepsilon K'_1,$$

$$K'^* = K'^*_0 + \varepsilon K'^*_1, \qquad \vartheta = \vartheta_0 + \varepsilon \vartheta_1 + \varepsilon^2 \vartheta_2 + \ldots,$$

where $\varepsilon > 0$ is a small parameter, and the operators K'_0, K'_1, K'^*_0 and K'^*_1 have the following form:

$$K'_0 = \begin{pmatrix} 0 & \partial & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\partial & 0 \end{pmatrix}, \quad K'_1 = \begin{pmatrix} -p & -u & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ q & \nu & u & p \end{pmatrix},$$

$$K'^*_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\partial & 0 & 0 & 0 \\ 0 & 0 & 0 & \partial \\ -1 & 0 & 1 & 0 \end{pmatrix}, \quad K'^*_1 = \begin{pmatrix} -p & 0 & 0 & q \\ -u & 0 & 0 & \nu \\ 0 & 0 & 0 & u \\ 0 & 0 & 0 & p \end{pmatrix}.$$

Substituting the above into equation (6.182), we obtain the recurrence for finding the operators such as $\vartheta_0, \vartheta_1, \vartheta_2 : T^*(M) \to T(M)$,

$$\vartheta_0 K'^*_0 + K'_0 \vartheta_0 = 0,$$

$$\vartheta_1 K'^*_1 + \vartheta_0 K'^*_1 + K'_1 \vartheta_0 + K'_0 \vartheta_1 = d\vartheta_1/dt_0, \tag{6.183}$$

$$\vartheta_2 K'^*_0 + \vartheta_1 K'^*_1 + K'_1 \vartheta_1 + K'_0 \vartheta_2 = d\vartheta_1/dt_1 + d\vartheta_2/dt_0,$$

$$\ldots \quad \cdots \quad \cdots$$

$$\vartheta_k K'^*_0 + \vartheta_{k-1} K'^*_1 + K'_1 \vartheta_{k-1} + K'_0 \vartheta_k = d\vartheta_{k-1}/dt_1 + d\vartheta_k/dt_0, \ldots,$$

and so on. If we denote the elements of the matrix ϑ_0 by $\{\vartheta_0^{(ij)} : 1 \leq i, j \leq 4\}$, the first relation of (6.183) becomes equivalent to the following system of equations:

$$\begin{gathered}
\partial\vartheta_0^{(21)} - \vartheta_0^{(41)} - \vartheta_0^{(12)}\partial - \vartheta_0^{(14)} = 0, \quad \vartheta_0^{(11)} - \vartheta_0^{(22)}\partial - \vartheta_0^{(24)} = 0, \\
\vartheta_0^{(41)} - \vartheta_0^{(32)}\partial - \vartheta_0^{(34)} = 0, \quad \partial\vartheta_0^{(31)} + \vartheta_0^{(42)}\partial + \vartheta_0^{(44)} = 0, \\
\vartheta_0^{(11)} + \partial\vartheta_0^{(22)} - \vartheta_0^{(42)} = 0, \quad \vartheta_0^{(21)} + \vartheta_0^{(12)} = 0, \quad \vartheta_0^{(42)} + \vartheta_0^{(31)} = 0, \\
\vartheta_0^{(41)} - \partial\vartheta_0^{(32)} = 0, \quad \vartheta_0^{(13)}\partial + \partial\vartheta_0^{(24)} - \vartheta_0^{(44)} = 0, \qquad (6.184) \\
\vartheta_0^{(23)}\partial + \vartheta_0^{(14)} = 0, \quad \vartheta_0^{(33)}\partial + \vartheta_0^{(44)} = 0, \\
\vartheta_0^{(43)}\partial + \partial\vartheta_0^{(34)} = 0, \quad \vartheta_0^{(14)} + \partial\vartheta_0^{(23)} - \vartheta_0^{(43)} = 0, \\
\vartheta_0^{(24)} + \vartheta_0^{(13)} = 0, \quad \vartheta_0^{(34)} + \vartheta_0^{(43)} = 0, \quad \vartheta_0^{(44)} - \partial\vartheta_0^{(33)} = 0.
\end{gathered}$$

System (6.184) has two algebraically independent solutions. They are associated with the matrices η_0 and $\vartheta_0 : T^*(M) \to T(M)$, which provide the two independent operator matrices:

$$\vartheta_0 = \begin{pmatrix} 0 & -1 & 0 & \partial \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ \partial & 0 & 0 & 0 \end{pmatrix}, \quad \eta_0 = \begin{pmatrix} \partial & 0 & -\partial & 0 \\ 0 & 0 & 0 & \partial \\ -\partial & 0 & 0 & 0 \\ 0 & \partial & 0 & 0 \end{pmatrix}.$$

Further, let us find the operator $\vartheta_1 : T^*(M) \to T(M)$ from equation (6.182). To do this, we multiply equation (6.183) by $\varphi^{(0)} \in T(M)$, $\varphi^{(0)} = (\varphi_1^{(0)}, \varphi_2^{(0)}, \varphi_3^{(0)}, \varphi_4^{(0)})^\intercal$, which satisfies the condition:

$$d\varphi^{(0)}/dt_0 + K_0'^*\varphi^{(0)} = 0.$$

This condition is equivalent to the relations

$$\begin{gathered}
\varphi_{1,t_0}^{(0)} + \varphi_2^{(0)} = 0, \qquad \varphi_{2,t_0}^{(0)} - \varphi_{1,x}^{(0)} = 0, \\
\varphi_{3,t_0}^{(0)} + \varphi_{4,x}^{(0)} = 0, \qquad \varphi_{4,t_0}^{(0)} - \varphi_1^{(0)} + \varphi_3^{(0)} = 0. \qquad (6.185)
\end{gathered}$$

Equation (6.183) multiplied by $\varphi^{(0)}$ has the form

$$d(\vartheta_1\varphi^{(0)})/dt_0 - K_0'(\vartheta_1\varphi^{(0)}) = \vartheta_0(K_1'^*\varphi^{(0)}) + K_1'(\vartheta_0\varphi^{(0)}). \qquad (6.186)$$

The terms containing the unknown operator $\vartheta_1 : T^*(M) \to T(M)$ are gathered on the left-hand side, while those with the known operator ϑ_0 :

$T^*(M) \to T(M)$ are placed to the right. Taking into account the explicit form of the operators ϑ_0, K_1' and $K_1'^*$ and introducing the notation

$$\vartheta_0(K_1'^*\varphi^{(0)}) + K_1'(\vartheta_0\varphi^{(0)}) = (g_1, g_2, g_3, g_4)^\intercal,$$

$$\vartheta_1\varphi^{(0)} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)^\intercal,$$

we can rewrite system (6.186) in the form

$$\frac{d\alpha_1}{dt_0} - (\alpha_{2,x} - \alpha_4) = g_1, \quad \frac{d\alpha_2}{dt_0} - \alpha_1 = g_2,$$

$$\frac{d\alpha_3}{dt_0} - \alpha_4 = g_3, \quad \frac{d\alpha_4}{dt_0} + \alpha_{3,x} = g_4, \tag{6.187}$$

where

$$g_1 = u\varphi_3^{(0)} - \nu\varphi_4^{(0)} + p_x\varphi_4^{(0)} + p\varphi_2^{(0)},$$

$$g_2 = q\varphi_4^{(0)} - p\varphi_1^{(0)} - u\varphi_4^{(0)},$$

$$g_3 = \nu\varphi_4^{(0)} - u\varphi_1^{(0)},$$

$$g_4 = q_x\varphi_4^{(0)} + u\varphi_2^{(0)} + 2q\varphi_{4,x}^{(0)} - p_x\varphi_1^{(0)} - q\varphi_2^{(0)} - \nu\varphi_1^{(0)} - \nu\varphi_3^{(0)}.$$

System (6.187) is equivalent to

$$\frac{d^2\alpha_3}{dt_0^2} + \alpha_{3,x} = g_4 + \frac{dg_3}{dt_0}, \quad \alpha_4 = \frac{d\alpha_3}{dt_0} - g_3,$$

$$\frac{d^2\alpha_2}{dt_0^2} - \alpha_{2,x} = \frac{dg_2}{dt_0} + g_1 - \alpha_4, \quad \alpha_1 = \frac{d\alpha_2}{dt_0} - g_2.$$

We solve this system recursively, by using relations (6.185) and the relations $u_{t_0} = p_x - \nu$, $p_{t_0} = u$, $q_{t_0} = \nu$, $\nu_{t_0} = -q_x$. As a result, we obtain

$$\alpha_1 = p\varphi_3 - q\varphi_4, \quad \alpha_2 = -p\varphi_4,$$

$$\alpha_3 = q\varphi_4 - p\varphi_1, \quad \alpha_4 = q(\varphi_1 - \varphi_3) + p\varphi_2.$$

Hence, the expressions for the actions of components of the required operator $\vartheta_1 : T^*(M) \to T(M)$ have the form

$$\begin{aligned}
\vartheta_1^{(11)}\varphi_1^{(0)} &= 0, & \vartheta_1^{(21)}\varphi_1^{(0)} &= 0,\\
\vartheta_1^{(12)}\varphi_2^{(0)} &= 0, & \vartheta_1^{(22)}\varphi_2^{(0)} &= 0,\\
\vartheta_1^{(13)}\varphi_3^{(0)} &= p\varphi_3^{(0)}, & \vartheta_1^{(23)}\varphi_3^{(0)} &= 0,\\
\vartheta_1^{(14)}\varphi_4^{(0)} &= -q\varphi_4^{(0)}, & \vartheta_1^{(24)}\varphi_4^{(0)} &= -p\varphi_4^{(0)},
\end{aligned}$$

$$\begin{aligned}
\vartheta_1^{(31)}\varphi_1^{(0)} &= -p\varphi_1^{(0)}, & \vartheta_1^{(41)}\varphi_1^{(0)} &= q\varphi_1^{(0)},\\
\vartheta_1^{(32)}\varphi_2^{(0)} &= 0, & \vartheta_1^{(42)}\varphi_2^{(0)} &= p\varphi_2^{(0)},\\
\vartheta_1^{(33)}\varphi_3^{(0)} &= 0, & \vartheta_1^{(43)}\varphi_3^{(0)} &= -q\varphi_3^{(0)},\\
\vartheta_1^{(34)}\varphi_4^{(0)} &= 0, & \vartheta_1^{(44)}\varphi_4^{(0)} &= 0.
\end{aligned}$$

We can now easily construct the operator $\vartheta_1 : T^*(M) \to T(M)$:

$$\vartheta_1 = \begin{pmatrix} 0 & 0 & p & -q \\ 0 & 0 & 0 & -p \\ -p & 0 & 0 & q \\ q & p & -q & 0 \end{pmatrix}.$$

Similarly, we can find the operator $\vartheta_2 : T^*(M) \to T(M)$ from equation (6.184). We have, respectively:

$$\bar{g}_1 = -p^2\varphi_3^{(0)} + 2pu\varphi_4^{(0)}, \quad \bar{g}_2 = -p^2\varphi_4^{(0)},$$
$$\bar{g}_3 = p^2\varphi_1^{(0)}, \quad \bar{g}_4 = -2pu\varphi_1^{(0)} + p^2\varphi_2^{(0)},$$
$$\overline{\alpha}_1 = p^2\varphi_4^{(0)}, \quad \overline{\alpha}_2 = 0, \quad \overline{\alpha}_3 = 0, \quad \overline{\alpha}_4 = -p^2\varphi_1^{(0)},$$

that is

$$\vartheta_2 = \begin{pmatrix} 0 & 0 & 0 & p^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -p^2 & 0 & 0 & 0 \end{pmatrix}.$$

The operator $\vartheta_3 : T^*(M) \to T(M)$ is equal to zero by virtue of the equality

$$\vartheta_2 K_1'^* + K_1'\vartheta_2 = d\vartheta_2/dt_1.$$

We finally obtain

$$\vartheta = \begin{pmatrix} 0 & -1 & p & \partial - q + p^2 \\ 1 & 0 & -1 & -p \\ -p & 1 & 0 & q \\ \partial + q - p^2 & p & -q & 0 \end{pmatrix}.$$

By similar computations, we can find another implectic Nötherian operator $\eta : T^*(M) \to T(M)$ connected with the operator $\eta_0 : T^*(M) \to T(M)$; namely,

$$\eta = \begin{pmatrix} 2\partial & p & 2\partial - p^2 & (-2p\partial - \partial p - \nu + qp - pu - p^3) \\ -p & 0 & p & 2\partial + u + p^2 \\ -2\partial + p^2 & -p & 0 & -qp + \nu \\ -2\partial p - p\partial + \nu - qp + pu + p^3 & (2\partial - u - p^2) & qp - \nu & \partial q + q\partial \end{pmatrix}.$$

Hence, the original dynamical system (6.126) can be rewritten in the bi-Hamiltonian form

$$w_t = -\vartheta \operatorname{grad} H = -\eta, \operatorname{grad}, \bar{H},$$

where $w = (u, p, q, \nu)^\intercal$,

$$H = 4\gamma_2 = \int_{x_0}^{x_0+2\pi} dx(p^2 q + uq - p\nu + q^2/2)$$

and

$$\bar{H} = -2\gamma_1 = \int_{x_0}^{x_0+2\pi} dx(pq - \nu),$$

in turn, $\{H, \gamma_j\}_\vartheta = 0 = \{\bar{H}, \gamma_j\}_\vartheta$, $j \in \mathbb{Z}_+$.

We now show that dynamical system (6.175) possesses a Lax representation. To this end, we first express an $L[u, p, q, \nu; \lambda]$-operator of the Lax representation in the $(r \times r)$-matrix form, $r \in \mathbb{Z}_+$:

$$L = \frac{\partial}{\partial x} - l[u, p, q, \nu; \lambda], \quad l[u, p, q, \nu; \lambda] = 0, \tag{6.188}$$

where $l[u, p, q, \nu; \lambda]$, $\lambda \in \mathbb{C}$, is a local matrix-valued smooth functional on M. Then, for a gradient of the form $\varphi(\lambda) = \operatorname{grad} \Delta(x_0; \lambda)$, where $\Delta(x_0; \lambda)$ is the trace of the monodromy matrix $S(x; \lambda)$, $\lambda \in \mathbb{C}$, of the operator L, the following equation holds:

$$z^q(\lambda)\vartheta\varphi(\lambda) = \eta\varphi(\lambda). \tag{6.189}$$

In the case under consideration $z^q(\lambda) = \lambda$, so $q = 1$. The matrix $S(x; \lambda)$, $\lambda \in \mathbb{C}$, satisfies the Novikov–Marchenko equation

$$\frac{\partial}{\partial x} S = [l, S]. \tag{6.190}$$

It is easy to show that the order of the matrix $l[u, p, q, \nu; \lambda]$ in (6.188) is equal to $r = 2$. Equations (6.188) – (6.190) yield the system

$$\lambda \left[-l_p + pl_q - ql_\nu + p^2 l_\nu + [l_\nu, l] + \frac{\partial}{\partial x} l_\nu \right]$$

$$= 2[l_u, l] + 2\frac{\partial}{\partial x} l_u + pl_p - 2[l_q, l] - 2\frac{\partial}{\partial x} l_q - p^2 l_q$$

$$-3p[l_\nu, l] - 3p\frac{\partial}{\partial x} l_\nu - p_x l_\nu - \nu l_\nu + qpl_\nu - pul_\nu - p^3 l_\nu,$$

$$\lambda[l_u - l_q - pl_\nu] = -pl_u + pl_q + 2[l_\nu, l] + 2\frac{\partial}{\partial x} l_\nu + ul_\nu + p^2 l_\nu, \tag{6.191}$$

$$\lambda[-pl_u + l_p + ql_\nu] = -2[l_u, l] - 2\frac{\partial}{\partial x}l_u + p^2 l_u - pl_p + \nu l_\nu - qpl_\nu,$$

$$\lambda\{[l_u, l] + \frac{\partial}{\partial x}l_u + ql_u - p^2 l_u u + pl_q - ql_q\}$$

$$= -3p[l_u, l] - 3p\frac{\partial}{\partial x}l_u - 2p_x l_u + \nu l_u + pul_u - qpl_u + p^3 l_u + 2[l_p, l]$$

$$+2\frac{\partial}{\partial x}l_p - ul_p - p^2 l_p - \nu l_q + qpl_q + 2q[l_\nu, l] + 2q\frac{\partial}{\partial x}l_\nu + q_x l_\nu,$$

where we have put $l[u,p,q,\nu;\lambda] = l(u,p,q,\nu;\lambda)$, $\lambda \in \mathbb{C}$, owing to the form of equation (6.189). In deriving these equations, we used the equality

$$z^q(\lambda)\vartheta : \mathrm{tr} : (Sl'_w) = \eta : \mathrm{tr} : (Sl'_w), \tag{6.192}$$

where $l'_w = (l_u, l_p, l_q, l_\nu)^\intercal$. As the matrix $l[w;\lambda]$ does not contain first derivatives of the phase variables, we conclude from the second and third equations in system (6.191) that $\frac{\partial}{\partial x}l_\nu = 0$ and $\frac{\partial}{\partial x}l_u = 0$. Hence the matrices l_u and l_ν may depend only on the parameter $\lambda \in \mathbb{C}$ and do not depend on the function w. It follows from the first and fourth equations that the matrices l_p and l_q have the form

$$l_p = c_1 - l_\nu q/2 + l_u q, \quad l_q = c_2 - l_\nu p/2.$$

We can now express the general form of the matrix $l[u,p,q,\nu;\lambda]$ as

$$l = c + c_1 p + c_2 q + l_u u + l_\nu \nu + l_u p^2/2 - l_\nu qp/2, \tag{6.193}$$

where c, c_1, c_2, l_u and l_ν are constant matrices that depend on the parameter $\lambda \in \mathbb{C}$. By substituting the matrix l in the form (6.193) in equation (6.192), we obtain the relations

$$2[l_\nu, l_u] + l_\nu = 0, \quad [l_\nu, c_2] = 0, \quad 2[c_1, l_u] = c_1, \quad -\lambda l_\nu = 4[l_u, c_2],$$

$$\lambda(l_u - c_2) = 2[l_\nu, c], \quad \lambda[l_\nu, c] = -2[c_2, c], \quad \lambda c_1 = -2[l_u, c], \tag{6.194}$$

$$\lambda[l_u, c] = 2[c_1, c], \quad \lambda[l_u, c_2] = 2[c_1, c_2] - [l_\nu, c],$$

$$2[l_u, c_2] = c_2 - l_u + 2[l_\nu, c_1].$$

To determine the unknown matrices from relations (6.194), we use the representation theory of Lie algebras [155, 172, 179, 274]. Assume that the matrix l belongs to a Lie subalgebra $\mathfrak{g}$ of the Lie algebra $\mathrm{sl}(2;\mathbb{C})$. Then, the matrix $l(w;\lambda)$ can be decomposed using the standard basis $\{\sigma_3, \sigma_\pm\}$, where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Clearly, the matrices c, c_1, c_2, l_u and l_ν can also be decomposed in terms of the basis $\{\sigma_3, \sigma_\pm\}$. It follows from relations (6.194) that these matrices can be taken in the form $l_u = \bar{a}\sigma_3$, $l_\nu = \bar{b}\sigma_+$, $c_2 = \bar{c}\sigma_+$, $c_1 = \bar{f}\sigma_-$, and $c = \bar{e}\sigma_3 + \bar{k}\sigma_-$. Inserting this form of the matrices in relations (6.193), we obtain the unknown coefficients $\bar{a}, \bar{b}, \bar{c}, \bar{f}, \bar{e}$ and $\bar{k}$: $\bar{a} = \bar{b} = 1/4$, $\bar{c} = -\lambda/8$, $\bar{f} = 1/2$, $\bar{e} = -\lambda^2/8$, and $\bar{k} = \lambda/2$. Consequently, the matrix $l[u,p,q,v;\lambda]$, $\lambda \in \mathbb{C}$, can be written as

$$l[u,p,q,\nu;\lambda] = \begin{pmatrix} \frac{1}{8}\,p^2 + \frac{1}{4}\,u - \frac{1}{8}\,\lambda^2 & \frac{1}{4}\,\nu - \frac{1}{8}qp - \frac{1}{8}\,\lambda q \\ \frac{1}{2}\,\lambda + \frac{1}{2}p & \frac{1}{8}\lambda^2 - \frac{1}{8}p^2 - \frac{1}{4}u \end{pmatrix}.$$

Thus, it is easy to see that system (6.175) possesses the Lax representation

$$dL/dt = [L, p(l)],$$

where for all $\lambda \in \mathbb{C}$

$$L = \frac{\partial}{\partial x} - l[u,p,q,v;\lambda], \quad p(l) = \begin{pmatrix} \frac{1}{4}p - \frac{\lambda}{4} & -\frac{1}{4}q \\ 1 & -\frac{1}{4}p + \frac{\lambda}{4} \end{pmatrix}.$$

This Lax representation enables us to find, via the inverse scattering transform, a wide class of exact solutions to the dynamical system (6.175); in particular, finite-zone and soliton solutions, which have many applications.

6.3 Analysis of a Whitham type nonlocal dynamical system for a relaxing medium with spatial memory

6.3.1 *Introduction*

Many important wave propagation problems in nonlinear media with distributed parameters can be described by means of evolution differential equations of special types. For example, if the nonlinear medium has spatial memory, the propagation of the corresponding waves can be modeled by means of the generalized Whitham type evolution equations [292, 398]. The problems of propagating waves in nonlinear media with distributed parameters, for instance invisible non-dissipative dark matter, which play a key role [168, 169] in the formation of large scale structure in the Universe such as galaxies, clusters of galaxies, and super-clusters, can be modeled by evolution differential equations of a dispersive hydrodynamical type.

Moreover, it is well known [68, 269, 328, 391, 398] that shortwave perturbations in a relaxing one-dimensional medium can be described by reduced Whitham type equations in the form of the evolution equation

$$du/dt = 2uu_x + \int_{\mathbb{R}} \mathcal{K}(x,s)u_s ds, \tag{6.195}$$

first discussed in [398]. Here the kernel $\mathcal{K} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ depends on the elasticity and spatial memory properties of the medium and can, in general, be a function of the pressure gradient $u_x \in C^\infty(\mathbb{R};\mathbb{R})$, evolving according to equation (6.195). In particular, if $\mathcal{K}(x,s) = \frac{1}{2} \mid x - s \mid,\ x, s \in \mathbb{R}$, then equation (6.195) reduces to

$$du/dt = 2uu_x + \partial^{-1}u, \tag{6.196}$$

which has been studied in [269, 300, 301, 391, 398].

For cases where the elasticity medium depends strongly on the spatial pressure gradient $u_x,\ x \in \mathbb{R}$, the corresponding Whitham kernel has the form

$$\mathcal{K}(x,s) := -\theta(x-s)u_s \tag{6.197}$$

for $x, s \in \mathbb{R}$, which models the relaxing spatial memory effects in a natural way. The resulting equation (6.195) with the kernel (6.197) becomes

$$du/dt = K[u] := 2uu_x - \partial^{-1}u_x^2, \tag{6.198}$$

which has very interesting mathematical properties. This equation shall be the focus of our investigation in the section.

6.3.2 *Lagrangian analysis*

A more mathematically precise form of equation (6.198) is

$$u_{xt} = 2(uu_x)_x - u_x^2, \tag{6.199}$$

which represents a nonlinear hyperbolic flow on $\mathbb{R}$. Concerning the preceding form of (6.198), it is necessary to define the operation $\partial^{-1} : C^\infty(\mathbb{R};\mathbb{R}) \to C^\infty(\mathbb{R};\mathbb{R})$, which is not an easy problem. As equation (6.199) is well defined in the space of 2π-periodic functions $C^\infty_{2\pi}(\mathbb{R};\mathbb{R})$, one can determine on its subspace $\bar{C}^\infty_{2\pi}(\mathbb{R};\mathbb{R}) \subset C^\infty_{2\pi}(\mathbb{R};\mathbb{R})$ of functions under the condition $\int_0^{2\pi} f(s)ds = 0$ for any $f \in \bar{C}^\infty_{2\pi}(\mathbb{R};\mathbb{R})$ the inverse operation

$$\partial^{-1}(\cdot) := \frac{1}{2}[\int_0^x (\cdot)ds - \int_x^{2\pi} (\cdot)ds], \tag{6.200}$$

which is accordingly well defined for all $x \in \mathbb{R}$ and satisfies $\partial \cdot \partial^{-1} = 1$. Hence, we consider the flow (6.198) to be on the smooth functional submanifold $M := \bar{C}^{\infty}_{2\pi}(\mathbb{R};\mathbb{R})$. The corresponding vector field $K : M \to T(M)$ defines on M a dynamical system, which, as we shall see, has both Lagrangian and Hamiltonian properties.

To this end, we first consider the partial differential equation (6.199) and prove that it has the Lagrangian form

$$u_{xt} = -\frac{\delta H_{\vartheta}}{\delta u} := \xi[u], \tag{6.201}$$

where $H_{\vartheta} : M \to \mathbb{R}$ is a Fréchet smooth Lagrangian function. To prove (6.201), following the scheme in [137, 290, 326], it suffices to verify that the Volterrian identity $\xi^{'} = \xi^{'*}$ holds, that is

$$[2(uu_x)_x - u_x^2]' = [2(uu_x)_x - u_x^2]'^{*}, \tag{6.202}$$

where the prime is the Fréchet derivative with respect to the variable $u \in M$ and $*$ denotes the corresponding conjugation with respect to the natural scalar product on the tangent space $T(M) \simeq T^{*}(M)$. Whence, there exists a Lagrangian function $H_{\vartheta} : M \to \mathbb{R}$ having the explicit form

$$H_{\vartheta} := \int_0^{2\pi} \mathcal{H}_{\vartheta} dx = \int_0^{2\pi} u u_x^2 dx, \tag{6.203}$$

where we used the standard [173, 290, 326] homotopy formula $H_{\vartheta} = \int_0^1 d\lambda (\text{grad} H_{\vartheta}[u\lambda\}, u)$. Thus, expression (6.201) can be presented as the Euler equation

$$\delta \mathcal{L}/\delta u = 0, \tag{6.204}$$

where

$$\mathcal{L} := \int_0^t \int_0^{2\pi} (\frac{1}{2} u_x u_{\tau} - \mathcal{H}_{\vartheta}) dx d\tau. \tag{6.205}$$

Recall now that owing to the standard results [3, 14, 137, 173, 290, 326], any Lagrangian system in the form (6.204) is Hamiltonian. To show this, we recast the action functional (6.205) as

$$\mathcal{L} = \int_0^t [(\varphi, u_{\tau}) - H_{\vartheta}] d\tau, \tag{6.206}$$

where $\varphi := (1/2)u_x \in T^{*}(M)$. Then, it follows from the condition (6.204) that

$$u_t = -\vartheta \ \text{grad} \ H_{\vartheta}[u] = K[u], \tag{6.207}$$

where

$$\vartheta^{-1} := \varphi' - \varphi'^{*} = \partial/\partial x. \tag{6.208}$$

It is easy to see that the operator $\vartheta := \partial^{-1} : T^{*}(M) \to T(M)$ is implectic [121, 137, 326] and also Nötherian with respect to the flow (6.207). Thus we have proved the following [328] theorem.

Theorem 6.7. *The partial differential equation (6.199) is equivalent on the functional manifold M to the Hamiltonian flow (6.207) with the Hamiltonian function (6.203) and co-implectic structure (6.208).*

This result means that the flow (6.198) is Hamiltonian (and conservative), so one might expect that it also possesses an infinite hierarchy of conservation laws, which is very important [2, 121, 290, 326, 406] for its integrability analysis. This expectation is verified in the sequel.

6.3.3 *Gradient-holonomic analysis*

Since any conservation law $\gamma \in \mathcal{D}(M)$ satisfies the linear Lax equation

$$d\psi/dt + K'^{*}\psi = 0, \tag{6.209}$$

where $\psi = \text{grad } \gamma \in T^{*}(M)$ as a consequence of its existence as a local functional on M, it can be found, for instance, using the asymptotic small parameter method [173, 265, 326, 365]. In particular, one easily sees that

$$\psi_{\vartheta} = u_{xx}, \quad \psi_{\eta_{-1}} = \frac{1}{2}(u_x^2 - (u)_{xx}^2) \tag{6.210}$$

satisfy the Lax equation (6.209) and are the gradients of the corresponding functionals on M, that is

$$\psi_{\vartheta} = \text{grad } \gamma_{\vartheta} \quad \psi_{\eta_{-1}} = \text{grad } \gamma_{\eta_{-1}}, \tag{6.211}$$

where

$$\gamma_{\vartheta} = \frac{1}{2}\int_0^{2\pi} u_x^2 dx \quad \gamma_{\eta_{-1}} = \frac{1}{2}\int_0^{2\pi} uu_x^2 dx. \tag{6.212}$$

Thus, we have shown that the dynamical system (6.198) allows additional invariants (conservation laws), which can be used in the gradient-holonomic algorithm [173, 262, 326, 406] for finding new associated nontrivial implectic structures on the manifold M. Let us represent conservation laws (6.210) in the scalar product form on M as

$$\gamma_{\vartheta} = (\varphi_{\vartheta}, u_x) \quad \gamma_{\eta_{-1}} = (\varphi_{\eta_{-1}}, u_x), \tag{6.213}$$

where

$$\varphi_\vartheta = \frac{1}{2}u_x, \quad \varphi_{\eta_{-1}} = -\frac{1}{2}\partial^{-1}u_x^2 \in T^*(M). \tag{6.214}$$

Then, the operators

$$\vartheta^{-1} = \varphi_\vartheta^{'} - \varphi_\eta^{'*} = \frac{1}{2}\partial - (-\frac{1}{2}\partial) = \partial,$$

$$\eta_{-1}^{-1} = \varphi_{\eta_{-1}}^{'} - \varphi_{\eta_{-1}}^{'*} = \partial^{-1}u_{xx} + u_{xx}\partial^{-1} \tag{6.215}$$

are co-implectic [137, 290, 326] on M, and, as it is easy to check, also Nötherian with respect to the dynamical system (6.198). Moreover, via direct calculations one can show that the corresponding implectic operators $\vartheta,\ \eta_{-1} : T^*(M) \to T(M)$ are compatible on M; that is, for any $\lambda \in \mathbb{R}$ the expression $\vartheta + \lambda\eta_{-1}$ is also implectic on M [137, 247, 326]. In this vein, it is enough to show [137, 326] that the operator $\vartheta^{-1}\eta_{-1}\vartheta^{-1} : T^*(M) \to T(M)$ is symplectic on M or equivalently that the differential two-form $\Omega^{(2)} := \int_0^{2\pi} dx(du \wedge \vartheta^{-1}\eta_{-1}\vartheta^{-1}du) \in \Lambda^2(M)$ is closed, or $d\Omega^{(2)} = 0$, which is easily checked by direct calculations. This means, in particular, that all operators of the form

$$\eta_n = \vartheta(\eta_{-1}^{-1}\vartheta)^n \tag{6.216}$$

for $n \in \mathbb{Z}$ are implectic on M. Another consequence of this fact is the existence of an infinite hierarchy of invariants $\gamma_n \in \mathcal{D}(M)$, $n \in \mathbb{Z}$, satisfying

$$K[u] = -\eta_n \text{ grad } \gamma_n. \tag{6.217}$$

As a particular case, one can define an implectic operator $\eta : T^*(M) \to T(M)$ in the form

$$\eta = \vartheta\eta_{-1}^{-1}\vartheta = \partial^{-2}u_{xx}\partial^{-1} + \partial^{-1}u_{xx}\partial^{-2}. \tag{6.218}$$

Whence, from (6.217) we obtain that

$$u_t = K[u] = -\vartheta\text{grad } H_\vartheta = -\eta\text{grad } H_\eta, \tag{6.219}$$

where

$$H_\vartheta = \int_0^{2\pi} uu_x^2 dx, \quad H_\eta = \int_0^{2\pi} u_x^2 dx.$$

The set of expressions (6.216) can also be written as

$$\lambda\vartheta \text{ grad } \gamma(\lambda) = \eta \text{ grad } \gamma(\lambda), \tag{6.220}$$

which is in a sense equivalent [137, 290, 326] together with equation (6.209) to the adjoint Lax representation

$$d\Lambda/dt = [\Lambda, K'^*] \tag{6.221}$$

for the dynamical system (6.198), where $\Lambda := \vartheta^{-1}\eta : T^*(M) \to T^*(M)$ is a recursion operator [137, 290, 326] and $\gamma(\lambda) \in \mathcal{D}(M)$, $\lambda \in \mathbb{C}$, is a generating function of the infinite hierarchy of conservation laws (6.198). In particular, as $\|\lambda| \to \infty$ the asymptotic expansion

$$\text{grad } \gamma(\lambda)_{|_{|\lambda| \to \infty}} \sim \sum_{j \in \mathbb{Z}_+} \lambda^{-j} \text{ grad } \gamma_j \tag{6.222}$$

holds, where

$$\text{grad } \gamma_n = \Lambda^n \text{ grad } \gamma_0, \quad \gamma_0 := H_\eta, \tag{6.223}$$

for all $n \in \mathbb{Z}_+$. Concerning this infinite hierarchy of conservation laws, it is easy to show one that all of them are dispersionless. The result obtained above can be formulated as the next theorem.

Theorem 6.8. *The dynamical system (6.198) on the functional manifold M is a compatible bi-Hamiltonian flow, possessing an infinite hierarchy of commuting functionally independent dispersionless conservation laws, satisfying the fundamental gradient identity (6.220). This identity is equivalent together with the relationship (6.209) to the adjoint Lax type representation (6.221).*

As mentioned above, the hierarchy of commuting flows $K_n := -\vartheta$ grad γ_n, $n \in \mathbb{Z}_+$, shows an interesting property in that they are dispersionless. In particular, this means that they cannot be treated directly by the gradient-holonomic algorithm [173, 262, 326, 406]. This is because the corresponding asymptotic solutions to the Lax equations

$$d\varphi/d\tau_n + K_n'^* \varphi = 0, \quad \varphi' \neq \varphi'^+, \tag{6.224}$$

as $|\lambda| \to \infty$ and $du/d\tau_n = K_n[u]$, $\tau_n \in \mathbb{R}$, $n \in \mathbb{Z}$, do not generate explicit functional expressions defining a new associated hierarchy of conservation laws for the dynamical system (6.101). Nonetheless, the corresponding hierarchy of dispersive commuting flows on M does exist for (6.198), and is associated with the trivial flow $du/dt_0 := 0$ on M. Namely, let $H_0 \in \mathcal{D}(M)$ be a conservation law of (6.198) satisfying the kernel condition for the operator $\eta : T^*(M) \to T(M)$, that is

$$du/dt_0 = 0 := \eta \text{ grad } H_0. \tag{6.225}$$

It is easy to find from (6.225) and (6.218) that grad $H_0 = [2(u_{xx})^{-1/2}]_{xx} \in T^*(M)$; whence

$$H_0 = 4 \int_0^{2\pi} \sqrt{u_{xx}} dx. \tag{6.226}$$

The invariant (6.226) allows one to construct a new commuting flow associated with (6.101), which is given as

$$du/d\tau = -\vartheta \text{ grad } H_0 = u_{xxx}(u_{xx})^{-3/2} := \tilde{K}[u], \tag{6.227}$$

$\tau \in \mathbb{R}$, and already possesses a nontrivial dispersion. This means that the Lax equation

$$d\varphi/d\tau + \tilde{K}'^{*}\varphi = 0, \tag{6.228}$$

has as $|\lambda| \to \infty$ an asymptotic solution $\varphi := \varphi(\tau, x; \lambda) \in T^*(M) \otimes \mathbb{C}$, where

$$\varphi(\tau, x; \lambda) \sim \exp(\lambda^3 \tau + \int_{x_0}^{x} \sigma(y; \lambda) dy), \tag{6.229}$$
$$\sigma(x; \lambda) \sim \sum_{j \in \mathbb{Z}_+} \sigma_{j-1}[u] \lambda^{-j+1}.$$

The nontrivial functionals $\tilde{\gamma}_{j-1} := \int_0^{2\pi} \sigma_{j-1}[u]dx$, $j \in \mathbb{Z}_+$, are obviously functionally independent and commuting conservation laws for both dynamical systems (6.227) and (6.198). As a result of some simple but tedious calculations, one finds that

$$\sigma_{-1} = \sqrt{u_{xx}}, \quad \sigma_0 = \frac{1}{2} u_{xx}^{-1} u_{xxx}, \quad \sigma_1 = \frac{1}{8}(u_{xx})^{-5/2} u_{xxx}^2, \dots, \tag{6.230}$$

and the corresponding hierarchy of dispersive invariants is given as

$$\tilde{\gamma}_{-1} = \int_0^{2\pi} \sqrt{u_{xx}}\, dx, \quad \tilde{\gamma}_0 = 0, \tag{6.231}$$

$$\tilde{\gamma}_1 = \tfrac{1}{8} \int_0^{2\pi} u_{xx}^{-5/2} u_{xxx}^2\, dx, \quad \tilde{\gamma}_2 = 0,$$

and so on. Then, owing to conditions (6.227) and 6.228), the generating functional $\tilde{\gamma}(\lambda) := \int_0^{2\pi} \sigma(x; \lambda)dx$, $\lambda \in \mathbb{C}$, satisfies [137, 262, 326] the following gradient relationship

$$\lambda^2 \vartheta \text{ grad } \tilde{\gamma}(\lambda) = \eta \text{ grad } \tilde{\gamma}(\lambda), \tag{6.232}$$

suitably modifying the relationship (6.220).

The above results are very important for further investigation of the Lax integrability of the dynamical system (6.198) and finding, in particular, a wide class of its special soliton like and quasiperiodic solutions by means of analytical quadratures. Some of these aspects of the integrability problem are presented in the next section.

6.3.4 *Lax form and finite-dimensional reductions*

Since the functional solution (6.229) satisfies the Lax equation (6.228), it can be considered [121, 262, 406] as a Bloch type eigenfunction of the adjoint Lax representation (6.221), that is

$$\Lambda\varphi(x;\lambda) = \lambda^2\varphi(x;\lambda) \tag{6.233}$$

for all $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$. This gives rise, following the gradient-holonomic algorithm [262, 324, 406], to the existence of a standard Lax representation for the associated dynamical system (6.227) and, thereby, for the Whitham dynamical system (6.198). Omitting here the related calculations, we find surprisingly that this, adjoint to (6.233), Lax spectral problem for the flow (6.198) is equal to

$$\ell f := \begin{pmatrix} -i\lambda & \lambda(u_{xx}-1) \\ -\lambda & i\lambda \end{pmatrix} f, \tag{6.234}$$

where an eigenfunction $f \in L^\infty(\mathbb{R};\mathbb{C}^2)$ and $\lambda \in \mathbb{C}$ is a time independent spectral parameter. The result (6.234) can be used to solve our nonlinear equation 6.198), making use either of the inverse spectral transform method [2, 111, 121, 262, 358, 406] or of the dual Bogoyavlensky–Novikov method [326, 406] of finite-dimensional reductions. For the latter case, we need to construct finite-dimensional invariant symplectic functional submanifolds $M^{2N} \subset M$, $N \in \mathbb{Z}_+$, and to represent the main vector fields d/dx and d/dt on them as the corresponding commuting Hamiltonian flows. Moreover, since these flows on M^{2N} are Liouville–Arnold integrable, we obtain both the complete integrability of our dynamical system (6.198) in quadratures and their exact solutions, expressed, in general, by means of Riemann theta-functions [111, 121, 262, 406] on specially constructed algebraic Riemann surfaces.

We consider, for simplicity, the following invariant two- and four-dimensional functional submanifolds:

$$i) \quad M^2 := \{u \in M : \mathrm{grad}\mathcal{L}_2[u] = 0\}, \tag{6.235}$$

where $\mathcal{L}_2 := H_\vartheta + c_\eta H_\eta \in \mathcal{D}(M)$, and

$$ii) \quad M^4 := \{u \in M : \mathrm{grad}\mathcal{L}_4[u] = 0\}, \tag{6.236}$$

where $\mathcal{L}_4 := \gamma_{-1} + c_\vartheta H_\vartheta + c_\eta H_\eta \in \mathcal{D}(M)$.

Case i) We have [101, 111, 326, 406] on the invariant manifold M^2 commuting Hamiltonian vector fields d/dx and d/dt with respect to the canonical symplectic structure

$$\omega^{(2)} := d\alpha^{(1)}, \tag{6.237}$$

where the 1-form $\alpha^{(1)} \in \Lambda^1(M)$ is determined by the Gelfand-Dickey [101, 326] relationship

$$d\mathcal{L}_2[u] = \mathrm{grad}\mathcal{L}_2[u]du + d\alpha^{(1)}/dx, \tag{6.238}$$

on M. Now it is easy to show that for all $u \in M^2 \subset M$

$$\begin{aligned} \mathrm{grad}\mathcal{L}_2[u] &= u_x^2 - 2(u_x u)_x - 2c_\eta u_{xx} = 0, \\ \alpha^{(1)} &= 2(u + c_\eta)u_x du, \\ \omega^{(2)} &= d[2(u + c_\eta)u_x] \wedge du := dp \wedge dq, \end{aligned} \tag{6.239}$$

where $p := 2(u + c_\eta)u_x$ and $q := u$. The corresponding Hamiltonian functions $h^{(x)}$ and $h^{(t)} \in \mathcal{D}(M^2)$ for the flows

$$\begin{aligned} dq/dx &= \partial h^{(x)}/\partial p, \qquad dp/dx = -\partial h^{(x)}/\partial q, \\ dq/dt &= \partial h^{(t)}/\partial p, \qquad dp/dt = -\partial h^{(t)}/\partial q \end{aligned} \tag{6.240}$$

are found [173, 326], respectively, from the determining relationships

$$\mathrm{grad}\mathcal{L}_2[u]\ u_x := -dh^{(x)}/dx, \qquad \mathrm{grad}\mathcal{L}_2[u]\ u_t := -dh^{(t)}/dx, \tag{6.241}$$

which imply that

$$h^{(x)} = (u + c_\eta)u_x = \frac{p^2}{4(q + c_\eta)}, \qquad h^{(t)} = -2c_\eta(u + c_\eta)u_x = \frac{-c_\eta p^2}{2(q + c_\eta)}. \tag{6.242}$$

One can readily verify that the two flows d/dx and d/dt on the two-dimensional invariant submanifold $M^2 \subset M$ of infinite period are proportional, confirming the classical fact [3, 14] that there only can be one such flow on a symplectic surface.

The set of Hamiltonian equations (6.241) for the flows d/dx and d/dt are

$$\begin{aligned} \frac{dq}{dx} &= \frac{p}{2(q + c_\eta)}, \qquad \frac{dp}{dx} = \frac{p^2}{4(q + c_\eta)^2}, \\ \frac{dq}{dt} &= \frac{-c_\eta p}{(q + c_\eta)}, \qquad \frac{dp}{dt} = \frac{-c_\eta p^2}{2(q + c_\eta)^2}, \end{aligned} \tag{6.243}$$

with solutions

$$q(x,t) = -c + [\frac{3}{2}\sqrt{\bar{h}^{(x)}}(x - 2c_\eta t) + \bar{k}]^{2/3} \Rightarrow u(x,t), \tag{6.244}$$

with $\bar{k} \in \mathbb{R}$ a real constant, supplying us with an exact partial one-parameter solution to the Whitham equation (6.198).

Case ii). Similarly to the first case, we find the quantities

$$\operatorname{grad}\mathcal{L}_4[u] = (\frac{1}{2\sqrt{u_{xx}}})_{xx} + c_\vartheta[u_x^2 - 2(uu_x)_x] - 2c_\eta u_{xx} = 0, \tag{6.245}$$

$$\alpha^{(1)} = [2(c_\vartheta u + c_\eta)u_x - (\frac{1}{2\sqrt{u_{xx}}})_x]du + \frac{1}{2\sqrt{u_{xx}}}du_x,$$

providing the symplectic structure

$$\begin{aligned}\omega^{(2)} &= d[2(c_\vartheta u + c_\eta)u_x - (\frac{1}{2\sqrt{u_{xx}}})_x] \wedge du + d(\frac{1}{2\sqrt{u_{xx}}}) \wedge du_x \\ &= dp_1 \wedge dq_1 + dp_2 \wedge dq_2,\end{aligned} \tag{6.246}$$

where $(q_1 := u,\ q_2 : u_x,\ \ p_1 := 2(c_\vartheta u + c_\eta)u_x - (\frac{1}{2\sqrt{u_{xx}}})_x,\ p_2 := \frac{1}{2\sqrt{u_{xx}}}) \in M^4$, which are canonical symplectic coordinates on the invariant functional submanifold $M^4 \subset M$. The commuting Hamiltonian functions related to the flows d/dx and d/dt are

$$\begin{aligned}h^{(x)} &= u_x^2(c_\vartheta u + c_\eta) - (\frac{1}{2\sqrt{u_{xx}}})_x u_x - \sqrt{u_{xx}} \\ &= 3q_2^2(c_\vartheta q_1 + c_{\eta)} - q_2 p_1 - 1/(2p_2)\end{aligned} \tag{6.247}$$

and

$$h^{(t)} = q_1/(2p_2) - 2q_1q_2[p_1 - q_2(c_\vartheta q_1 + c_\eta)]. \tag{6.248}$$

As a result, we have reduced our Whitham type dynamical system (6.198) on the constructed four-dimensional invariant submanifold $M^4 \subset M$ so that it is exactly equivalent to two commuting canonical Hamiltonian flows

$$\begin{aligned} dq_j/dx &= \partial h^{(x)}/\partial p_j, \quad & dp_j/dx &= -\partial h^{(x)}/\partial q_j, \\ dq_j/dt &= \partial h^{(t)}/\partial p_j, \quad & dp_j/dt &= -\partial h^{(t)}/\partial q_j \end{aligned} \tag{6.249}$$

for $j = 1, 2$, where the corresponding Poisson bracket $\{h^{(x)}, h^{(t)}\} = 0$ on M^4. Hence, owing to the classical Liouville–Arnold theorem [3, 14, 326], our Whitham type dynamical system (6.198), reduced invariantly on the four-dimensional invariant submanifold $M^4 \subset M$, is completely integrable by quadratures. This result can be summarized as follows.

Theorem 6.9. *The Whitham type dynamical system (6.198), reduced on the invariant two-parametric four-dimensional functional submanifold $M^4 \subset M$ is exactly equivalent to the set of two commuting canonical Hamiltonian flows (6.249) that are completely Liouville–Arnold integrable by quadratures systems. The corresponding Hamiltonian functions are given by expressions (6.247) and (6.248).*

Unfortunately, the above reduction scheme is nonlocal, so it does not lend itself to the construction of many exact two-parameter solutions to the Whitham nonlinear equation (6.198) by means of quadratures. That is why we revisit this problem using a regularization scheme in the next section.

6.4 A regularization scheme for a generalized Riemann hydrodynamic equation and integrability analysis

6.4.1 *Differential-geometric integrability analysis*

Consider again the Whitham type equation

$$du/dt = K[u] := 2uu_x - \partial^{-1}u_x^2, \tag{6.250}$$

which, as we have seen, possesses very interesting mathematical properties. We focus here on the geometric and Hamiltonian analysis of the dynamical system (6.250) and its special regularizations.

Owing to the results obtained in [327], the dynamical system (6.250) appears to be a Lax integrable bi-Hamiltonian flow, but with an ill-posed temporal evolution. As indicated above, the nonlocality of this dynamical system limits the effectiveness of using suitable finite-dimensional [54, 326] reductions to construct solutions by quadratures. Some of these integrability aspects [68] were presented above, where a suitable regularization scheme for treating this nonlocality problem was proposed. We shall treat the well-posed integrability problem for the Whitham type nonlinear and nonlocal dynamical system (6.250) and reanalyze it in detail making use of this regularization scheme and its generalization.

Define a smooth periodic function $v \in C_{2\pi}^{\infty}(\mathbb{R};\mathbb{R})$, such that

$$v := \partial^{-1}u_x^2 \tag{6.251}$$

for any $x, t \in \mathbb{R}$, where the function $u \in C_{2\pi}^{\infty}(\mathbb{R};\mathbb{R})$ solves equation (6.250). Then it is easy to see that the nonlinear dynamical system

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = K[u, v] := \begin{pmatrix} 2uu_x - v \\ 2uv_x \end{pmatrix} \tag{6.252}$$

of hydrodynamic type, which is well defined on the extended 2π-periodic functional space $\mathcal{M} := C_{2\pi}^{\infty}(\mathbb{R};\mathbb{R}^2)$, is completely equivalent to that given by expression (6.250). The system (6.252) is equivalent to the system

$$(u_t, v_t, u_x, v_x, w_t) = K[u, v, w] := (2uu_x - v, 2uv_x, w, u_xw, v_x + 2uw_x) \tag{6.253}$$

and further to the set of 2-forms

$$\{\alpha\} := \Big\{\alpha^{(1)} = du \wedge dx + 2u\, du \wedge dt - v\, dx \wedge dt;\ \alpha^{(2)} = dv \wedge dx$$
$$+2u\, dv \wedge dt;\ \alpha^{(3)} = du \wedge dt - w\, dx \wedge dt;\ \alpha^{(4)} = dv \wedge dt$$
$$-w\, du \wedge dt;\ \alpha^{(5)} = dv \wedge dx + dv \wedge dt + 2u\, dw \wedge dt\Big\}.$$

This set of two-forms generates the closed ideal $\mathcal{I}(\alpha)$, since

$$d\,\alpha^{(1)} = -\alpha^{(2)} \wedge dt; \quad d\,\alpha^{(2)} = 2du \wedge \alpha^{(4)}; \quad d\,\alpha^{(3)} = -\alpha^{(5)} \wedge dt;$$
$$d\,\alpha^{(4)} = -dw \wedge \alpha^{(3)} - w\, dt \wedge \alpha^{(5)}; \quad d\,\alpha^{(5)} = -2dw \wedge \alpha^{(3)} - 2w\, dt \wedge \alpha^{(5)}.$$

The integral submanifold $\bar{M}$ is defined by the condition $\mathcal{I}(\alpha) = 0$. Making use of the differential-geometric method devised in [326], we look for a reduced connection 1-form Γ on $\bar{M}$, belonging to a not yet determined Lie algebra $\mathcal{G}$. This 1-form can be represented as

$$\Gamma = \mathcal{A}(u,v,w)\, dx + \mathcal{B}(u,v,w)\, dt, \tag{6.254}$$

where the elements $\mathcal{A},\ \mathcal{B} \in \mathcal{G}$ satisfy the determining equations

$$\Omega = \frac{\partial \mathcal{A}}{\partial u}\, du \wedge dx + \frac{\partial \mathcal{A}}{\partial v}\, dv \wedge dx + \frac{\partial \mathcal{A}}{\partial w}\, dw \wedge dx + \frac{\partial \mathcal{B}}{\partial u}\, du \wedge dt$$
$$\frac{\partial \mathcal{B}}{\partial v}\, dv \wedge dt + \frac{\partial \mathcal{B}}{\partial w}\, dw \wedge dt + [\mathcal{A},\mathcal{B}] dx \wedge dt \tag{6.255}$$

$$\Rightarrow \quad g_1(du \wedge dx + 2u\, du \wedge dt - v\, dx \wedge dt) \ + \ g_2(dv \wedge dx + 2u\, dv \wedge dt)$$
$$+g_3(du \wedge dt - w\, dx \wedge dt) \ + \ g_4(dv \wedge dt - w\, du \wedge dt) +$$
$$+g_5(dw \wedge dx + 2u\, dw \wedge dt + dv \wedge dt) \in \mathcal{I}(\alpha) \otimes \mathcal{G}$$

for $\mathcal{G}$-valued functions $g_1, ..., g_5 \in \mathcal{G}$ on M. From (6.255), one finds that

$$\frac{\partial \mathcal{A}}{\partial u} = g_1, \quad \frac{\partial \mathcal{A}}{\partial v} = g_2, \quad \frac{\partial \mathcal{A}}{\partial w} = g_5, \tag{6.256}$$
$$\frac{\partial \mathcal{B}}{\partial u} = 2u\, g_1 + g_3 - w\, g_4, \quad \frac{\partial \mathcal{B}}{\partial v} = 2u\, g_2 + g_4 + g_5,$$
$$\frac{\partial \mathcal{B}}{\partial w} = 2u\, g_5, \quad [\mathcal{A},\mathcal{B}] = -v\, g_1 - w\, g_3.$$

Thus, it follows from the relationships (6.256) that

$$\mathcal{B} = 2u\mathcal{A} + \mathcal{C}(u,v), \quad g_4 = \frac{\partial \mathcal{C}}{\partial v} - \frac{\partial \mathcal{A}}{\partial w},$$
$$g_3 = 2\mathcal{A} + \frac{\partial \mathcal{C}}{\partial u} + w\frac{\partial \mathcal{C}}{\partial v} - w\frac{\partial \mathcal{A}}{\partial w}, \tag{6.257}$$
$$[\mathcal{A},\mathcal{C}] = -v\frac{\partial \mathcal{A}}{\partial u} - 2w\mathcal{A} - w\frac{\partial \mathcal{C}}{\partial u} - w^2\frac{\partial \mathcal{C}}{\partial v} + w^2\frac{\partial \mathcal{A}}{\partial w},$$

which can be used to determine that connection (6.254) by employing the differential-geometric scheme devised in [326], which is based on analyzing the related holonomy Lie algebra.

6.4.2 *Bi-Hamiltonian structure and Lax representation*

Consider the following polynomial expansion of the element $\mathcal{A}(u,v;w) \in \mathcal{G}$ with respect to the variable w :

$$\mathcal{A} = \mathcal{A}_0(u,v) + \mathcal{A}_1(u,v)w + \mathcal{A}_2(u,v)w^2 \tag{6.258}$$

and substitute it into the last equation of (6.256). This yields

$$\begin{aligned} &[\mathcal{A}_0, C] = -v\frac{\partial \mathcal{A}_0}{\partial u}, [\mathcal{A}_1, C] = -v\frac{\partial \mathcal{A}_1}{\partial u} - 2\mathcal{A}_0 - \frac{\partial C}{\partial u}, \\ &[\mathcal{A}_2, C] = -v\frac{\partial \mathcal{A}_2}{\partial u} - \frac{\partial C}{\partial v} - \mathcal{A}_1, \end{aligned} \tag{6.259}$$

or

$$\mathcal{A}_1 = [C, \mathcal{A}_2] - v\frac{\partial \mathcal{A}_2}{\partial u} - \frac{\partial C}{\partial v}, \tag{6.260}$$

which can be substituted into the second equation of (6.259):

$$\begin{aligned} [[C,\mathcal{A}_2],C] - 2v[\tfrac{\partial \mathcal{A}_2}{\partial u}, C] - [\tfrac{\partial C}{\partial v}, C] = -v[\tfrac{\partial C}{\partial u}, \mathcal{A}_2] \\ -v^2\tfrac{\partial^2 \mathcal{A}_2}{\partial u^2} - v\tfrac{\partial^2 C}{\partial u \partial v} - 2\mathcal{A}_0 - \tfrac{\partial C}{\partial u}. \end{aligned} \tag{6.261}$$

Thus, recalling (6.259) and (6.260), we have that

$$\begin{aligned} 2\mathcal{A}_0 &= [C,[C,\mathcal{A}_2]] + 2v[\frac{\partial \mathcal{A}_2}{\partial u}, C] \\ &+ [\frac{\partial C}{\partial v}, C] - v[\frac{\partial C}{\partial u}, \mathcal{A}_2] \\ &- v^2\frac{\partial^2 \mathcal{A}_2}{\partial u^2} - v\frac{\partial^2 C}{\partial u \partial v} - \frac{\partial C}{\partial u}, \\ [\mathcal{A}_0, C] &= -v\frac{\partial \mathcal{A}_0}{\partial u}, \quad \mathcal{A}_1 = [C,\mathcal{A}_2] - v\frac{\partial \mathcal{A}_2}{\partial u} - \frac{\partial C}{\partial v}. \end{aligned} \tag{6.262}$$

Now we assume that the element $C := C_0 \in \mathbb{C}$ is constant and the elements $\mathcal{A}_0$ and $\mathcal{A}_2$ are linear with respect to variables u and v, that is

$$\begin{aligned} \mathcal{A}_0 &= \mathcal{A}_0^{(0)} + \mathcal{A}_0^{(1)}u + \mathcal{A}_0^{(2)}v, \\ \mathcal{A}_2 &= \mathcal{A}_2^{(0)} + \mathcal{A}_2^{(1)}u + \mathcal{A}_2^{(2)}v. \end{aligned} \tag{6.263}$$

This and (6.262) imply that

$$\begin{aligned} &2\mathcal{A}_0^{(0)} = [C_0,[C_0,\mathcal{A}_2^{(0)}]], [\mathcal{A}_0^{(1)}, C_0] = 0, [\mathcal{A}_0^{(2)}, C_0] = -\mathcal{A}_0^{(1)}, \\ &2\mathcal{A}_0^{(1)} = [C_0,[C_0,\mathcal{A}_2^{(1)}]], 2\mathcal{A}_0^{(2)} = [C_0,[C_0,\mathcal{A}_2^{(2)}]] + 2[\mathcal{A}_2^{(1)}, C_0]. \end{aligned} \tag{6.264}$$

To solve the closed system (6.264), we need to calculate the corresponding holonomy Lie algebra of the connection (6.254). As a result of simple but lengthy calculations, we derive that elements $\mathcal{A}_2^{(j)}, 0 \leq j \leq 2$, and C_0 belonging to the Lie algebra $\mathrm{sl}(2;\mathbb{C})$, whose basis L_0, L_+ and L_- can be taken to satisfy the following canonical commutation relations:

$$[L_0, L_\pm] = \pm L_\pm, [L_+, L_-] = 2L_0. \tag{6.265}$$

Then, making use of the standard determining expansions

$$\mathcal{A}_2^{(j)} = \sum_\pm c_\pm^{(j)} L_\pm + c_0^{(j)} L_0, \tag{6.266}$$

$$C_0 = \sum_\pm k_\pm L_\pm + k_0 L_0,$$

and substituting (6.266) into (6.264), we obtain relationships among the values $c_\pm^{(j)}, c_0^{(j)} \in \mathbb{C}, 0 \leq j \leq 2$, and $k_\pm, k_0 \in \mathbb{C}$. After some straightforward calculations with these relationships, depending on a spectral parameter $\lambda \in \mathbb{C}$, we find the desired basic elements $\mathcal{A}$ and $\mathcal{B}$ of the connection Γ. Whence, we have the corresponding Lax commutative spectral representation of the dynamical system (6.252) in the following (2×2)-matrix form:

$$\begin{gathered} \tfrac{df}{dx} - \ell[u,v;\lambda]f = 0, \quad \ell[u,v;\lambda] := \begin{pmatrix} -\lambda u_x & v_x \\ -\lambda^2 & \lambda u_x \end{pmatrix}, \\ \tfrac{df}{dt} = p(\ell)f, \quad p(\ell) := \begin{pmatrix} \lambda u_x u & -v_x u \\ \lambda/2 + \lambda^2 u & -\lambda u_x u \end{pmatrix}, \end{gathered} \tag{6.267}$$

defining the generalized time-independent spectrum $\mathrm{Spec}(\ell) \subset \mathbb{C} : \ \lambda \in \mathrm{Spec}(\ell)$, if the corresponding solution $f \in L^\infty(\mathbb{R};\mathbb{C}^2)$.

It follows directly from the standard Riccati equation, derived from (6.267), that we have an infinite hierarchy of local conservation laws:

$$\hat{\gamma}_{-1} := \int_0^{2\pi} \sqrt{u_x^2 - v_x}\, dx, \quad \hat{\gamma}_0 := \int_0^{2\pi} \frac{(u_x v_{xx} - v_x u_{xx})}{2v_x\sqrt{u_x^2 - v_x}} dx, ..., \tag{6.268}$$

and so on. All of conservation laws (6.268), except γ_{-1}, are singular at the Cauchy condition (6.253). Therefore, we need to construct another hierarchy of polynomial conservation laws that are regular on the functional submanifold

$$\mathcal{M}_{red} := \{(u,v) \in \mathcal{M} : u_x^2 - v_x = 0, \ \ x \in \mathbb{R}/2\pi\mathbb{Z}\}, \tag{6.269}$$

which exists owing to the results of [298, 326]. The simplest way to find them consists in determining the bi-Hamiltonian structure of flow (6.252). As is easy to check, the dynamical system (6.252) is canonically Hamiltonian, that is

$$\frac{d}{dt}(u,v)^\intercal = \hat{K}[u,v] := -\hat{\vartheta}\,\mathrm{grad}\hat{H}_\vartheta, \tag{6.270}$$

where the corresponding co-symplectic structure $\hat{\vartheta} : T^*(\mathcal{M}) \to T(\mathcal{M})$ is canonical and equals

$$\hat{\vartheta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{6.271}$$

and satisfies the Nöther equation

$$L_{\hat{K}}\hat{\vartheta} = 0 = d\hat{\vartheta}/dt - \hat{\vartheta}\hat{K}^{\prime,*} - \hat{K}'\hat{\vartheta}.$$

To prove this, it suffices to find by means of the small parameter method [326] a non-symmetric ($\varphi' \neq \varphi'^{,*}$) solution $\varphi \in T(\mathcal{M})$ to the following Lie–Lax equation:

$$d\varphi/dt + \hat{K'}^{*}\varphi = \mathrm{grad}L \tag{6.272}$$

for a suitably chosen smooth functional $L \in \mathcal{D}(M)$. One readily computes that

$$\varphi = (-v, 0)^{\intercal}, \quad L = -\int_0^{2\pi} uvdx. \tag{6.273}$$

Making use of (6.273) and the classical Legendre relationships for the corresponding symplectic structure

$$\hat{\vartheta}^{-1} := \varphi' - \varphi'^{,*} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{6.274}$$

and the Hamiltonian functional

$$H := (\varphi, \hat{K}) - L, \tag{6.275}$$

we determine the implectic structure (6.271) and the corresponding non-singular Hamiltonian function

$$\hat{H}_{\vartheta} := \int_0^{2\pi} (v^2/2 + v_x u^2)dx. \tag{6.276}$$

We note here that the determining Lie–Lax equation (6.272) has yet another solution:

$$\varphi = (\frac{u_x}{2}, -\frac{u_x^2}{2v_x}), \ L = \frac{1}{4}\int_0^{2\pi} uv_x dx, \tag{6.277}$$

giving rise to, owing to formulas (6.274) and (6.275), the new co-implectic (singular symplectic) structure

$$\hat{\vartheta}_s^{-1} := \varphi' - \varphi'^{,*} = \begin{pmatrix} \partial/2 & -\partial u_x v_x^{-2} \\ -u_x v_x^{-2}\partial & \frac{1}{2}(u_x^2 v_x^2 \partial + \partial u_x^2 v_x^2) \end{pmatrix} \tag{6.278}$$

and the Hamiltonian functional

$$\hat{H}_\eta := \frac{1}{2}\int_0^{2\pi}(uv_x - vu_x)dx.$$

Consequently the co-implectic structure (6.278) is, evidently, singular since $\hat{\vartheta}_s^{-1}(u_x, v_x)^\intercal = 0$. It is surprising that there exists a third implectic structure $\hat{\eta} : T^*(\mathcal{M}) \to T^*(\mathcal{M})$ satisfying the determining Nöther equation

$$L_{\hat{K}}\hat{\eta} = 0 = d\hat{\eta}/dt - \hat{\eta}\hat{K}'^{,*} - \hat{K}'\hat{\eta} \tag{6.279}$$

and whose solutions can also be obtained by means of the small parameter method [326]. As a result, the second implectic operator has the form

$$\hat{\eta} := \begin{pmatrix} \partial^{-1}/2 & u_x\partial^{-1} \\ \partial^{-1}u_x & v_x\partial^{-1} + \partial^{-1}v_x \end{pmatrix}, \tag{6.280}$$

giving rise to the new infinite hierarchy of polynomial conservation laws

$$\hat{\gamma}_n := \int_0^1 d\lambda < (\hat{\vartheta}^{-1}\hat{\eta})^n \text{grad}\hat{H}_\vartheta[u\lambda\}, u > \tag{6.281}$$

for all $n \in \mathbb{Z}_+$.

Remark 6.1. The Lax pair (6.267), (6.254) for the dynamical system (6.252) first appeared in [76]. It coincides with those found later in [298], making use of a very special bi-Lagrangian representation of the dynamical system (6.252). However, the existence of the singular co-implectic structure (6.278) in these references was not proved.

In particular, it is easy to see that we have the representations

$$\frac{d}{dt}(u,v)^\intercal = -\hat{\eta}\text{grad}\hat{H}_\eta, \qquad \frac{d}{dx}(u,v)^\intercal = -\hat{\vartheta}\text{grad}\hat{H}_\eta, \tag{6.282}$$

where

$$\hat{H}_\eta := \frac{1}{2}\int_0^{2\pi}(uv_x - vu_x)dx. \tag{6.283}$$

Now, making use of (6.281), one can apply the standard reduction procedure on the corresponding finite-dimensional functional subspaces $\mathcal{M}^{2n} \subset \mathcal{M}$, $n \in \mathbb{Z}_+$, and obtain a large set of exact solutions to the dynamical system (6.252) on the functional submanifold $\mathcal{M}_{red}$, if the Cauchy data are taken to satisfy the constraint (6.269).

6.5 The generalized Riemann hydrodynamic regularization

6.5.1 *Introduction*

Nonlinear hydrodynamic equations have captured the interest of scientists and engineers ever since the classical work of Riemann. He studied them in the general three-dimensional case and paid special attention to their one-dimensional spatial reduction, for which he devised the generalized method of characteristics and Riemann invariants. These methods have proved to be very effective [315, 326, 398] in investigating many types of nonlinear spatially one-dimensional systems of hydrodynamic type. In particular, the method of characteristics in the form of a reciprocal transformation of variables has been used recently in studying the Gurevich–Zybin system [168, 169] in [298] and a Whitham type system in [363]. Moreover, this method has also been applied to solutions of a generalized Riemann type hydrodynamic system [151] of Holm and Pavlov having the form

$$D_t^N u = 0, \quad D_t := \partial/\partial t + u\partial/\partial x, \quad N \in \mathbb{Z}_+, \tag{6.284}$$

where $dx/dt = u \in C^\infty(\mathbb{R};\mathbb{R})$ is the corresponding characteristic flow velocity along $\mathbb{R}$.

We shall consider, for convenience, the hydrodynamic equation (6.284) on the 2π-periodic space of functions $\mathcal{M} := C^\infty(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R})$, which can obviously be recast as the following nonlinear dynamical system on the augmented functional manifold $\mathcal{M} := C^\infty(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^N)$ on the vector $\hat{u} := (u^{(0)} := u,\ u^{(1)} := D_t u^{(0)},\ u^{(2)} := D_t u^{(1)}, ..., u^{(N-1)} := D_t u^{(N-2)})^\intercal \in \mathcal{M}$:

$$\begin{aligned}
u_t^{(0)} &= u^{(1)} - u^{(0)} u_x^{(1)}, \\
u_t^{(1)} &= u^{(2)} - u^{(0)} u_x^{(2)}, \\
&\vdots \\
u_t^{(N-2)} &= u^{(N-1)} - u^{(0)} u_x^{(N-1)} \\
u_t^{(N-1)} &= -u^{(1)} u_x^{(N)}.
\end{aligned} \tag{6.285}$$

The dynamical system (6.285) has a very interesting and important property: all partial flows of the velocity components $u^{(j)}, 0 \leq j \leq N-1$, are realized along $\mathbb{R}$ with the same characteristic velocity $dx/dt = u^{(0)}$. This is exactly the case studied by Riemann (see, for example, [398]) when one can introduce *Riemann invariants* that allow a separation of dependent variables that simplify integration. Actually, we can observe that the system (6.285) is equivalent along the characteristics $dx/dt = u^{(0)}$ to the following set of differential equations in total differential form:

$$
\begin{aligned}
du^{(0)} &= u^{(1)}dt, \\
du^{(1)} &= u^{(2)}dt, \\
&\vdots \\
du^{(N-2)} &= u^{(N-1)}dt, \\
du^{(N-1)} &= 0,
\end{aligned}
\tag{6.286}
$$

whose solution in parametric form readily yields

$$
\begin{aligned}
u^{(N-1)} &:= z,\ z = \beta_N(x - \frac{zt^N}{N!} - \sum_{j=1}^{N-1} \frac{t^j}{j!}\beta_{N-j}(z)), \\
u^{(N-2)} &= zt + \beta_1(z),\ u^{(N-3)} = \frac{zt^2}{2!} + t\beta_1(z) + \beta_2(z), \\
u^{(N-4)} &= \frac{zt^3}{3!} + \frac{t^2}{2!}\beta_1(z) + t\beta_2(z) + \beta_3, \\
&\vdots \\
u^{(1)} &= \frac{zt^{N-2}}{(N-2)!} + \sum_{j=0}^{N-3} \frac{t^j}{j!}\beta_{N-2-j}(z), \\
u^{(0)} &= \frac{zt^{N-1}}{(N-1)!} + \sum_{j=0}^{N-2} \frac{t^j}{j!}\beta_{N-1-j}(z),
\end{aligned}
\tag{6.287}
$$

where $\beta_j \in C^\infty(\mathbb{R};\mathbb{R}), 1 \leq j \leq N-1$, and $\beta_N \in C^\infty(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R})$ are arbitrary smooth functions depending on a suitable first integral $z \in C^\infty(\mathbb{R}^N;\mathbb{R})$ of the system (6.286). The above result coincides to some extent with those obtained by Pavlov [299], and can be used to construct special solutions in analytical form to the generalized Riemann type hydrodynamic equation (6.284).

As shown in [76, 71, 298, 299, 327, 328], the Riemann hydrodynamic system (6.285) for $N = 2$ and $N = 3$ possesses additional very interesting properties related to their complete bi-Hamiltonian integrability. In particular, they have infinite hierarchies of dispersionless and dispersive conservation laws, which have an important hydrodynamic interpretations and may be used to construct many special quasiperiodic and soliton solutions.

The next section below is devoted to the Hamiltonian analysis of the hydrodynamic equation (6.285) for $N=2$, $N=3$ and $N=4$, as well as to the description of their new hierarchies of conservation laws, with their related symplectic structures and Lax representations.

6.5.2 *The generalized Riemann hydrodynamical equation for $N = 2$: Conservation laws, bi-Hamiltonian structure and Lax representation*

Consider the generalized Riemann hydrodynamic equation (6.284) for $N = 2$:

$$D_t^2 u = 0, \tag{6.288}$$

where $D_t = \partial/\partial t + u\partial/\partial x$, which is equivalent to the dynamical system

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = K[u,v] := \begin{pmatrix} v - uu_x \\ -uv_x \end{pmatrix}, \tag{6.289}$$

where $K : \mathcal{M} \to T(\mathcal{M})$ is a vector field on the 2π-periodic smooth nonsingular functional phase space $\mathcal{M} := \{(u,v)^\intercal \in C^\infty(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^2) : u_x^2 - 2v_x \neq 0, x \in \mathbb{R}\}$. As we are interested primarily in the conservation laws for the system (6.289), we first prove the following result.

Proposition 6.1. *Let $H(\lambda) := \int_0^{2\pi} h(x;\lambda)dx \in \mathcal{D}(\mathcal{M})$ be an almost everywhere smooth functional of the parameter $\lambda \in \mathbb{C}$ on the manifold $\mathcal{M}$, whose density satisfies the differential condition*

$$h_t = \lambda(uh)_x \tag{6.290}$$

for all $t \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ on the solution set of dynamical system (6.289). Then the following iterative differential relationship

$$(f/h)_t = \lambda(uf/h)_x \tag{6.291}$$

holds, if a smooth function $f \in C^\infty(\mathbb{R};\mathbb{R})$ (parametrically depending on $\lambda \in \mathbb{C}$) satisfies for all $t \in \mathbb{R}$ the linear equation

$$f_t = 2\lambda u_x f + \lambda u f_x. \tag{6.292}$$

Proof. It follows directly from (6.290)-(6.292) that

$$\begin{aligned}(f/h)_t &= f_t/h - fh_t/h^2 = f_t/h - \lambda f u_x/h - \lambda f u h_x/h^2 \\ &= f_t/h + \lambda f u(1/h)_x - \lambda u_x f/h \\ &= \lambda(uf)_x/h + \lambda uf(1/h)_x = \lambda(uf/h)_x,\end{aligned} \tag{6.293}$$

which proves the proposition. $\square$

There is a rather obvious generalization of this result having, except for some very minor modifications, essentially the same proof; namely,

Proposition 6.2. *If a smooth function* $h \in C^{\infty}(\mathbb{R};\mathbb{R})$ *satisfies*

$$h_t = ku_x h + uh_x, \tag{6.294}$$

where $k \in \mathbb{R}$, *then*

$$H = \int_0^{2\pi} h^{1/k} dx \tag{6.295}$$

is a conservation law for the Riemann system (6.289).

Remark 6.2. Let $\hat{h} \in C^{\infty}(\mathbb{R};\mathbb{R})$ satisfy

$$\hat{h}_t = (\hat{h}u)_x,$$

then $f = \hat{h}^2$ is a solution to equation (6.291).

Remark 6.3. If functions $h_j \in C^{\infty}(\mathbb{R};\mathbb{R}), j \in \mathbb{Z}_+$, satisfy the relationships

$$h_{j,t} = \lambda(h_j u)_x, \quad j \in \mathbb{Z}_+, \quad \lambda \in \mathbb{C},$$

then the functionals

$$H_{(i,j)} = \sum_{n \in \mathbb{Z}_+} k_n^{(i,j)} \int_0^{2\pi} h_i^{2^n} h_j^{(1-2^n)} \tag{6.296}$$

with $k_n^{(i,j)} \in \mathbb{R}$, $n \in \mathbb{Z}_+$, $i, j \in \mathbb{Z}_+$, arbitrary constants, are conserved quantities for equation (6.289). This formula, in particular, makes it possible to construct an infinite hierarchy of non-polynomial conservation laws for the Riemann hydrodynamic system (6.289). Moreover, it is worth noting that there are several ways of modifying this approach to obtain infinite hierarchies of non-polynomial constants of motion for other types of infinite-dimensional Hamiltonian dynamical systems.

Example 6.1. The non-polynomial functionals

$$H_4^{\frac{1}{3}} = \int_0^{2\pi} \sqrt{u_x^2 - 2v_x}dx, \qquad H_7^{\frac{1}{3}} = \int_0^{2\pi} (u_x v_{xx} - u_{xx} v_x)^{1/3}\, dx,$$

$$H_7^{\frac{1}{2}} = \int_0^{2\pi} \sqrt{v(u_x^2 - 2v_x)}dx,$$

$$H_8^{\frac{1}{3}} = \int_0^{2\pi} \left(k_1 u(u_{xx}v_x - u_x v_{xx}) + k_1 v_{xx} v_x + k_2(u_x^2 v - 2v_x^2)\right)^{1/3} dx, \tag{6.297}$$

$$H_9^{\frac{1}{6}} = \int_0^{2\pi} (u_{xx}v_{xxx} - u_{xxx}v_{xx})^{\frac{1}{6}}\, dx,$$

$$H_9^{\frac{1}{4}} = \int_0^{2\pi} (u_x(u_{xx}v_x - u_x v_{xx}) + v_{xx}v_x))^{\frac{1}{4}}\, ,$$

$$H_{10}^{\frac{1}{6}} = \int_0^{2\pi} \left(2u_{xx}(u_x v_{xx} - u_{xx}v_x) - v_{xx}^2\right)^{\frac{1}{6}}$$

are conservation laws for the Riemann dynamical system (6.289).

Now to our analysis of the Hamiltonian properties of the dynamical system (6.289), for which we will search for solutions to the determining [54, 326] Nöther equation

$$L_K\vartheta = \vartheta_t - \vartheta K^{\prime,*} - K^\prime \vartheta = 0, \tag{6.298}$$

where L_K denotes the corresponding Lie derivative on $\mathcal{M}$ subject to the vector field $K : \mathcal{M} \to T(\mathcal{M})$, $K' : T(\mathcal{M}) \to T(\mathcal{M})$ is its Fréchet derivative, $K'^* : T^*(\mathcal{M}) \to T^*(\mathcal{M})$ is its conjugation with respect to the standard bilinear form $(\cdot,\cdot)$ on $T^*(\mathcal{M})\times T(\mathcal{M})$, and $\vartheta : T^*(\mathcal{M}) \to T(\mathcal{M})$ is a suitable implectic operator on $\mathcal{M}$, with respect to which the following Hamiltonian representation

$$K = -\vartheta\, \operatorname{grad} H_\vartheta \tag{6.299}$$

holds for some smooth functional $H_\vartheta \in \mathcal{D}(\mathcal{M})$. To show this, it suffices to find, via the small parameter method [174, 326], a non-symmetric ($\psi' \neq \psi^{\prime,*}$) solution $\psi \in T^*(\mathcal{M})$ to the Lie–Lax equation

$$\psi_t + K^{\prime,*}\psi = \operatorname{grad} \mathcal{L} \tag{6.300}$$

for a suitably chosen smooth functional $L \in \mathcal{D}(M)$. It is easy to compute that

$$\psi = (v, 0)^{\intercal}, \quad L = \frac{1}{2}\int_0^{2\pi} v^2 dx. \tag{6.301}$$

Making use of (6.300) together with the classical Legendre relationship

$$H_\vartheta := (\psi, K) - \mathcal{L} \tag{6.302}$$

for the suitable Hamiltonian function, one easily obtains the corresponding symplectic structure

$$\vartheta^{-1} := \psi' - \psi'^{,*} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{6.303}$$

and the nonsingular Hamiltonian function

$$H_\vartheta := \frac{1}{2}\int_0^{2\pi} (v^2 + v_x u^2) dx. \tag{6.304}$$

Since the operator (6.303) is nonsingular, we obtain the corresponding implectic operator

$$\vartheta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{6.305}$$

which necessarily satisfies the Nöther equation (6.298).

Observe that the Lie–Lax equation (6.300) has another solution,

$$\psi = (\frac{u_x}{2}, -\frac{u_x^2}{2v_x}), \ \mathcal{L} = \frac{1}{4}\int_0^{2\pi} u v_x dx, \tag{6.306}$$

giving rise to, owing to expressions (6.303) and (6.302), the new co-implectic (singular symplectic) structure

$$\eta^{-1} := \psi' - \psi'^{,*} = \begin{pmatrix} \partial & -\partial u_x v_x^{-1} \\ -u_x v_x^{-1}\partial & u_x^2 v_x^{-2}\partial + \partial u_x^2 v_x^{-2} \end{pmatrix} \tag{6.307}$$

on the manifold $\mathcal{M}$, subject to which the Hamiltonian functional equals

$$H_\eta := \frac{1}{2}\int_0^{2\pi} (u_x v - v_x u) dx, \tag{6.308}$$

which supplies the second Hamiltonian representation

$$K = -\eta \operatorname{grad} H_\eta \tag{6.309}$$

of the Riemann type hydrodynamic system (6.289). The co-implectic structure (6.307) is singular, since $\hat{\eta}^{-1}(u_x, v_x)^{\intercal} = 0$, nonetheless one can calculate its inverse expression

$$\eta := \begin{pmatrix} -\partial^{-1} & u_x \partial^{-1} \\ \partial^{-1} u_x & v_x \partial^{-1} + \partial^{-1} v_x \end{pmatrix}. \tag{6.310}$$

Moreover, the corresponding implectic structure $\eta : T^*(\mathcal{M}) \to T^*(\mathcal{M})$ satisfies the determining Nöther equation

$$L_K\eta = \eta_t - \eta K'^{,*} - K'\eta = 0, \tag{6.311}$$

whose solutions can also be obtained by means of the small parameter method [261, 326]. We remark also that, owing to the general symplectic theory results [54, 174, 261, 326] for nonlinear dynamical systems on smooth functional manifolds, the operator (6.307) defines on the manifold $\mathcal{M}$ a closed functional differential two-form. Hence, it is co-implectic (in general, singular symplectic), satisfying the standard Jacobi identity on $\mathcal{M}$.

As a result, the second implectic operator (6.310), since it is compatible [54, 326] with the implectic operator (6.305), gives rise to a new infinite hierarchy of polynomial conservation laws, namely

$$\gamma_n := \int_0^1 d\lambda < (\vartheta^{-1}\eta)^n \text{grad } H_\vartheta[u\lambda\}, u > \tag{6.312}$$

for all $n \in \mathbb{Z}_+$. Defining, as usual, the recursion operator $\Lambda := \vartheta^{-1}\eta$, one also finds from (6.312), (6.298) and (6.311) that the Lax relationship

$$L_K\Lambda = \Lambda_t - [\Lambda, K^{',*}] = 0 \tag{6.313}$$

holds. It is now easy to find the asymptotic expansion

$$\varphi(x;\lambda) \sim \sum_{j\in\mathbb{Z}_+} \lambda^{1-2j} \text{grad}\gamma_{j-1}[u,v]$$

as $\lambda \to \infty$, and then to see from (6.312) that

$$\lambda^2\vartheta\varphi(x;\lambda) = \eta\varphi(x;\lambda) \tag{6.314}$$

holds. This, making use of the implectic operators (6.305) and (6.310), can be represented in the following two factorized forms:

$$\varphi(x;\lambda) := \begin{pmatrix} \varphi_1(x;\lambda) \\ \varphi_2(x;\lambda) \end{pmatrix} = \begin{pmatrix} -4\lambda^3 f_1^2 + 2\lambda v_x f_2^2 \\ -4\lambda^2 f_1 f_2 - 2\lambda u_x f_2^2 \end{pmatrix} = \begin{pmatrix} -2\lambda(f_1 f_2)_x \\ -(f_2^2)_x \end{pmatrix}, \tag{6.315}$$

where a vector $f \in C^\infty(\mathbb{R};\mathbb{C}^2)$ lies in a vector bundle $\mathcal{L}(\mathcal{M};\mathbb{E}^2)$, (associated to a manifold $\mathcal{M}$) whose fibers are isomorphic to the complex Euclidean vector space $\mathbb{E}^2$. Now we note [174, 326] that the Lie–Lax equation

$$L_K\varphi(x;\lambda) = d\varphi(x;\lambda)/dt + K'^{,*}\varphi(x;\lambda) = 0 \tag{6.316}$$

can be transformed for all $x \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ into the evolution system

$$D_t\varphi = \begin{pmatrix} 0 & v_x \\ -1 & -u_x \end{pmatrix}\varphi, \quad D_t = \partial/\partial t + u\partial/\partial x. \tag{6.317}$$

The equation (6.317), owing to the relationship (6.314) and the obvious identity

$$D_t f_x + u_x f_x = (D_t f)_x, \tag{6.318}$$

can be further split into the adjoint to (6.335) system

$$D_t f = q(\lambda) f, \;\; q(\lambda) := \begin{pmatrix} 0 & 0 \\ -\lambda & 0 \end{pmatrix}, \tag{6.319}$$

where a vector $f \in C^\infty(\mathbb{R}^2; \mathbb{C}^2)$, satisfies the following linear equation

$$f_x = \ell[u,v;\lambda] f, \;\; \ell[u,v;\lambda] := \begin{pmatrix} -\lambda u_x & -v_x \\ 2\lambda^2 & \lambda u_x \end{pmatrix}, \tag{6.320}$$

compatible with (6.319). Moreover, as a result of (6.319) and (6.287), the general solution to (6.320) has the following functional representation:

$$\begin{aligned} f_1(x,t) &= \tilde{g}_1(u - tv, x - tu + vt^2/2), \\ f_2(x,t) &= -t\lambda \tilde{g}_1(u - tv, x - tu + vt^2/2) \\ &\quad + \tilde{g}_2(u - tv, x - tu + vt^2/2), \end{aligned} \tag{6.321}$$

where $\tilde{g}_j \in C^\infty(\mathbb{R}^2; \mathbb{C}), j = 1, 2$, are arbitrary smooth complex-valued functions. Now combining together the relationships (6.319) and (6.320), we have the following proposition.

Proposition 6.3. *The Riemann hydrodynamic system (6.289) is a completely integrable bi-Hamiltonian flow on the functional manifold $\mathcal{M}$, with Lax representation*

$$\begin{gathered} f_x = \ell[u,v;\lambda] f, \quad f_t = p(\ell) f, \qquad p(\ell) := -u\ell[u,v;\lambda] + q(\lambda), \\ \ell[u,v;\lambda] := \begin{pmatrix} -\lambda u_x & -v_x \\ 2\lambda^2 & \lambda u_x \end{pmatrix}, \;\; q(\lambda) := \begin{pmatrix} 0 & 0 \\ -\lambda & 0 \end{pmatrix}, \\ p(\ell) = \begin{pmatrix} \lambda u_x u & v_x u \\ -\lambda - 2\lambda^2 u & -\lambda u_x u \end{pmatrix}, \end{gathered} \tag{6.322}$$

where $f \in C^\infty(\mathbb{R}^2; \mathbb{C}^2)$ and $\lambda \in \mathbb{C}$ is an arbitrary spectral parameter.

Remark 6.4. We mention here that equation (6.319) is equivalent on the solution set of the Riemann type hydrodynamic system (6.289) to the single equation

$$D_t^2 f_2 = 0 \Longleftrightarrow D_t f_1 = 0, D_t f_2 = -\lambda f_1, \tag{6.323}$$

where the vector $f \in C^\infty(\mathbb{R}^2; \mathbb{C}^2)$ satisfies for all $\lambda \in \mathbb{C}$ the compatibility condition (6.320) and whose general solution is represented in the functional form (6.321).

The conservation laws $\{H_0^{(1/2)}, H_1^{(1/2)}\}$ constructed above can be extended to an infinite hierarchy of invariants

$$\{H_j^{(1/2)} \in \mathcal{D}(\mathcal{M}) : j \in \mathbb{Z}_+ \cup \{-1\}\},$$

where

$$H_j^{(1/2)} := \int_0^{2\pi} \sigma_{2j+1}[u,v]dx, \tag{6.324}$$

and the affine generating function

$$\sigma(x;\lambda) := \frac{d}{dx}\log f_2(x;\lambda) \sim \sum_{j=\mathbb{Z}_+\cup\{-1\}} \sigma_j[u,v]\lambda^{-j},$$

as $\lambda \to \infty$, satisfies the functional equation

$$(\sigma - \lambda u_x)_x + \sigma^2 + \lambda^2(2v_x - u_x^2) = 0. \tag{6.325}$$

In addition, the gradient functional

$$\varphi(x;\lambda) := \operatorname{grad}\gamma(x;\lambda) \in T^*(\mathcal{M}),$$

where

$$\gamma(\lambda) := \int_0^{2\pi} \sigma(x;\lambda)dx,$$

satisfies the gradient relationship (6.314) for all $\lambda \in \mathbb{C}$.

6.5.3 *The generalized Riemann hydrodynamic equation for $N = 3$: Conservation laws, bi-Hamiltonian structure and Lax representation*

Next we conduct a detailed analysis of the conservation laws and bi-Hamiltonian structure of the generalized Riemann hydrodynamic equation (6.284) for $N = 3$, which has the form

$$\begin{pmatrix} u_t \\ v_t \\ z_t \end{pmatrix} = K[u,v,z] := \begin{pmatrix} v - uu_x \\ z - uv_x \\ -uz_x \end{pmatrix}, \tag{6.326}$$

where $K : \mathcal{M} \to T(\mathcal{M})$ is a vector field on the periodic functional manifold $\mathcal{M} := C^\infty(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^3)$ and $t \in \mathbb{R}$ is an evolution parameter. The system (6.326) also possesses infinite hierarchies of polynomial conservation laws, which strongly suggest complete and Lax integrability.

In particular, the following polynomial functionals are conserved with respect to the flow (6.326):

$$
\begin{aligned}
H_n^{(1)} &:= \int_0^{2\pi} dx z^n (vu_x - v_x u - \frac{n+2}{n+1} z), \qquad (6.327)\\
H^{(4)} &:= \int_0^{2\pi} dx[-7v_x v^2 u + z(6zu + 2v_x u^2 - 3v^2 - 4vuu_x)],\\
H^{(5)} &:= \int_0^{2\pi} dx(z^2 u_x - 2zvv_x), H^{(6)} := \int_0^{2\pi} dx(z_z v^3 + 3z^2 v_x u + z^3),\\
H^{(7)} &:= \int_0^{2\pi} dx(z_x v^3 + 3z^2 vu_x - 3z^3),\\
H^{(8)} &:= \int_0^{2\pi} dxz(6z^2 u + 3zv_x u^2 - 3zv^2 - 4zvu_x - 2v_x v^2 u + 2v^3 u_x),\\
H^{(9)} &:= \int_0^{2\pi} dx[1001 v_x v^4 u + (1092 z^2 u^2 + 364 z v_x u^3\\
&\quad - 1092 z v^2 u - 728 z v u_x u^2 - 364 v_x v^2 u^2 + 273 v^4 + 728 v^3 u_x u]),\\
H_n^{(2)} &:= \int_0^{2\pi} dx z_x v z^n, \quad H_n^{(3)} := \int_0^{2\pi} dx z_x (v^2 - 2zu)^n,
\end{aligned}
$$

where $n \in \mathbb{Z}_+$. Hence, for $n = 1, 2, ...,$ from (6.327) one computes that

$$
\begin{aligned}
H_0^{(2)} &:= \int_0^{2\pi} dx z_x v, \quad H_1^{(2)} := \int_0^{2\pi} dx z_x z v, ..., \qquad (6.328)\\
H_1^{(3)} &:= \int_0^{2\pi} dx z_x (v^2 - 2uz),\\
H_2^{(3)} &:= \int_0^{2\pi} dx z_x (v^4 + 4z^2 u^2 - 4zv^2 u), ...,
\end{aligned}
$$

and so on.

Making use of the recursion, as above in Proposition 6.1, one can construct the following hierarchy of non-polynomial dispersive and dispersionless conservation laws:

$$
\begin{aligned}
H_1^{(1/4)} &= \int_0^{2\pi} dx(-2u_{xx}u_x z_x + u_{xx}v_x^2 + 2u_x^2 z_{xx} \\
&\quad - u_x v_{xx} v_x + 3v_{xx}z_x - 3v_x z_{xx})^{1/4}, \\
H_2^{(1/3)} &= \int_0^{2\pi} dx(-v_{xx}z_x + v_x z_{xx})^{1/3}, \\
H_3^{(1/3)} &= \int_0^{2\pi} dx(v_{xx}u_x - v_x u_{xx} - z_{xx})^{1/3}, \\
H_1^{(1/2)} &= \int_0^{2\pi} dx[-2vu_x z_x + v_x^2 + z(-u_x v_x + 3z_x)]^{1/2}, \\
H_2^{(1/2)} &= \int_0^{2\pi} dx(8u_x^3 z_x - 3u_x^2 v_x^2 - 18u_x v_x z_x + 6v_x^3 + 9z_x)^{1/2}, \qquad (6.329)\\
H_1^{(1/5)} &= \int_0^{2\pi} dx(-2u_{xxx}u_x z_x + u_{xxx}v_x^2 + 6u_{xx}^2 z_x - 6u_{xx}u_x z_{xx} \\
&\quad - 3u_{xx}v_{xx}v_x + 2u_x^2 z_{xxx} - u_x v_{xxx} v_x \\
&\quad + 3u_x v_{xx}^2 + 3v_{xxx}z_x - 3v_x z_{xxx})^{1/5}, \\
H_{3,k}^{(1/3)} &= \int_0^{2\pi} dx[k_1 u(-v_{xx}z_x + v_x z_{xx}) + k_1 v(u_{xx}z_x - u_x z_{xx}) \\
&\quad + z(k_2 u_{xx}v_x - k_2 u_x v_{xx} + k_1 z_{xx} + k_2 z_{xx}) \\
&\quad + k_3(-3u_x v_x z_x + v_x^3 + 3z_x^2)]^{1/3},
\end{aligned}
$$

where $k_j \in \mathbb{R}, j = 1,2,3$, are arbitrary real numbers. Next, we shall generalize the crucial relationship (6.319) to the Riemann hydrodynamic system (6.326). Namely, we assume, based on Remark 6.2, that there exists the linearization

$$D_t^3 f_3(\lambda) = 0, \qquad (6.330)$$

modeling the generalized Riemann hydrodynamical equation (6.284) for $N = 3$, and where $f_3(\lambda) \in C^\infty(\mathbb{R}^2;\mathbb{C})$ for all values of the parameter $\lambda \in \mathbb{C}$. The scalar equation (6.330) can be easily rewritten as the system of three linear equations

$$D_t f_1 = 0, \quad D_t f_2 = \mu_1 f_1, \quad D_t f_3 = \mu_2 f_2 \qquad (6.331)$$

where we have defined a vector $f := (f_1, f_2, f_2)^\intercal \in C^\infty(\mathbb{R}^2;\mathbb{C}^3)$ and naturally introduced constants $\mu_j := \mu_j(\lambda) \in \mathbb{C},\ \ j = 1,2$. It is easy to observe now that, owing to the result (6.287), the system of equations (6.331) allows the following solution representation:

$$
\begin{aligned}
f_1(x,t) &= \tilde{g}_1(u - tv + zt^2/2, v - zt, x - tu + vt^2/2 - zt^3/6), \\
f_2(x,t) &= t\mu_1\tilde{g}_1(u - tv + zt^2/2, v - zt, x - tu + vt^2/2 - zt^3/6) \\
&\quad + \tilde{g}_2(u - tv + zt^2/2, v - zt, x - tu + vt^2/2 - zt^3/6), \qquad (6.332) \\
f_3(x,t) &= \mu_1\mu_2\frac{t^2}{2}\tilde{g}_1(u - tv + zt^2/2, v - zt, x - tu + vt^2/2 - zt^3/6) \\
&\quad + t\mu_2\tilde{g}_2(u - tv + zt^2/2, v - zt, x - tu + vt^2/2 - zt^3/6) \\
&\quad + \tilde{g}_3(u - tv + zt^2/2, v - zt, x - tu + vt^2/2 - zt^3/6),
\end{aligned}
$$

where $\tilde{g}_j \in C^\infty(\mathbb{R}^3;\mathbb{C}), j = 1,2,3,$ are arbitrary smooth complex-valued functions. The system (6.331) transforms into the equivalent vector equation

$$
D_t f = q(\mu)f, \quad q(\lambda) := \begin{pmatrix} 0 & 0 & 0 \\ \mu_1(\lambda) & 0 & 0 \\ 0 & \mu_2(\lambda) & 0 \end{pmatrix}, \qquad (6.333)
$$

which ought to be compatible both with a suitably chosen equation for the derivative

$$
f_x = \ell[u,v,z;\lambda]f \qquad (6.334)
$$

with some matrix $\ell[u,v,z;\lambda] \in SL(3;\mathbb{C})$, defined on the functional manifold $\mathcal{M}$, and with the Lie-Lax equation (6.316), rewritten as

$$
D_t\varphi = \begin{pmatrix} 0 & v_x & z_x \\ -1 & -u_x & 0 \\ 0 & -1 & -u_x \end{pmatrix}\varphi, \quad D_t = \partial/\partial t + u\partial/\partial x, \qquad (6.335)
$$

where the vector $\varphi := \varphi(x;\lambda) \in T^*(\mathcal{M})$ is considered as the one factorized by means of a solution $f \in C^\infty(\mathbb{R};\mathbb{C}^3)$ to (6.334), satisfying the identity (6.318). Namely, it is assumed that the following trace-relationship

$$
\varphi = \operatorname{tr}(\Phi[\lambda;u,v,z]\ f \otimes f^\intercal) \qquad (6.336)
$$

holds for some vector valued matrix $\Phi[\lambda;u,v,z] \in \mathbb{E}^3 \otimes End\,\mathbb{E}^3$, defined on the manifold $\mathcal{M}$, where $\otimes$ denotes the standard tensor product.

Now, using the determining expressions (6.318), (6.336) and (6.333), one can find by straightforward calculations that $\mu_1(\lambda) = \lambda$, $\mu_2(\lambda) = 1$, $\lambda \in \mathbb{C}$, and the matrix representation of the derivative (6.334)

$$
\ell[u,v,z;\lambda] = \begin{pmatrix} \lambda u_x & -v_x & z_x \\ 3\lambda^2 & -2\lambda u_x & \lambda v_x \\ \lambda^2 r[u,v,z] & -3\lambda & \lambda u_x \end{pmatrix}, \qquad (6.337)
$$

compatible with equation (6.335), where a smooth map $r : \mathcal{M} \to \mathbb{R}$ satisfies the differential relationship

$$D_t r + u_x r = 6. \tag{6.338}$$

This has an extensive set $\mathcal{R}$ of different solutions, amongst which there are the following:

$$r \in \mathcal{R} := \{(v^4/z^3)_x/4, 3(2v_x - u_x^2)z_x^{-1}, \frac{2u_x^3 - 6u_x v_x + 9z_x}{2u_x z_x - v_x^2}, \tag{6.339}$$
$$(v_x v^3 - 3u_x v^2 z + u z_x(uz - v^2) + 6vz^2)z^{-3}\}.$$

Note that the third element from the set (6.339) allows the reduction $z = 0$ to the case $N = 2$. Thus, the resulting Lax representation for the Riemann type system (6.326) ensues in the form

$$f_x = \ell[u, v, z; \lambda]f, \quad f_t = p(\ell)f, \quad p(\ell) := -u\ell[u, v, z; \lambda] + q(\lambda),$$

$$\ell[u, v, z; \lambda] = \begin{pmatrix} \lambda u_x & -v_x & z_x \\ 3\lambda^2 & -2\lambda u_x & \lambda v_x \\ \lambda^2 r[u, v, z] & -3\lambda & \lambda u_x \end{pmatrix}, \quad q(\lambda) := \begin{pmatrix} 0 & 0 & 0 \\ \lambda & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{6.340}$$

$$p(\ell) = \begin{pmatrix} -\lambda u u_x & u v_x & -u z_x \\ -3u\lambda^2 + \lambda & 2\lambda u u_x & -\lambda u v_x \\ -\lambda^2 r[u, v, z]u & 1 + 3u\lambda & -\lambda u u_x \end{pmatrix},$$

where $f \in C^\infty(\mathbb{R}^2; \mathbb{C}^3)$ and $\lambda \in \mathbb{C}$ is a spectral parameter.

The next problem, which is of great interest, consists in proving that the generalized hydrodynamic system (6.326) is a completely integrable bi-Hamiltonian flow on the periodic functional manifold $\mathcal{M}$, as it was proved above for the system (6.289).

To tackle this along with determining the Hamiltonian structure of the dynamical system (6.326), it suffices, as in Section 6.2, to construct [326, 174] exact non-symmetric solutions to the Lie-Lax equation

$$\psi_t + K^{',*}\psi = \text{grad } \mathcal{L}, \quad \psi^{'} \neq \psi^{',*}, \tag{6.341}$$

for a functional $L \in D(\mathcal{M})$, where $\psi \in T^*(\mathcal{M})$ is, in general, a quasi-local vector, such that the system (6.326) allows the following Hamiltonian representation:

$$\begin{gathered} K[u, v, z] = -\eta \text{ grad } H[u, v, z], \\ H_\eta = (\psi, K) - \mathcal{L}, \quad \eta^{-1} = \psi^{'} - \psi^{',*}. \end{gathered} \tag{6.342}$$

As a test solution to (6.341), one can take

$$\psi = (u_x/2, 0, -z_x^{-1}u_x^2/2 + z_x^{-1}v_x)^\intercal, \qquad \mathcal{L} = \frac{1}{2}\int_0^{2\pi}(2z + vu_x)dx,$$

which gives rise to the co-implectic operator

$$\eta^{-1} := \psi' - \psi'^{,*} = \begin{pmatrix} \partial & 0 & -\partial u_x z_x^{-1} \\ 0 & 0 & \partial z_x \\ -u_x z_x^{-1}\partial & z_x\partial & \begin{array}{c} \frac{1}{2}(u_x^2 z_x^{-2}\partial + \partial u_x^2 z_x^{-2}) \\ -(v_x z_x^{-2}\partial + \partial v_x z_x^{-2}) \end{array} \end{pmatrix}. \tag{6.343}$$

This expression is not strictly invertible, as its kernel includes the translation vector field $d/dx : \mathcal{M} \to T(\mathcal{M})$ with components $(u_x, v_x, z_x)^\intercal \in T(\mathcal{M})$, that is $\eta^{-1}(u_x, v_x, z_x)^\intercal = 0$.

Nonetheless, upon formally inverting the operator expression (6.343), it is straightforward to compute that the Hamiltonian function is

$$H_\eta := \int_0^{2\pi} dx(u_x v - z) \tag{6.344}$$

and the implectic η-operator has the form

$$\eta := \begin{pmatrix} \partial^{-1} & u_x\partial^{-1} & 0 \\ \partial^{-1}u_x & v_x\partial^{-1} + \partial^{-1}v_x & \partial^{-1}z_x \\ 0 & z_x\partial^{-1} & 0 \end{pmatrix}. \tag{6.345}$$

In the same way, representing the Hamiltonian function (6.344) in the scalar form

$$H_\eta = (\psi, (u_x, v_x, z_x)^\intercal), \quad \psi = \frac{1}{2}(-v,\ u + ..., -\frac{1}{\sqrt{z}}\partial^{-1}\sqrt{z})^\intercal, \tag{6.346}$$

one can construct a second implectic (co-symplectic) operator $\vartheta : T^*(\mathcal{M}) \to T(\mathcal{M})$, given up to $O(\mu^2)$ terms, as follows:

$$\vartheta = \begin{pmatrix} \mu(\frac{(u^{(1)})^2}{z^{(1)}}\partial + \partial\frac{(u^{(1)})^2}{z^{(1)}}) & 1 + \frac{2\mu}{3}(\partial\frac{u^{(1)}v^{(1)}}{z^{(1)}} + 2\partial\frac{u^{(1)}v^{(1)}}{z^{(1)}}) & \frac{2\mu}{3}(\partial\frac{(v^{(1)})^2}{z^{(1)}} + \partial u^{(1)}) \\ -1 + \frac{2\mu}{3}(\partial\frac{u^{(1)}v^{(1)}}{z^{(1)}} + 2\frac{u^{(1)}v^{(1)}}{z^{(1)}}\partial) & \frac{2\mu}{3}\left((\frac{(v^{(1)})^2}{z^{(1)}} + u^{(1)})\partial + \partial(\frac{(v^{(1)})^2}{z^{(1)}} + u^{(1)})\right) & 2\mu v^{(1)}\partial \\ \frac{2\mu}{3}(\frac{(v^{(1)})^2}{z^{(1)}}\partial + u^{(1)}\partial) & 2\mu v^{(1)}\partial & \mu(\partial z^{(1)} + z^{(1)}\partial) \end{pmatrix} + O\left(\mu^2\right), \tag{6.347}$$

where $\vartheta^{-1} := (\psi' - \psi'^{,*})$, $u := \mu u^{(1)}, v := \mu v^{(1)}, z := \mu z^{(1)}$ as $\mu \to 0$, and whose exact form needs some additional simple but lengthy calculations, which we omit.

The operator (6.347) satisfies the Hamiltonian vector field condition

$$(u_x, v_x, z_x)^{\intercal} = -\vartheta \text{ grad } H_\eta, \tag{6.348}$$

which follows directly from (6.346).

Remark 6.5. The infinite hierarchy of conservation laws (6.329) and related recursive relationships can be regularly reconstructed from the asymptotic solutions to the Lie–Lax equation

$$\begin{aligned} L_{\tilde{K}}\tilde{\varphi} &= \tilde{\varphi}_\tau + \tilde{K}^{\prime,*}\tilde{\varphi} = 0, \\ \tilde{\varphi} &\sim \tilde{a}(x;\lambda)\exp\{\lambda^2\tau + \partial^{-1}\tilde{\sigma}(x;\lambda)\}, \end{aligned} \tag{6.349}$$

where $\tilde{a}(x;\lambda) \sim \sum_{j\in\mathbb{Z}_+} \tilde{a}_j[u,v,z]\lambda^{-j}$, $\tilde{\sigma}(x;\lambda) \sim \sum_{j\in\mathbb{Z}_+\cup\{-1\}} \tilde{\sigma}_j[u,v,z]\lambda^{-j}$ as $\lambda \to \infty$, and

$$\begin{aligned} \frac{d}{d\tau}(u,v,z)^{\intercal} &= \tilde{K}[u,v,z] := -3\eta \text{ grad } H_3^{(1/3)}[u,v,z] \\ &= \left(-\left(\left(\frac{u_x}{h}\right)^2\right)_x + v_x^{-1}\left(\left(\frac{v_x}{h}\right)^2\right)_x, -\frac{v_x}{u_x}\left(\left(\frac{u_x}{h}\right)^2\right)_x \right. \\ &\qquad \left. + z_x^{-1}\left(\left(\frac{z_x}{h}\right)^2\right)_x, -\frac{z_x}{u_x}\left(\left(\frac{z_x}{h}\right)^2\right)_x\right)^\top \\ H_3^{(1/3)} &:= \int_0^{2\pi} h[u,v,z]dx, \quad h[u,v,z] = (v_{xx}u_x - u_{xx}v_x - z_{xx})^{1/3}, \end{aligned} \tag{6.350}$$

is a Hamiltonian vector field on the functional manifold $\mathcal{M}$ with respect to a suitable evolution parameter $\tau \in \mathbb{R}$. Since the vector fields (6.350) and (6.326) commute on $\mathcal{M}$, the functionals $\tilde{H}_j^{(1/3)} := \int_0^{2\pi} \tilde{\sigma}_{j-2}[u,v,z]dx$, $j \in \mathbb{Z}_+$, will be functionally independent conservation laws for both systems.

The Lax integrability of the Riemann type hydrodynamical equation for $N = 2$ and $N = 3$, proved above, suggest that it is also integrable for arbitrary $N \in \mathbb{Z}_+$. To support this hypothesis, we will prove next that for $N = 4$ it is also equivalent to a Lax integrable bi-Hamiltonian dynamical system on the smooth 2π-periodic functional manifold $\mathcal{M} := C^\infty(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^4)$, and possesses infinite hierarchies of polynomial dispersionless and dispersive non-polynomial conservation laws.

Before launching our analysis of the case $N = 4$, it will be helpful to delve a bit more deeply into the cases that we have already treated. Making use of the expressions (6.342) and (6.346) in (6.348), one can readily derive the equations

$$\vartheta^{-1}\eta \text{ grad } H_\eta = -\vartheta^{-1}K := \varphi_\vartheta, \qquad \varphi_\vartheta' = \varphi_\vartheta^{\prime,*}. \tag{6.351}$$

Owing to the second equality of (6.351) and the classical homology relationship $\varphi_\vartheta = \text{grad } H_\vartheta$ for some function $H_\vartheta \in \mathcal{D}(\mathcal{M})$, one computes that

$$H_\vartheta = \int_0^1 (\varphi_\vartheta[su, sv, sz], (u, v, z)^\intercal) ds, \tag{6.352}$$

satisfying the Hamiltonian condition

$$K = -\vartheta \text{ grad } H_\vartheta. \tag{6.353}$$

Remark 6.6. The expression for the Hamiltonian function (6.352) can be easily calculated modulo the exact form of the element $\varphi_\vartheta \in T^*(\mathcal{M})$ and the co-implectic operator $\vartheta^{-1} : T(\mathcal{M}) \to T^*(\mathcal{M})$, constructed by formulae (6.351) and (6.347), respectively.

Therefore, we have the following result.

Proposition 6.4. *The Riemann hydrodynamical system for $N = 3$ is equivalent to a completely integrable bi-Hamiltonian flow on the functional manifold $\mathcal{M}$, allowing the Lax type representation (6.340) and the compatible pair of co-symplectic structures (6.345) and (6.347).*

6.5.4 *The hierarchies of conservation laws and their analysis*

Let us return to equations (6.349) and (6.350) of Remark 6.5 in which we concluded that the

$$\tilde{H}_j^{(1/3)} := \int_0^{2\pi} \tilde{\sigma}_{j-2}[u, v, z] dx, \tag{6.354}$$

$j \in \mathbb{Z}_+$, are functionally independent conservation laws for both these dynamical systems. Moreover, as one can verify by tedious but straightforward calculations, the $\tilde{H}_j^{(1/3)}, j \in \mathbb{Z}_+$, coincide up to constant coefficients with the conservation laws $H_j^{(1/3)}, j \in \mathbb{Z}_+$, given by suitable elements of (6.329). But here a question arises: how are they related to the Lax pair (6.340), which depends strongly on the r-solutions to the differential-functional equation (6.338)? To answer this question, it suffices to construct the corresponding hierarchy of conservation laws by making use of the standard Riccati procedure applied to the first equation of (6.340).

Namely, define

$$\partial f_3/\partial x := \sigma(x; \lambda) f_3, \quad f_2 := b(x; \lambda) f_3, \quad f_1 := a(x; \lambda) f_3, \tag{6.355}$$

where the following asymptotic expansions

$$\sigma(x;\lambda) \sim \sum_{j\geq -2}^{\infty} \sigma_j[u,v,z;r]\lambda^{-j}, a(x;\lambda) \sim \sum_{j\geq 2}^{\infty} a_j[u,v,z;r]\lambda^{-j}, \tag{6.356}$$
$$b(x;\lambda) \sim \sum_{j\geq 1}^{\infty} b_j[u,v,z;r]\lambda^{-j},$$

hold as $|\lambda| \to \infty$ and whose coefficients satisfy the sequences of differential-functional recurrence equations

$$\begin{gathered} \partial a_j/\partial x + \sum_k a_{j-k}\sigma_k = u_x a_{j+2} - v_x b_{j+1} + z_x \delta_{j,0}, \\ \partial b_j/\partial x + \sum_k b_{j-k}\sigma_k = 3a_{j+3} - 2u_x b_{j+2} + v_x \delta_{j,-1}, \\ \sigma_j = r a_{j+4} - 3b_{j+3} + u_x \delta_{j,-2}, \end{gathered} \tag{6.357}$$

for all integers $j+4 \in \mathbb{Z}_+$. Whence, we readily find the initial local functionals $\sigma_{-2}[u,v,z;r], a_2[u,v,z;r]$ and $b_1[u,v,z;r]$ by solving

$$\begin{aligned} \sigma_{-2} + 3b_1 - ra_2 &= u_x, \\ b_1(3u_x + ra_2 - 3b_1) - 3a_2 &= v_x, \\ a_2(ra_2 - 3b_1) + v_x b_1 &= z_x, \end{aligned} \tag{6.358}$$

which reduces to a single cubic equation on the local functional $\sigma_{-2}[u,v,z;r]$. Consequently, owing to (6.357), it is easy to recursively determine all the other functionals $\sigma_j[u,v,z;r], j \in \mathbb{Z}_+$. We obtain in this way an infinite hierarchy of functionals

$$\gamma_j^{(1/3)} := \int_0^{2\pi} \sigma_{j-2}[u,v,z;r]dx \tag{6.359}$$

for all $j \in \mathbb{Z}_+$, that lead, by virtue of the first equation of (6.355) and the second of (6.340), to the conservation laws of the dynamical system (6.326). Moreover, these conservation laws for $r := (v_x - u_x^2/6)z_x^{-1}$ coincide, up to constant coefficients, with those (6.354) constructed above. Similar calculations can also be performed for other r-solutions of (6.338), but we shall not present them here due to their complexity.

Remark 6.7. The Lax representation (6.355) and direct calculations imply that on the manifold $\mathcal{M}$ the well-known gradient-like identity (6.314)

$$\lambda^2 \vartheta\varphi(x;\lambda) = \eta\varphi(x;\lambda) \tag{6.360}$$

holds for the gradient functional $\varphi(x;\lambda) := \operatorname{grad} \lambda[u,v,z;r] \in T^*(\mathcal{M})$, where the implectic operators $\eta, \vartheta : T^*(\mathcal{M}) \to T(\mathcal{M})$ coincide, for an $r \in \mathcal{R}$, with those given by expressions (6.345) and (6.347).

The considerable difference between the analytical properties of the cases $N = 2$ and $N = 3$ is clearly related to the structures of the corresponding Lax operators (6.320) and (6.340): In the first case, the corresponding r-equation (6.338) is trivial (empty), but in the second case it is nontrivial and has many different solutions. This situation generalizes, as we shall see in the following sections, to the cases $N \geq 4$, which essentially explains the diversity of related Lax representations.

6.5.5 *Generalized Riemann hydrodynamic equation for $N = 4$: Conservation laws, bi-Hamiltonian structure and Lax representation*

The Riemann hydrodynamic equation (6.284) for $N = 4$ is equivalent to the nonlinear dynamical system

$$\begin{pmatrix} u_t \\ v_t \\ w_t \\ z_t \end{pmatrix} = K[u,v,w,z] := \begin{pmatrix} v - uu_x \\ w - uv_x \\ z - uw_x \\ -uz_x \end{pmatrix}, \tag{6.361}$$

where $K : \mathcal{M} \to T(\mathcal{M})$ is a vector field on the smooth 2π-periodic functional manifold $\mathcal{M} := C^\infty(R/2\pi\mathbb{Z}; \mathbb{R}^4)$. To determine its Hamiltonian structure, we need to find an exact nonsymmetric functional solution $\psi \in T^*(\mathcal{M})$ to the Lie–Lax equation (6.341):

$$\psi_t + K^{',*}\psi = \text{grad } \mathcal{L} \tag{6.362}$$

for some smooth functional $\mathcal{L} \in \mathcal{D}(\mathcal{M})$, where

$$K' = \begin{pmatrix} -\partial u & 1 & 0 & 0 \\ -v_x & -u\partial & 1 & 0 \\ -w_x & 0 & -u\partial & 1 \\ -z_x & 0 & 0 & -u\partial \end{pmatrix}, \quad K'^* = \begin{pmatrix} u\partial & -v_x & -w_x & -z_x \\ 1 & \partial u & 0 & 0 \\ 0 & 1 & \partial u & 0 \\ 0 & 0 & 1 & \partial u \end{pmatrix} \tag{6.363}$$

are, respectively, the Fréchet derivative of the mapping $K : \mathcal{M} \to T(\mathcal{M})$ and its conjugate. Applying the small parameter method [326] to equation (6.362), we find that

$$\psi = (-w_x, v_x/2, 0, -\frac{v_x^2}{2z_x} + \frac{u_x w_x}{z_x})^\intercal, \tag{6.364}$$

$$\mathcal{L} = \int_0^{2\pi} (zu_x - vw_x/2)dx.$$

Accordingly it follows directly from (6.362) that the dynamical system (6.361) is a Hamiltonian on the functional manifold $\mathcal{M}$, that is

$$K = -\vartheta \text{ grad } H, \tag{6.365}$$

where the Hamiltonian functional is

$$H := (\psi, K) - \mathcal{L} = \int_0^{2\pi} (uz_x - vw_x)dx \tag{6.366}$$

and the co-implectic operator has the form

$$\vartheta^{-1} := \psi^{'} - \psi^{',*} = \begin{pmatrix} 0 & 0 & -\partial & \partial \frac{w_x}{z_x} \\ 0 & -u\partial & 0 & -\partial \frac{v_x}{z_x} \\ -\partial & 0 & 0 & \partial \frac{u_x}{z_x} \\ \frac{w_x}{z_x}\partial & -\frac{v_x}{z_x}\partial & \frac{u_x}{z_x}\partial & \begin{matrix} \frac{1}{2}[z_x^{-2}(v_x^2 - 2u_xw_x)\partial \\ +\partial(v_x^2 - 2u_xw_x)z_x^{-2}] \end{matrix} \end{pmatrix}. \tag{6.367}$$

This operator is degenerate, for $\vartheta^{-1}(u_x, v_x, w_x, z_x)^{\intercal}$ vanishes $\mathcal{M}$, but the inverse to (6.367) exists and can be calculated analytically.

To prove the Lax integrability of the Hamiltonian system (6.361), we apply the standard gradient-holonomic scheme of [174, 326] and find its linearization

$$D_t^4 f_4(\lambda) = 0, \tag{6.368}$$

where $f_4(\lambda) \in C^\infty(\mathbb{R}^2; \mathbb{C})$ for all $\lambda \in \mathbb{C}$. Recasting (6.368) in the form of the linear system

$$D_t f = q(\mu)f, \quad q(\lambda) := \begin{pmatrix} 0 & 0 & 0 & 0 \\ \mu_1(\lambda) & 0 & 0 & 0 \\ 0 & \mu_2(\lambda) & 0 & 0 \\ 0 & 0 & \mu_3(\lambda) & 0 \end{pmatrix}, \tag{6.369}$$

with $\mu_j(\lambda) \in \mathbb{C}, j = 1, 2, 3$ constant, it is easy to find, owing to the relationships (6.287), the following representation for $f \in C^\infty(\mathbb{R}^2;\mathbb{C}^4)$:

$$
\begin{aligned}
f_1(x,t) &= \tilde{g}_1(u - tv + wt^2/2 - xt^3/3!, v - wt + zt^2/2, w - zt,\\
&\quad x - tu + vt^2/2 - wt^3/3! + zt^4/4!),\\
f_2(x,t) &= t\mu_1\tilde{g}_1(u - tv + wt^2/2 - xt^3/3!, v - wt + zt^2/2, w - zt,\\
&\quad x - tu + vt^2/2 - wt^3/3! + zt^4/4!)\\
&\quad + \tilde{g}_2(u - tv + wt^2/2 - xt^3/3!, v - wt + zt^2/2, w - zt,\\
&\quad x - tu + vt^2/2 - wt^3/3! + zt^4/4!),\\
f_3(x,t) &= \mu_1\mu_2\frac{t^2}{2}\tilde{g}_1(u - tv + wt^2/2 - xt^3/3!, v - wt + zt^2/2, w - zt,\\
&\quad x - tu + vt^2/2 - wt^3/3! + zt^4/4!)\\
&\quad + t\mu_2\tilde{g}_2(u - tv + wt^2/2 - xt^3/3!, v - wt + zt^2/2, w - zt,\\
&\quad x - tu + vt^2/2 - wt^3/3! + zt^4/4!)\\
&\quad + \tilde{g}_3(u - tv + wt^2/2 - xt^3/3!, v - wt + zt^2/2, w - zt, \qquad (6.370)\\
&\quad x - tu + vt^2/2 - wt^3/3! + zt^4/4!),\\
f_4(x,t) &= \mu_1\mu_2\mu_3\frac{t^3}{3!}\tilde{g}_1(u - tv + wt^2/2 - xt^3/3!, v - wt + zt^2/2, w - zt,\\
&\quad x - tu + vt^2/2 - wt^3/3! + zt^4/4!)\\
&\quad + \mu_2\mu_3\frac{t^2}{2}\tilde{g}_2(u - tv + wt^2/2 - xt^3/3!, v - wt + zt^2/2, w - zt,\\
&\quad x - tu + vt^2/2 - wt^3/3! + zt^4/4!)\\
&\quad + t\mu_3\tilde{g}_3(u - tv + wt^2/2 - xt^3/3!, v - wt + zt^2/2, w - zt,\\
&\quad x - tu + vt^2/2 - wt^3/3! + zt^4/4!)\\
&\quad + \tilde{g}_4(u - tv + wt^2/2 - xt^3/3!, v - wt + zt^2/2, w - zt,\\
&\quad x - tu + vt^2/2 - wt^3/3! + zt^4/4!),
\end{aligned}
$$

where $\tilde{g}_j \in C^\infty(\mathbb{R}^4;\mathbb{C}), 1 \le j \le 4$, are arbitrary smooth complex-valued functions.

From the expressions (6.369) and (6.370), one can readily construct the related Lax representation for dynamical system (6.361) in the compatible form

$$f_x = \ell[u,v,w,z;\lambda]f,\ f_t = p(\ell)f,\ p(\ell) := -u\ell[u,v,w,z;\lambda] + q(\lambda), \quad (6.371)$$

where $f_x = \ell[u,v,w,z;\lambda]f$ for $f \in C^\infty(\mathbb{R}^2;\mathbb{C}^4)$ is a suitable matrix x-derivative representation; one whose exact form requires some simple but lengthy calculations that we omit. Owing to the existence of the Lax representation (6.371) and the related gradient-like relationship (6.314), we can easily show that the Hamiltonian system (6.361) is also a bi-Hamiltonian flow on $\mathcal{M}$. In addition, we can easily construct, using the results and approach above, both the dispersionless polynomial and dispersive non-polynomial infinite hierarchies of conservation laws for (6.361):

a) polynomial conservation laws:

$$H^{(9)} = \int_0^{2\pi} dx(vw_x - uz_x), \quad H^{(13)} = \int_0^{2\pi} dx(z_x w - zw_x), \tag{6.372}$$

$$H^{(14)} = \int_0^{2\pi} dx(k_1\Big(z_x(v^2 - 2uw) - z^2\Big) + k_2\Big(- z_x v^2 + 2w_x(vw - uz) - z^2\Big) + k_3\Big(2z_x v^2 + 4w_x(4z - vw) + 2z^2\Big)),$$

$$H^{(16)} = \int_0^{2\pi} dx(3uz - vw)z_x, \quad H^{18} = \int_0^{2\pi} dx z_x(w^2 - 2vz),$$

$$H^{(17)} = \int_0^{2\pi} dx[12v_x uzw + z_x(9u^2 z + 16uvw - 2v^3) + 6ww_x(v^2 - 2uw) + 6z(2vz - w^2)],$$

$$H^{(19)} = \int_0^{2\pi} dx[k_1\Big(10v_x uz^2 + z_x(12uvz - uw^2 - 2v^2 w) + 5ww_x(vw - 2uz)\Big) + k_2\Big(z_x(6uvz - 3uw^2 - v^2 w) + 5w_x v(w^2 - vz)\Big)];$$

b) non-polynomial conservation laws:

$$H^{(11)} = \int_0^{2\pi} dx\Big(u_{xx}z_x - u_x z_{xx} + v_x w_{xx} - v_{xx}w_{xx}\Big)^{\frac{1}{3}}, \tag{6.373}$$

$$H_2^{(12)} = \int_0^{2\pi} dx\sqrt{w_x^2 - 2v_x z_x},$$

$$H_1^{(12)} = \int_0^{2\pi} dx\Big(9u_x^2 z_x - 6u_x v_x w_x + 2v_x^3 - 12v_x z_x + 6w_x^2\Big)^{\frac{1}{3}},$$

$$H^{(13)} = \int_0^{2\pi} dx\Big(u(2v_x z_x - w_x^2) + v(v_x w_x - 3u_x z_x) + w(u_x w_x - v_x^2 + 2z_x) + z(u_x v_x - 2w_x)\Big)^{\frac{1}{3}},$$

$$H_2^{(15)} = \int_0^{2\pi} dx\Big(z_x w_{xx} - z_{xx} w_x\Big)^{\frac{1}{3}},$$

$$H_3^{(15)} = \int_0^{2\pi} dx\Big(k_1\big(v(2v_x z_x - w_x^2) + z(4z_x - u_x w_x) + w(v_x w_x - 3u_x z_x)\big) + k_2 z\big(2z_x + v_x^2 - u_x w_x\big)\Big)^{\frac{1}{2}},$$

$$H_1^{(16)} = \int_0^{2\pi} dx\Big(u_{xxx}(2v_x z_x - w_x^2) + v_{xxx}(v_x w_w - 3u_x z_x) + z_{xxx}(u_x v_x - 2w_x) + w_{xxx}(u_x w_x - v_x^2 + 2z_x) +3u_{xx}(v_{xx} z_x - 3v_x z_{xx} + w_{xx} w_x) + 3v_{xx}(2u_x z_{xx} - v_{xx} w_x + v_x w_{xx}) - 3w_{xx}^2 u_x\Big)^{\frac{1}{5}},$$

$$H_2^{(16)} = \int_0^{2\pi} dx\Big(4u_x^2 w_x^2 - 4u_x v_x^2 w_x - 8u_x z_x w_x + v_x^4 + 4v_x^2 z_x + 4z_x^2\Big)^{\frac{1}{4}},$$

$$H_3^{(16)} = \int_0^{2\pi} dx\Big(k_1\big(u(z_x w_{xx} - z_{xx} w_x) + v(v_x z_{xx} - v_{xx} z_x) + z z_{xx} + w(u_{xx} z_x - u_x z_{xx})\big) +k_2\Big(z(u_{xx} w_x - u_x w_{xx} + 2z_{xx}) + w(u_{xx} z_x - u_x z_{xx} - v_{xx} w_x + v_x w_{xx})\Big) + k_3 z_x(v_x^2 - 2u_x w_x + 2z_x)\Big)^{\frac{1}{3}},$$

where $k_j, j = 1, 2, 3$, are arbitrary constants. We observe also that the Hamiltonian functional (6.366) coincides exactly up the sign with the polynomial conservation law $H^{(9)} \in \mathcal{D}(\mathcal{M})$.

Concerning the general case $N \in \mathbb{Z}_+$, applying the method devised above, one can also obtain for the Riemann type hydrodynamic system (6.284) both infinite hierarchies of dispersive and dispersionless conservation laws, their symplectic structures and the related Lax representations.

6.5.6 *Summary conclusions*

As follows from the results obtained in this section, the generalized Riemann type hydrodynamic equation (6.284) possesses many infinite hierarchies of conservation laws, both dispersive non-polynomial and dispersionless polynomial. This can be easily explained by the fact that the corresponding dynamical system (6.285) allows many (plausibly infinitely many) infinite sets of algebraically independent compatible implectic structures, which generate via the standard gradient like relationship (6.312) the related infinite hierarchies of conservation laws, and as a by-product, infinite hierarchies of associated Lax representations. Such richness of structures and artifacts is quite rare in the theory of Lax integrable nonlinear dynamical systems, and is definitely worth further investigation.

6.6 Differential-algebraic integrability analysis of the generalized Riemann and KdV hydrodynamic equations

6.6.1 *Introduction*

As mentioned above, nonlinear hydrodynamic equations continue to be an active area of research. The recent approaches developed in the literature have proven to be quite useful. For example, much progress [150] has been made in analyzing the Holm–Pavlov generalized hydrodynamic system

$$D_t^N u = 0, \quad D_t := \partial/\partial t + u\partial/\partial x, \quad N \in \mathbb{Z}_+, \tag{6.374}$$

where $u \in \mathcal{M} \subset C^\infty(\mathbb{R}^2;\mathbb{R})$ is a smooth function. In addition, using spectral and symplectic techniques devised in [174, 261, 326], Lax representations in explicit form were obtained for the case $N = 3$ [150].

In this section, a new and very simple differential-algebraic approach [312] to analyzing Lax integrability of the generalized Riemann type hydrodynamic equations at $N = 3, 4$ is devised. It can be easily generalized for treating the equations with arbitrary $N \in \mathbb{Z}_+$. The approach is also applied to studying the Lax integrability of the KdV dynamical system. So, we shall revisit both the Riemann hydrodynamic and the KdV dynamical systems with this new approach.

6.6.2 *Differential-algebraic description of the Lax integrability of the generalized Riemann hydrodynamic equation for $N = 3$ and $N = 4$*

Before dealing with the details, we give a brief description of some fundamentals of the approach.

6.6.2.1 *Differential-algebraic preliminaries*

Consider the ring $\mathcal{K} := \mathbb{R}\{\{x,t\}\}$, $(x,t) \in \mathbb{R}^2$, of convergent germs of real-valued smooth functions from $C^\infty(\mathbb{R}^2;\mathbb{R})$ and construct [91, 189, 199, 361, 394] the associated differential polynomial ring $\mathcal{K}\{u\} := \mathcal{K}[\Theta u]$ with respect to a functional variable u, where Θ denotes the standard monoid of all operators generated by commuting differentiations $\partial/\partial x := D_x$ and $\partial/\partial t$. The ideal $I\{u\} \subset \mathcal{K}\{u\}$ is said to be [189, 199, 361] *differential* if the condition $I\{u\} = \Theta I\{u\}$ holds.

We add now the differentiation

$$D_t : \mathcal{K}\{u\} \to \mathcal{K}\{u\}, \tag{6.375}$$

depending on the functional variable u, which satisfies the Lie-algebraic commutator condition

$$[D_x, D_t] = (D_x u)D_x, \tag{6.376}$$

for all $(x,t) \in \mathbb{R}^2$. As a simple consequence of (6.376), the following general (suitably normalized) *representation* of the differentiation (6.375)

$$D_t = \partial/\partial t + u\partial/\partial x \tag{6.377}$$

in the differential ring $\mathcal{K}\{u\}$ holds. Next, we impose on the differentiation (6.375) a new algebraic constraint

$$D_t^N u = 0, \tag{6.378}$$

defining a smooth functional space (or *manifold*) $\mathcal{M}^{(N)}$ of functions $u \in \mathbb{R}\{\{x,t\}\}$, which allows a natural reduction of the initial ring $\mathcal{K}\{u\}$ to the basic ring $\mathcal{K}\{u\}|_{\mathcal{M}_{(N)}} \subseteq \mathbb{R}\{\{x,t\}\}$. In this case, the following problem of constructing the corresponding representation of the differentiation (6.375) arises: find an equivalent *equivalent* linear representation of the reduced differentiation $D_t|_{\mathcal{M}_{(N)}} : \mathbb{R}^{p(N)}\{\{x,t\}\} \to \mathbb{R}^{p(N)}\{\{x,t\}\}$ in the functional vector space $\mathbb{R}^{p(N)}\{\{x,t\}\}$ for some specially chosen integer dimension $p(N) \in \mathbb{Z}_+$.

As we shall show for the cases $N = 3$ and $N = 4$, this problem is completely analytically solvable, giving rise to the corresponding Lax integrability of the generalized Riemann hydrodynamic system (6.374). Moreover, the same problem is also solvable for the more complicated constraint

$$D_t u - D_x^3 u = 0, \tag{6.379}$$

which is equivalent to the Lax integrable nonlinear KdV dynamical system.

6.6.2.2 *The generalized Riemann hydrodynamic equation: the case $N = 3$*

To proceed with analyzing the representation problem for the generalized Riemann type equation (6.378) for $N = 3$, we first construct an adjoint to the differential ring $\mathcal{K}\{u\}$ that is invariant with respect to the differentiation (6.377), called the *Riemann differential ideal* $R\{u\} \subset \mathcal{K}\{u\}$ as

$$R\{u\} := \{\lambda^2 \sum_{n \in \mathbb{Z}_+} f_n^{(1)} D_x^n u - \lambda \sum_{n \in \mathbb{Z}_+} f_n^{(2)} D_t D_x^n u + \sum_{n \in \mathbb{Z}_+} f_n^{(3)} D_t^2 D_x^n u$$
$$: D_t^3 u = 0, f_n^{(k)} \in \mathcal{K}\{u\}, k = 1,2,3, n \in \mathbb{Z}_+\} \subset \mathcal{K}\{u\}, \tag{6.380}$$

where $\lambda \in \mathbb{R}$ is an arbitrary parameter, and state the easily proved but important result.

Lemma 6.5. *The kernel* $\ker D_t \subset R\{u\}$ *of the differentiation* $D_t : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$, *reduced modulo the Riemann differential ideal* $R\{u\} \subset \mathcal{K}\{u\}$, *is generated by elements satisfying the following linear functional-differential relationships:*

$$D_t f^{(1)} = 0, \quad D_t f^{(2)} = \lambda f^{(1)}, \quad D_t f^{(3)} = \lambda f^{(2)}, \tag{6.381}$$

where $f^{(k)} := f^{(k)}(\lambda) = \sum_{n \in \mathbb{Z}_+} f_n^{(k)} \lambda^n \in \mathcal{K}\{u\}|_{\mathcal{M}_{(3)}} = \mathbb{R}\{\{x,t\}\}$, $k = 1,2,3$, *and* $\lambda \in \mathbb{R}$ *is arbitrary.*

It is easy to see that equations (6.381) can be recast both in the matrix form as

$$D_t f = q(\lambda) f, \quad q(\lambda) := \begin{pmatrix} 0 & 0 & 0 \\ \lambda & 0 & 0 \\ 0 & \lambda & 0 \end{pmatrix}, \tag{6.382}$$

where $f := (f^{(1)}, f^{(2)}, f^{(3)})^\intercal \in \mathcal{K}^3\{u\}|_{\mathcal{M}_{(3)}}$, $\lambda \in \mathbb{R}$ is an arbitrary spectral parameter, and in the compact scalar form as

$$D_t^3 f_3 = 0 \tag{6.383}$$

for an element $f_3 \in \mathcal{K}\{u\}|_{\mathcal{M}_{(3)}}$. Note here that the Riemann differential ideal (6.380), satisfying the D_t-invariance condition, is in this case maximal. Now, using the relationship (6.383), we can construct by means of a new invariant, the so-called *Lax differential ideal* $L\{u\} \subset \mathcal{K}\{u\}$, isomorphic to the Riemann differential ideal $R\{u\} \subset \mathcal{K}\{u\}$ and realizing the Lax integrability condition of the Riemann type hydrodynamic equation (6.284). Namely, it follows directly from Lemma 6.5 that we have the result.

Proposition 6.5. *The expression (6.382) is an adjoint linear matrix representation in the space* $\mathbb{R}^3\{\{x,t\}\}$ *of the differentiation* $D_t : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$, *reduced to the ideal* $R\{u\} \subset \mathcal{K}\{u\}$. *The related* D_x- *and* D_t-*invariant Lax differential ideal* $L\{u\} \subset \mathcal{K}\{u\}$, *which is isomorphic to the invariant Riemann differential ideal* $R\{u\} \subset \mathcal{K}\{u\}$, *is generated by the element* $f_3(\lambda) \in \mathcal{K}\{u\}, \lambda \in \mathbb{R}$, *satisfying condition (6.383), and equals*

$$\begin{aligned} L\{u\} := \{ & g_1 f_3(\lambda) + g_2 D_t f_3(\lambda) + g_3 D_t^2 f_3(\lambda) : D_t^3 f_3(\lambda) = 0, \\ & \lambda \in \mathbb{R}, g_j \in \mathcal{K}\{u\}, j = 1,2,3\} \subset \mathcal{K}\{u\}. \end{aligned} \tag{6.384}$$

We now construct a related adjoint linear matrix representation in the functional vector space $\mathbb{R}^3\{\{x,t\}\}$ for the differentiation $D_x : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$, reduced modulo the Lax differential ideal $L\{u\} \subset \mathcal{K}\{u\}$. For this, we need to take into account the commutator relationship (6.376) and the important invariance condition of the Lax differential ideal $L\{u\} \subset \mathcal{K}\{u\}$ with respect to the differentiation $D_x : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$. As a result of simple but lengthy calculations, one obtains the following matrix representation.

$$D_x f = \ell[u,v,z;\lambda] f, \quad \ell[u,v,z;\lambda] := \begin{pmatrix} \lambda u_x & -v_x & z_x \\ 3\lambda^2 & -2\lambda u_x & \lambda v_x \\ 6\lambda^2 r[u,v,z] & -3\lambda & \lambda u_x \end{pmatrix}, \tag{6.385}$$

where $v := D_t u, z := D_t v$, $(...)_x := D_x(...)$, a vector $f \in \mathbb{R}^3\{\{x,t\}\}$, $\lambda \in \mathbb{R}$ is an arbitrary spectral parameter and a smooth functional mapping $r : \tilde{\mathcal{M}}_{(3)} \to \mathbb{R}\{\{x,t\}\}, \tilde{\mathcal{M}}_{(3)} := \sqcap_{j=1}^3 D_t^j \mathcal{M}_{(3)}$, solves the following functional-differential equation

$$D_t r + r D_x u = 1. \tag{6.386}$$

Moreover, the matrix $\ell := \ell[u,v,z;\lambda] : \mathbb{R}^3\{\{x,t\}\} \to \mathbb{R}^3\{\{x,t\}\}$ satisfies the determining functional-differential equation

$$D_t \ell + \ell D_x u = [q(\lambda), \ell], \tag{6.387}$$

where $[\cdot,\cdot]$ denotes the usual matrix commutator in the functional space $\mathbb{R}^3\{\{x,t\}\}$. The following proposition solving the representation problem can be easily verified.

Proposition 6.6. *The expression (6.385) is an adjoint linear matrix representation in the space* $\mathbb{R}^3\{\{x,t\}\}$ *of the differentiation* $D_x : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$, *reduced modulo the invariant Lax differential ideal* $L\{u\} \subset \mathcal{K}\{u\}$ *given by (6.384).*

Remark 6.8. It should be mentioned that the matrix representation (6.382) coincides with that in [150], which was derived by completely different methods, based mainly on the gradient-holonomic algorithm [174, 261, 326]. The derivation of these representations (6.382) and (6.385) is much easier and simpler and can be explained by a deeper insight into the integrability problem obtained using the differential algebraic approach.

Observe that the invariance condition for the Lax differential ideal $L\{u\} \subset \mathcal{K}\{u\}$ with respect to the differentiations $D_x, D_t : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$ is also equivalent to the related Lax representation for the generalized Riemann type equation (6.284) in the following dynamical system form:

$$\begin{pmatrix} u_t \\ v_t \\ z_t \end{pmatrix} = K[u,v,z] := \begin{pmatrix} v - uu_x \\ z - uv_x \\ -uz_x \end{pmatrix}. \tag{6.388}$$

The following theorem that can be readily proved using the above results, neatly summarizes the results so far in this section.

Theorem 6.10. *The linear differential-matrix expressions (6.382) and (6.385) in the space* $\mathbb{R}^3\{\{x,t\}\}$ *for differentiations* $D_t : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$ *and* $D_x : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$, *respectively, provide us with the standard Lax representation for the generalized Riemann type equation (6.284) in the equivalent dynamical system form (6.388), thereby implying its Lax integrability.*

Next we construct, making use of the differential-algebraic tools, the functional-differential solutions to the determining equation (6.391), and also the corresponding differential-algebraic analogs of the symplectic structures characterizing the differentiations $D_x, D_t : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$, as well as the local densities of the related conservation laws, which were derived in [150, 306].

6.6.2.3 *Solution set analysis of the functional-differential equation* $D_t r + r D_x u = 1$

We consider again the generalized Riemann type dynamical system (6.388) on a suitable 2π-periodic functional manifold $\mathcal{M}_{(3)} \subset \mathbb{R}^3\{\{x,t\}\}$:

$$\begin{pmatrix} u_t \\ v_t \\ z_t \end{pmatrix} = K[u,v,z] := \begin{pmatrix} v - uu_x \\ z - uv_x \\ -uz_x \end{pmatrix}, \tag{6.389}$$

which, as shown above and in [150, 306], has the Lax representation

$$f_x = \ell[u,v,z;\lambda]f, \quad f_t = p(\ell)f, \quad p(\ell) := -u\ell[u,v,z;\lambda] + q(\lambda),$$

$$\ell[u,v,z;\lambda] = \begin{pmatrix} \lambda^2 u_x & -\lambda v_x & z_x \\ 3\lambda^3 & -2\lambda^2 u_x & \lambda v_x \\ 6\lambda^4 r[u,v,z] & -3\lambda^3 & \lambda^2 u_x \end{pmatrix}, \quad q(\lambda) := \begin{pmatrix} 0 & 0 & 0 \\ \lambda & 0 & 0 \\ 0 & \lambda & 0 \end{pmatrix},$$

$$p(\ell) = \begin{pmatrix} -\lambda^2 uu_x & \lambda uv_x & -uz_x \\ -3u\lambda^3 + \lambda & 2\lambda^2 uu_x & -\lambda uv_x \\ -6\lambda^4 u\, r[u,v,z] & \lambda + 3u\lambda^3 & -\lambda^2 uu_x \end{pmatrix}, \tag{6.390}$$

where $f \in L_\infty(\mathbb{R};\mathbb{E}^3)$, $\lambda \in \mathbb{R}$ is an arbitrary spectral parameter and the function $r : \mathcal{M} \to \mathbb{R}$ satisfies the following functional-differential equation:

$$D_t r + r D_x u = 1 \tag{6.391}$$

under the commutator condition (6.376).

We now describe all functional solutions to equation (6.391), by making use of the lemma in [149, 306].

Lemma 6.6. *The functions*

$$B_0 = \xi(z), \ B_1 = u - tv + zt^2/2, \ B_2 = v - zt, \quad B_3 = x - tu + vt^2/2 - zt^3/6, \tag{6.392}$$

where $\xi : D_t^2 \mathcal{M} \to \mathbb{R}\{\{x,t\}\}$ *is an arbitrary smooth mapping, are the main invariants of the Riemann type dynamical system (6.389), satisfying the determining condition*

$$D_t B = 0. \tag{6.393}$$

As a simple consequence of the relationships (6.392), we obtain the following result.

Lemma 6.7. *The local functionals*

$$b_0 := \xi(z), b_1 := \frac{u}{z} - \frac{v^2}{2z^2}, b_2 := \frac{u_x}{z_x} - \frac{v_x^2}{2z_x^2}, b_3 := x - \frac{uv}{z} + \frac{v^3}{3z^2} \tag{6.394}$$

and

$$\tilde{b}_1 := \frac{v}{z}, \tilde{b}_2 := \frac{v_x}{z_x}$$

on the functional manifold $\tilde{\mathcal{M}}_{(3)}$ are, respectively, the basic functional solutions $b_j : \tilde{\mathcal{M}}_{(3)} \to \mathbb{R}\{\{x,t\}\}$, $0 \le j \le 3$, and $\tilde{b}_k : \tilde{\mathcal{M}}_{(3)} \to \mathbb{R}\{\{x,t\}\}, k = 1, 2$, to the determining functional-differential equations

$$D_t b = 0 \tag{6.395}$$

and

$$D_t \tilde{b} = 1. \tag{6.396}$$

Thus, we are led to the following theorem about the general solution set of the functional-differential equation (6.395).

Theorem 6.11. *The following infinite hierarchies*

$$\eta_{1,j}^{(n)} := (\alpha D_x)^n b_j, \quad \eta_{2,k}^{(n)} := (\alpha D_x)^{n+1} \tilde{b}_k, \tag{6.397}$$

where $\alpha := 1/z_x$, $0 \le j \le 3$, $k = 1, 2$ and $n \in \mathbb{Z}_+$, are the basic functional solutions to the functional-differential equation (6.395), that is

$$D_t \eta_{s,j}^{(n)} = 0 \tag{6.398}$$

for $s = 1, 2$, $0 \le j \le 3$ and all $n \in \mathbb{Z}_+$.

Proof. It suffices to observe that for any smooth solutions b and $\tilde{b}$: $\tilde{\mathcal{M}}_{(3)} \to \mathbb{R}\{\{x,t\}\}$ to functional-differential equations (6.395) and (6.396), respectively, the expressions $(\alpha D_x)b$ and $(\alpha D_x)\tilde{b}$ are solutions to the determining functional-differential equation (6.395). Iterating the operator αD_x, one obtains the desired result. □

Now we analyze the solution set to the functional-differential equation (6.391), making use of the transformation

$$r := \frac{a}{\alpha\eta}, \tag{6.399}$$

where $\eta : \tilde{\mathcal{M}}_{(3)} \to \mathbb{R}\{\{x,t\}\}$ is any solution to equation (6.398) and a smooth functional mapping $a : \mathcal{M}_{(3)} \to \mathbb{R}\{\{x,t\}\}$ satisfies the determining functional-differential equation

$$D_t a = \alpha\eta. \tag{6.400}$$

Then every solution to functional-differential equation (6.391) has the form

$$r = \frac{a}{\alpha\eta} + \eta_0, \tag{6.401}$$

where $\eta_0 : \tilde{\mathcal{M}}_{(3)} \to \mathbb{R}\{\{x,t\}\}$ is any smooth solution to the functional-differential equation (6.398).

To find solutions to equation (6.400), we use the following linear α-expansion in the corresponding Riemann differential ideal $R\{\alpha\} \subset \mathcal{K}\{\alpha\}$:

$$a = c_3 + c_0\alpha + c_1\dot{\alpha} + c_2\ddot{\alpha} \in R\{\alpha\}, \tag{6.402}$$

where $\dot{\alpha} := D_t\alpha, \ddot{\alpha} := D_t^2\alpha$ and take into account that all functions $\alpha, \dot{\alpha}$ and $\ddot{\alpha}$ are functionally independent owing to the fact that $\dddot{\alpha} := D_t^3\alpha = 0$. Substituting (6.402) into (6.400), we obtain

$$\dot{c}_1 + c_0 = 0,\ \dot{c}_0 = \eta,\ \dot{c}_2 + c_1 = 0,\ \dot{c}_3 + c_2 = 0. \tag{6.403}$$

Whence, by virtue of (6.396), we have for the special solution $\eta = 1$ to equation (6.398) two functional solutions for the map $c_0 : \tilde{\mathcal{M}}_{(3)} \to \mathbb{R}\{\{x,t\}\}$:

$$c_0^{(1)} = \frac{v}{z}, \quad c_0^{(2)} = \frac{v_x}{z_x}. \tag{6.404}$$

Consequently, solving the recurrent functional equations (6.403) yields

$$\begin{aligned} a_2^{(1)} &= [(xv - u^2/2)/z]_x,\ a_2 = \frac{v_x}{z_x^2} - \frac{u_x^2}{2z_x^2}, \\ a_2^{(1)} &= \frac{v_x v^3}{6z_x z^3} - \frac{u_x v^2}{2z_x z^2} + \frac{u(uz - v^2)}{6z^3} + \frac{v}{zz_x}, \end{aligned} \tag{6.405}$$

which lead to the following three functional solutions to (6.391):

$$\begin{aligned} r^{(1)} &= \frac{v_x v^3}{6z^3} - \frac{u_x v^2}{2z^2} + \frac{u(uz - v^2)z_x}{6z^3} + \frac{v}{z}, \\ r^{(2)} &= (xv - u^2/2)/z]_x,\ r^{(3)} = \frac{v_x}{z_x} - \frac{u_x^2}{2z_x}. \end{aligned} \tag{6.406}$$

Now, for the special solution $\eta := b_2 = \frac{u_x}{z_x} - \frac{v_x^2}{2z_x^2}$ to equation (6.398), one easily sees from (6.403) that the functional expression

$$r^{(4)} = (\frac{u_x^3}{6z_x^2} - \frac{u_x v_x}{2z_x^2} + \frac{3}{4z_x})/(\frac{u_x}{z_x} - \frac{v_x^2}{2z_x^2}) \tag{6.407}$$

also solves the functional-differential equation (6.403). Then, proceeding as above, one can construct an infinite set Ω of the desired solutions to the functional-differential equation (6.403) on the manifold $\tilde{\mathcal{M}}_{(3)}$. Thus, we have proved the following result.

Theorem 6.12. *The complete set $\mathcal{R}$ of functional-differential solutions to equation (6.391) on the manifold $\tilde{\mathcal{M}}_{(3)}$ is generated by functional solutions in the form (6.401) to the reduced functional-differential equations (6.398) and (6.400).*

In particular, the subset

$$\tilde{\mathcal{R}} = \{r^{(1)} = \frac{v_x v^3}{6z^3} - \frac{u_x v^2}{2z^2} + \frac{u(uz - v^2)z_x}{6z^3} + \frac{v}{z},\ r^{(2)} = [(xv - u^2/2)/z]_x,$$

$$r^{(3)} = \frac{v_x}{z_x} - \frac{u_x^2}{2z_x}, r^{(4)} = (\frac{u_x^3}{6z_x^2} - \frac{u_x v_x}{2z_x^2} + \frac{3}{4z_x})/(\frac{u_x}{z_x} - \frac{v_x^2}{2z_x^2})\} \subset \mathcal{R} \quad (6.408)$$

coincides with (6.339) found in [149, 150, 306].

6.6.2.4 *The generalized Riemann hydrodynamical equation for $N = 4$*

Now consider the generalized Riemann type differential equation (6.374) for $N = 4$, namely

$$D_t^4 u = 0 \qquad (6.409)$$

on an element $u \in \mathbb{R}\{\{x, t\}\}$ and construct the related invariant Riemann differential ideal $R\{u\} \subset \mathcal{K}\{u\}$ as follows:

$$\begin{aligned} R\{u\} := \{\lambda^3 \sum_{n\in\mathbb{Z}_+} f_n^{(1)} D_x^n u - \lambda^2 \sum_{n\in\mathbb{Z}_+} f_n^{(2)} D_t D_x^n u \\ + \lambda \sum_{n\in\mathbb{Z}_+} f_n^{(3)} D_t^2 D_x^n u - \sum_{n\in\mathbb{Z}_+} f_n^{(4)} D_t^3 D_x^n u \qquad (6.410) \\ : D_t^4 u = 0, \lambda \in \mathbb{R}, f_n^{(k)} \in \mathcal{K}\{u\}, 1 \le k \le 4, n \in \mathbb{Z}_+\} \end{aligned}$$

at a fixed function $u \in \mathbb{R}\{\{x, t\}\}$. The Riemann differential ideal (6.410), satisfying the D_t-invariance condition, is in this case also maximal. The corresponding kernel $\ker D_t \subset R\{u\}$ of the differentiation $D_t : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$, reduced modulo the Riemann differential ideal (6.410), is given by the following linear differential relationships:

$$D_t f^{(1)} = 0,\ D_t f^{(2)} = \lambda f^{(1)},\ D_t f^{(3)} = \lambda f^{(2)},\ D_t f^{(4)} = \lambda f^{(3)}, \qquad (6.411)$$

where $f^{(k)} := f^{(k)}(\lambda) = \sum_{n\in\mathbb{Z}_+} f_n^{(k)} \lambda^n \in \mathcal{K}\{u\}|_{\mathcal{M}_{(4)}} = \mathbb{R}\{\{x, t\}\}$, $1 \le k \le 4$ and $\lambda \in \mathbb{R}$ is arbitrary. The linear relationships (6.411) can be easily represented in the space $\mathbb{R}^4\{\{x, t\}\}$ in matrix form as

$$D_t f = q(\lambda) f, \quad q(\lambda) := \begin{pmatrix} 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \end{pmatrix}, \qquad (6.412)$$

where $f := (f^{(1)}, f^{(2)}, f^{(3)}, f^{(4)})^\intercal \in \mathbb{R}^4\{\{x,t\}\}$, and $\lambda \in \mathbb{R}$. Moreover, it is easy to see that relationships (6.411) is equivalent to the compact scalar equation

$$D_t^4 f^{(4)} = 0, \tag{6.413}$$

where $f_4 \in \mathcal{K}\{u\}$. Thus, we can now construct the invariant Lax differential ideal, isomorphically equivalent to (6.410), as

$$\begin{aligned} L\{u\} := \{g_1 f^{(4)} + g_2 D_t f^{(4)} + g_3 D_t^2 f^{(4)} + g_4 D_t^3 f^{(4)} : D_t^4 f^{(4)} = 0, \\ g_j \in \mathcal{K}\{u\}, 1 \le j \le 4\} \subset \mathcal{K}\{u\}, \end{aligned} \tag{6.414}$$

whose D_x-invariance can be readily checked. This gives rise to the representation

$$D_x f = \ell[u,v,w,z;\lambda] f; \tag{6.415}$$

with

$$\ell[u,v,w,z;\lambda] := \begin{pmatrix} -\lambda^3 u_x & \lambda^2 v_x & -\lambda w_x & z_x \\ -4\lambda^4 & 3\lambda^3 u_x & -2\lambda^2 v_x & \lambda w_x \\ -10\lambda^5 r_1 & 6\lambda^4 & -3\lambda^3 u_x & \lambda^2 v_x \\ -20\lambda^6 r_2 & 10\lambda^5 r_1 & -4\lambda^4 & \lambda^3 u_x \end{pmatrix},$$

where

$$D_t u := v, D_t v := w, D_t w := z, D_t z := 0, \tag{6.416}$$

$(u,v,w,z)^\intercal \in \tilde{\mathcal{M}}_{(4)} \subset \mathbb{R}^3\{\{x,t\}\}$, and the maps $r_j : \tilde{\mathcal{M}}_{(4)} \to \mathbb{R}\{\{x,t\}\}, j = 1,2$, satisfy the functional-differential equations

$$D_t r_1 + r_1 D_x u = 1, \qquad D_t r_2 + r_2 D_x u = r_1, \tag{6.417}$$

which are analogous to (6.386). The equations (6.417) have a plethora of solutions, amongst which are

$$\begin{aligned} r_1 &= D_x\Big(\frac{uw^2}{2z^2} - \frac{vw^3}{3z^3} + \frac{vw^4}{24z^4} + \frac{7w^5}{120z^4} - \frac{w^6}{144z^5}\Big), \\ r_2 &= D_x\Big(\frac{uw^3}{3z^3} - \frac{vw^4}{6z^4} + \frac{3w^6}{80z^5} + \frac{vw^5}{120z^5} - \frac{w^7}{420z^6}\Big); \end{aligned} \tag{6.418}$$

and

$$\begin{aligned} r_1 &= \Big(-\frac{2}{5}u_x v + \frac{1}{10}v_x u + w\Big)/z \\ &\quad + (-3u_x w^2 + 3v_x vw - w_x uw - w_x v^2 + z_x uv)/(10z^2), \\ r_2 &= (-u_x u + 2v)/(5z) \\ &\quad + (-3u_x vw/5 + 3v_x v^2/10 + 3v_x uw/10 - 2w_x uv/5 + z_x u^2/4 + 3w^2/10)/z^2, \end{aligned} \tag{6.419}$$

possessing the following important property:

If pairs $(r_1^{(s)}, r_2^{(s)}) \in \mathcal{R}$ *for* $1 \leq s \leq m$, *then for arbitrary real numbers* $\xi_s \in \mathbb{R}$, $1 \leq s \leq m$, $m \in \mathbb{Z}_+$, *such that* $\sum_{s=1}^{m} \xi_s = 1$,

$$(\sum_{s=1}^{m} \xi_s r_1^{(s)}, \sum_{s=1}^{m} \xi_s r_2^{(s)}) \in \mathcal{R}. \tag{6.420}$$

As a result, there exists a countable set (related to $\mathcal{R}$) of compatible implectic pairs $\eta^{(s)}, \vartheta^{(s)} : T^*(\mathcal{M}) \to T(\mathcal{M})$, $1 \leq s \leq m$, satisfying the corresponding gradient-like relationship (6.371) for all $\lambda \in \mathbb{R}$, and a countable set of related infinite hierarchies of conservation laws

$$\gamma_j^{(s)} := \int_0^{2\pi} \sigma_{j+2}^{(s)}[u, v, w, z; r]dx, \ j \in \mathbb{Z}_+, 1 \leq s \leq m,$$

satisfying the gradient-like relationships

$$\vartheta^{(s)} \mathrm{grad} \gamma_{j+2}^{(s)} = \eta^{(s)} \mathrm{grad} \gamma_j^{(s)} \tag{6.421}$$

for all $j \in \mathbb{Z}_+$ and each $s = 1, \ldots, m$. Thus, we have proved the following proposition.

Proposition 6.7. *The expressions (6.412) and (6.415) are the linear matrix representations in the space* $\mathbb{R}^4\{\{x,t\}\}$ *of the differentiations* $D_t : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$ *and* $D_x : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$, *respectively, reduced modulo the invariant Lax differential ideal* $L\{u\} \subset \mathcal{K}\{u\}$ *given by (6.384).*

It is now a simple matter to use the representations (6.412) and (6.415) to construct the Lax representation characterizing the integrability of

$$\begin{pmatrix} u_t \\ v_t \\ w_t \\ z_t \end{pmatrix} = K[u, v, w, z] := \begin{pmatrix} v - uu_x \\ w - uv_x \\ z - uw_x \\ -uz_x \end{pmatrix}, \tag{6.422}$$

which is equivalent to the generalized Riemann type hydrodynamic system (6.409). In particular, the following result follows readily from this approach.

Theorem 6.13. *The dynamical system (6.422), equivalent to the generalized Riemann hydrodynamical system (6.409), possesses the Lax representation*

$$f_x = \ell[u, v, z, w; \lambda]f, \quad f_t = p(\ell)f, \quad p(\ell) := -u\ell[u, v, w, z; \lambda] + q(\lambda), \tag{6.423}$$

where $f \in \mathbb{R}^4\{\{x,t\}\}$, $\lambda \in \mathbb{R}$ *is a spectral parameter and*

$$\ell[u,v,w,z;\lambda] := \begin{pmatrix} -\lambda^3 u_x & \lambda^2 v_x & -\lambda w_x & z_x \\ -4\lambda^4 & 3\lambda^3 u_x & -2\lambda^2 v_x & \lambda w_x \\ -10\lambda^5 r_1 & 6\lambda^4 & -3\lambda^3 u_x & \lambda^2 v_x \\ -20\lambda^6 r_2 & 10\lambda^5 r_1 & -4\lambda^4 & \lambda^3 u_x \end{pmatrix}, \quad q(\lambda) := \begin{pmatrix} 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \end{pmatrix},$$

$$p(\ell) = \begin{pmatrix} \lambda u u_x & -\lambda^2 u v_x & \lambda u w_x & -u z_x \\ \lambda + 4\lambda^4 u & -3\lambda^3 u u_x & 2\lambda^2 u v_x & -\lambda u w_x \\ 10\lambda^5 u r_1 & \lambda - 6\lambda^4 u & 3\lambda^3 u u_x & -\lambda^2 u v_x \\ 20\lambda^6 u r_2 & -10\lambda^5 u r_1 & \lambda + 4\lambda^4 u & -\lambda^3 u u_x \end{pmatrix}, \tag{6.424}$$

so it is a Lax integrable dynamical system on the functional manifold $\tilde{\mathcal{M}}_{(4)}$.

This result can be easily generalized to the case of an arbitrary integer $N \in \mathbb{Z}_+$, thereby proving the Lax integrability of the whole hierarchy of Riemann type hydrodynamic equations (6.284).

Remark 6.9. The Riemann hydrodynamic equation (6.374) as $N \to \infty$ is equivalent to the Benney type [27, 223, 315] chain

$$D_t u^{(n)} = u^{(n+1)}, \quad D_t := \partial/\partial t + u^{(0)}\partial/\partial x, \tag{6.425}$$

for suitably constructed moment functions $u^{(n)} := D_t^n u^{(0)}, u^{(0)} := u \in \mathbb{R}\{\{x,t\}\}$, $n \in \mathbb{Z}_+$.

6.6.3 *Differential-algebraic analysis of the Lax integrability of the KdV dynamical system*

6.6.3.1 *Differential-algebraic problem setting*

We consider the KdV equation in the following (6.379) differential-algebraic form:

$$D_t u - D_x^3 u = 0, \tag{6.426}$$

where $u \in \mathcal{K}\{u\}$ and the differentiations $D_t := \partial/\partial t + u\partial/\partial x, D_x := \partial/\partial x$ satisfy the commutation condition (6.376):

$$[D_x, D_t] = (D_x u) D_x. \tag{6.427}$$

It is convenient to also interpret the relationship (6.426) as a nonlinear dynamical system

$$D_t u = D_{xxx} u \tag{6.428}$$

on a suitably chosen functional manifold $\mathcal{M} \subset \mathbb{R}\{\{x,t\}\}$.

Using the expression (6.426), it is easy to construct a suitable invariant KdV-differential ideal $KdV\{u\} \subset \mathcal{K}\{u\}$ as follows:

$$KdV\{u\} := \{\sum_{k=\overline{0,2}}\sum_{n\in\mathbb{Z}_+} f_n^{(k)} D_x^k D_t^n u \in \mathcal{K}\{u\}: \; D_t u - D_x^3 u = 0,$$
$$f_n^{(k)} \in \mathcal{K}\{u\}, k = 0,1,2, n \in \mathbb{Z}_+\} \subset \mathcal{K}\{u\}. \tag{6.429}$$

The ideal (6.429) is not maximal, and this seriously affects the form of the reduced (modulo the ideal) representations of the derivatives D_x and $D_t : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$. Next, we need to find $\ker D_t \subset KdV\{u\}$ for the differentiation $D_t : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$, reduced modulo the KdV-differential ideal (6.429). We readily compute that it is generated by the following differential relationships:

$$\begin{aligned} D_t f^{(0)} &= -\lambda f^{(0)}, \quad D_t f^{(2)} = -\lambda f^{(2)} + 2f^{(2)} D_x u, \\ D_t f^{(1)} &= -\lambda f^{(1)} + f^{(1)} D_x u + f^{(2)} D_{xx} u, \end{aligned} \tag{6.430}$$

where $f^{(k)} := f^{(k)}(\lambda) = \sum_{n\in\mathbb{Z}_+} f_n^{(k)}\lambda^n \in \mathcal{K}\{u\}|_{\mathcal{M}} = \mathbb{R}\{\{x,t\}\}, k = 0,1,2,$ and $\lambda \in \mathbb{R}$ is an arbitrary parameter. The following result is a direct consequence of the relationships (6.430).

Proposition 6.8. *The differential relationships (6.430) can be recast in the following linear matrix form:*

$$D_t f = q(\lambda) f, \quad q(\lambda) := \begin{pmatrix} D_x u - \lambda & D_{xx} u \\ 0 & 2D_x u - \lambda \end{pmatrix}, \tag{6.431}$$

where $f := (f_1, f_2)^\intercal \in \mathbb{R}^2\{\{x,t\}\}$, $\lambda \in \mathbb{R}$, *giving rise to the corresponding linear matrix representation in the space* $\mathbb{R}^2\{\{x,t\}\}$ *of the differentiation* $D_t : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$, *reduced modulo the KdV-differential ideal (6.429).*

6.6.3.2 *The Lax representation*

Now, making use of the matrix differential relationship (6.431), we can construct the Lax differential ideal related to the ideal (6.429)

$$L\{u\} := \{\langle g, f\rangle_{\mathbb{E}^2} \in \mathcal{K}\{u\} : D_t f = q(\lambda) f,$$
$$f, g \in \mathcal{K}^2\{u\} \; \} \subset \mathcal{K}\{u\}. \tag{6.432}$$

Since the Lax differential ideal (6.432) is, by construction, D_t-invariant and isomorphic to the D_t- and D_x-invariant KdV-differential ideal (6.429), it is only necessary to check its D_x-invariance. As a result of this condition, the following differential relationship

$$D_x f = \ell[u;\lambda] f, \ \ell[u;\lambda] := \begin{pmatrix} D_x\tilde{a} & 2D_{xx}\tilde{a} \\ -1 & -D_x\tilde{a} \end{pmatrix} \tag{6.433}$$

holds, where the map $\tilde{a} : \mathcal{M} \to \mathbb{R}\{\{x,t\}\}$ satisfies the functional-differential relationships

$$D_t\tilde{a} = 1, D_t u - D_x^3 u = 0, \tag{6.434}$$

and the matrix $\ell := \ell[u;\lambda] : \mathbb{R}^2\{\{x,t\}\} \to \mathbb{R}^2\{\{x,t\}\}$ satisfies for all $\lambda \in \mathbb{R}$ the determining functional-differential equation

$$D_t\ell + \ell D_x u = [q(\lambda), \ell] + D_x q(\lambda), \tag{6.435}$$

generalizing the equation (6.387). This proves the following result.

Theorem 6.14. *The derivatives $D_t : \mathbb{R}\{\{x,t\}\} \to \mathbb{R}\{\{x,t\}\}$ and $D_x : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$ of the differential ring $\mathcal{K}\{u\}$, reduced modulo the Lax differential ideal $L\{u\} \subset \mathcal{K}\{u\}$, which is isomorphic to the KdV-differential ideal $KdV\{u\} \subset \mathcal{K}\{u\}$, allow the compatible Lax representation (generated by the invariant Lax differential ideal $L\{u\} \subset \mathcal{K}\{u\}$)*

$$D_t f = q(\lambda) f, \quad q(\lambda) := \begin{pmatrix} D_x u - \lambda & D_{xx}u \\ 0 & 2D_x u - \lambda \end{pmatrix},$$

$$D_x f = \ell[u;\lambda] f, \ \ell[u;\lambda] := \begin{pmatrix} D_x\tilde{a} & 2D_{xx}\tilde{a}. \\ -1 & -D_x\tilde{a} \end{pmatrix}, \tag{6.436}$$

where the map $\tilde{a} : \mathcal{M} \to \mathbb{R}\{\{x,t\}\}$ satisfies the functional-differential relationships (6.434), $f \in \mathbb{R}^2\{\{x,t\}\}$ and $\lambda \in \mathbb{R}$.

It is interesting to note that the Lax representation (6.436) is very different from that given by the well-known [406] classical expressions

$$D_t f = q_{cl}(\lambda) f, \quad q_{cl}(\lambda) := \begin{pmatrix} D_x u/6 & -(2u/3 - 4\lambda) \\ \begin{matrix} D_{xx}u/6 - (u/6 - \lambda) \\ \times(2u/3 - 4\lambda) \end{matrix} & -11D_x u/6 \end{pmatrix},$$

$$D_x f = \ell_{cl}[u;\lambda] f, \ \ell_{cl}[u;\lambda] := \begin{pmatrix} 0 & 1 \\ u/6 - \lambda & 0 \end{pmatrix}, \tag{6.437}$$

where, as above, the following functional-differential equation (equivalent to the nonlinear dynamical system (6.428) on the functional manifold $\mathcal{M}$)

$$D_t\ell_{cl} + \ell_{cl} D_x u = [q_{cl}(\lambda), \ell_{cl}] + D_x q_{cl}(\lambda), \tag{6.438}$$

holds for any $\lambda \in \mathbb{R}$. We suspect that this fact is related to the existence of different D_t-invariant KdV-differential ideals of the form (6.429), which are not maximal. Thus, a problem of constructing a suitable KdV-differential ideal $KdV\{u\} \subset \mathcal{K}\{u\}$ generating the corresponding invariant Lax differential ideal $L\{u\} \subset \mathcal{K}\{u\}$, invariant with respect to the differential representations (6.437), naturally arises, and this warrants further investigation. There is also the very interesting problem of the differential-algebraic analysis of the related symplectic structures on the functional manifold $\mathcal{M}$, with respect to which the dynamical system (6.428) is Hamiltonian and integrable. Here we must mention the interesting work [400], where the integrability structure of the KdV equation was analyzed from the differential-algebraic point of view.

6.6.4 *Summary remarks*

As demonstrated in this section, the differential-algebraic tools, when applied to a given set of differential relationships based on differentiations D_t and $D_x : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$ in the differential ring $\mathbb{R}\{\{x,t)\}\}$ and parametrized by a fixed element $u \in \mathcal{M}$, make it possible to construct the corresponding Lax representation as that realizing the linear matrix representations of the basic differentiations. This scheme was elaborated in detail for the generalized Riemann type differential equation (6.378) and for the classical KdV equation (6.428). As these equations are equivalent to the corresponding Hamiltonian systems with respect to suitable symplectic structures, this shows that there are still many questions in this vein that need to be answered.

6.7 Symplectic analysis of the Maxwell equations

6.7.1 *Introduction*

We consider Maxwell's electromagnetic equations to be given as

$$\begin{aligned} \partial E/\partial t = \nabla \times B - J, \qquad \partial B/\partial t = -\nabla \times E, \\ < \nabla, E >_{\mathbb{E}^3} = \rho, \qquad\qquad < \nabla, B >_{\mathbb{E}^3} = 0, \end{aligned} \tag{6.439}$$

on the cotangent phase space $T^*(N)$ to $N \subset T(D;\mathbb{E}^3)$, which is the smooth manifold of smooth vector fields on an open domain $D \subset \mathbb{R}^3$, all expressed in light speed units. Here $(E,B) \in T^*(N)$ is a vector of electric and magnetic fields, $\rho : D \to \mathbb{R}$ and $J : D \to \mathbb{E}^3$ are, simultaneously, fixed charge density and current functions on the domain D, satisfying the equation of continuity

$$\partial\rho/\partial t + <\nabla, J>_{\mathbb{E}^3} = 0 \tag{6.440}$$

for all $t \in \mathbb{R}$, where ∇ is the gradient operation with respect to a variable $x \in D$, and $\times$ is the usual vector product in $\mathbb{E}^3 := (\mathbb{R}^3, <\cdot,\cdot>_{\mathbb{E}^3})$, which is three-dimensional Euclidean vector space $\mathbb{R}^3$ endowed with the usual scalar product $<\cdot,\cdot>_{\mathbb{E}^3}$.

We shall represent equations (6.439) and (6.440) as those on a reduced symplectic space. To this end, we define an appropriate configuration space $M \subset \mathcal{T}(D;\mathbb{E}^3)$ with a vector potential field coordinate $A \in M$. The cotangent space $T^*(M)$ may be identified with pairs $(A;Y) \in T^*(M)$, where $Y \in \mathcal{T}^*(D;\mathbb{E}^3)$ is a suitable vector field density in D. On the space $T^*(M)$ there exists the canonical symplectic form $\omega^{(2)} \in \Lambda^2(T^*(M))$, allowing, owing to the definition of the Liouville form

$$\lambda(\alpha^{(1)})(A;Y) = \int_D d^3x(<Y, dA>_{\mathbb{E}^3}) := (Y, dA), \tag{6.441}$$

the canonical expression

$$\omega^{(2)} := d\lambda(\alpha^{(1)}) = d\, pr_R^*\alpha^{(1)} = (dY, \wedge dA). \tag{6.442}$$

Here d is the usual exterior derivative, $\wedge$ the wedge product, d^3x, $x \in D$, the Lebesgue measure in the domain D and by $pr_R : T^*(M) \to M$ is the standard projection upon the base space M. Define now a Hamiltonian function $\tilde{H} \in \mathcal{D}(T^*(M))$ as

$$\tilde{H}(A,Y) = 1/2[(Y,Y) + (\nabla\times A, \nabla\times A) + (<\nabla, A>_{\mathbb{E}^3}, <\nabla, A>_{\mathbb{E}^3})], \tag{6.443}$$

describing the Maxwell equations in vacuo, if the densities $\rho = 0$ and $J = 0$. Actually, owing to (6.442) one easily obtains from (6.443) that

$$\begin{aligned}\partial A/\partial t &:= \delta\tilde{H}/\delta Y = Y, \\ \partial Y/\partial t &:= -\delta\tilde{H}/\delta A = -\nabla\times B + \nabla <\nabla, A>_{\mathbb{E}^3},\end{aligned} \tag{6.444}$$

are true wave equations in vacuo, where

$$B := \nabla \times A, \tag{6.445}$$

is the corresponding magnetic field. Now, defining

$$E := -Y - \nabla W \tag{6.446}$$

for some function $W : D \to \mathbb{R}$ as the corresponding electric field, the system of equations (6.444) becomes, in light of definition (6.445),

$$\partial B/\partial t = -\nabla\times E, \quad \partial E/\partial t = \nabla\times B, \tag{6.447}$$

coinciding with the Maxwell equations in vacuo, if the Lorentz condition

$$\partial W/\partial t + <\nabla, A>_{\mathbb{E}^3} = 0 \tag{6.448}$$

is imposed.

6.7.2 *Symmetry properties*

Since definition (6.446) was not envisaged in the context of the Hamiltonian approach and our equations are valid only for a vacuum, we shall proceed with the analysis making use of the reduction approach devised above. Namely, we start with the Hamiltonian (6.443) and observe that it is invariant with respect to the following abelian symmetry group $G := \exp \mathcal{G}$, where $\mathcal{G} \simeq C^{(1)}(D;\mathbb{R})$, acting on the base manifold M naturally lifted to $T^*(M)$: for any $\psi \in \mathcal{G}$ and $(A, Y) \in T^*(M)$

$$\varphi_\psi(A) := A + \nabla\psi, \qquad \varphi_\psi(Y) = Y. \tag{6.449}$$

Under the transformation (6.449) the 1-form (6.441) is obviously invariant since

$$\begin{aligned} \varphi_\psi^* \lambda(\alpha^{(1)})(A, Y) = (Y, dA + \nabla d\psi) = \\ = (Y, dA) - (< \nabla, Y >_{\mathbb{E}^3}, d\psi) = \lambda(\alpha^{(1)})(A, Y), \end{aligned} \tag{6.450}$$

where we made use of the condition $d\psi = 0$ in $\Lambda^1(T^*(M))$ for any $\psi \in \mathcal{G}$. Thus, the corresponding momentum map is given as

$$l(A, Y) = - < \nabla, Y >_{\mathbb{E}^3} \tag{6.451}$$

for all $(A, Y) \in T^*(M)$. If $\rho \in \mathcal{G}^*$ is fixed, one can define the reduced phase space $\bar{\mathcal{M}}_\rho := l^{-1}(\rho)/G$, since the isotropy group $G_\rho = G$, owing to its commutativity and the condition (6.449). Consider now a principal fiber bundle $\pi : M \to N$ with the abelian structure group G and base manifold N taken as

$$N := \{B \in \mathcal{T}(D; \mathbb{E}^3) : \quad < \nabla,\ B >_{\mathbb{E}^3} = 0, \quad < \nabla, E(S) >_{\mathbb{E}^3} = \rho\}, \tag{6.452}$$

where

$$\pi(A) = B = \nabla \times A. \tag{6.453}$$

Over this bundle define a connection 1-form $\mathcal{A} \in \Lambda^1(M) \otimes \mathcal{G}$, where for all $A \in M$

$$\mathcal{A}(A) \cdot \hat{A}_*(l) = 1, \quad d < \mathcal{A}(A), \rho >_{\mathbb{E}^3 \mathcal{G}} = \Omega_\rho^{(2)}(A) \in H^2(M; \mathbb{Z}), \tag{6.454}$$

where $\mathcal{A}(A) \in \Lambda^1(M)$ is a differential 1-form that we choose as

$$\mathcal{A}(A) := -(W, d < \nabla, A >_{\mathbb{E}^3}), \tag{6.455}$$

where $W \in C^1(D; \mathbb{R})$ is a scalar function, still to be determined. As a result, the Liouville form (6.441) transforms into

$$\begin{aligned} \lambda(\tilde{\alpha}_\rho^{(1)}) := (Y, dA) - (W, d < \nabla, A >_{\mathbb{E}^3}) = (Y + \nabla W, dA) := (\tilde{Y},\ dA), \\ \tilde{Y} := Y + \nabla W, \end{aligned} \tag{6.456}$$

giving rise to the corresponding canonical symplectic structure on $T^*(M)$ as

$$\tilde{\omega}_\rho^{(2)} := d\lambda(\tilde{\alpha}_\rho^{(1)}) = (d\tilde{Y}, \wedge dA). \tag{6.457}$$

The Hamiltonian function (6.443), as a function on $T^*(M)$, transforms into

$$\tilde{H}_\rho(A, \tilde{Y}) = 1/2[(\tilde{Y}, \tilde{Y}) + (\nabla \times A, \nabla \times A) + (< \nabla, A >_{\mathbb{E}^3}, < \nabla, A >_{\mathbb{E}^3})], \tag{6.458}$$

coinciding with the well-known Dirac–Fock–Podolsky [57, 105] Hamiltonian expression. The corresponding Hamiltonian equations on the cotangent space $T^*(M)$

$$\begin{aligned} \partial A/\partial t &:= \delta \tilde{H}/\delta \tilde{Y} = \tilde{Y}, \quad \tilde{Y} := -E - \nabla W, \\ \partial \tilde{Y}/\partial t &:= -\delta \tilde{H}/\delta A = -\nabla \times (\nabla \times A) + \nabla < \nabla, A >_{\mathbb{E}^3}, \end{aligned}$$

describe true wave processes, related to Maxwell's equations in the vacuo, except for external charge and current density conditions. In fact, from (6.458), we obtain that

$$\partial^2 A/\partial t^2 - \nabla^2 A = 0 \Longrightarrow \partial E/\partial t + \nabla(\partial W/\partial t \ + \ < \nabla, A >_{\mathbb{E}^3}) = -\nabla \times B, \tag{6.459}$$

giving rise to the true vector potential wave equation, but the electromagnetic Faraday induction law equation requires the imposition of the Lorentz condition (6.448).

6.7.3 *Dirac–Fock–Podolsky problem analysis*

To remedy this situation (requiring the apparent addition of the Lorentz condition), we apply to this symplectic space the reduction technique devised above. Namely, owing to Theorem 2.3, the cotangent manifold $T^*(N)$ is symplectomorphic to the corresponding reduced phase space $\bar{\mathcal{M}}_\rho$; that is,

$$\bar{\mathcal{M}}_\rho \simeq \{(B; S) \in T^*(N) : < \nabla, E(S) >_{\mathbb{E}^3} = \rho, \quad < \nabla, B >_{\mathbb{E}^3} = 0\} \tag{6.460}$$

with the reduced canonical symplectic 2-form

$$\omega_\rho^{(2)}(B, S) = (dB, \wedge dS) = d\lambda(\alpha_\rho^{(1)})(B, S), \quad \lambda(\alpha_\rho^{(1)})(B, S) := -(S, dB), \tag{6.461}$$

where

$$\nabla \times S + F + \nabla W = -\tilde{Y} := E + \nabla W, \quad < \nabla, F >_{\mathbb{E}^3} := \rho, \tag{6.462}$$

for some fixed vector map $F \in C^1(D;\mathbb{E}^3)$, depending on the imposed external charge and current density conditions. The result (6.461) follows right away upon substituting the expression for the electric field $E = \nabla \times S + F$ into the symplectic structure (6.457), taking into account that $dF = 0$ in $\Lambda^1(M)$. The Hamiltonian function (6.458) reduces to the symbolic form

$$\begin{aligned} H_\rho(B,S) &= 1/2[(B,B) + (\nabla \times S + F + \nabla W, \nabla \times S + F + \nabla W) \\ &+(< \nabla, (\nabla\times)^{-1}B >_{\mathbb{E}^3}, < \nabla, (\nabla\times)^{-1}B >_{\mathbb{E}^3})], \end{aligned} \tag{6.463}$$

where $(\nabla\times)^{-1}$ denotes the corresponding inverse curl-operation mapping [251] the divergence-free subspace $C^1_{div}(D;\mathbb{E}^3) \subset C^1(D;\mathbb{E}^3)$ into itself. Thus, it follows from (6.463) that the Maxwell equations (6.439) become a canonical Hamiltonian system on the reduced phase space $T^*(N)$, endowed with the canonical symplectic structure (6.461) and the modified Hamiltonian function (6.463). Actually, one readily computes that

$$\begin{aligned} \partial S/\partial t &:= \delta H/\delta B = B - (\nabla\times)^{-1}\nabla < \nabla, (\nabla\times)^{-1}B >_{\mathbb{E}^3}, \\ \partial B/\partial t &:= -\delta H/\delta S = -\nabla \times (\nabla \times S + F + \nabla W) = -\nabla \times E, \end{aligned} \tag{6.464}$$

where we made use of the definition $E = \nabla \times S + F$ and the elementary identity $\nabla \times \nabla = 0$. Thus, the second equation of (6.464) coincides with the second Maxwell equation of (6.439) in the classical form

$$\partial B/\partial t = -\nabla \times E.$$

Moreover, from (6.462) and (6.464), one finds via the differentiation with respect to $t \in \mathbb{R}$ that

$$\begin{aligned} \partial E/\partial t &= \partial F/\partial t + \nabla \times \partial S/\partial t \\ &= \partial F/\partial t + \nabla \times B, \end{aligned} \tag{6.465}$$

as well as, by virtue of (6.440),

$$< \nabla, \partial F/\partial t >_{\mathbb{E}^3} = \partial\rho/\partial t = - < \nabla, J >_{\mathbb{E}^3} . \tag{6.466}$$

Now it follows directly from (6.466) that, up to non-essential curl-terms $\nabla \times (\cdot)$, the following relationship

$$\partial F/\partial t = -J \tag{6.467}$$

holds. In fact, the current vector $J \in C^1(D;\mathbb{E}^3)$ is defined up to curl-terms $\nabla \times (\cdot)$ which can be included into the definition of the right-hand side of (6.467), owing to the equation of continuity (6.440). Then, substituting (6.467) into (6.465), we obtain exactly the first Maxwell equation of (6.439):

$$\partial E/\partial t = \nabla \times B - J, \tag{6.468}$$

which is naturally supplemented with the external charge and current density conditions

$$
\begin{gathered}
<\nabla, B>_{\mathbb{E}^3}=0, \qquad <\nabla, E>_{\mathbb{E}^3}=\rho, \\
\partial\rho/\partial t+<\nabla, J>_{\mathbb{E}^3}=0,
\end{gathered} \tag{6.469}
$$

owing to the equation of continuity (6.440) and definition (6.460).

Concerning the wave equations, related to the Hamiltonian system (6.464), we obtain the following: the electric field E is recovered from the second equation as

$$
E:=-\partial A/\partial t-\nabla W, \tag{6.470}
$$

where $W \in C^{(1)}(D;\mathbb{R})$ is some smooth function, depending on the vector field $A \in M$. To retrieve this dependence, we substitute (6.467) into equation (6.468), having taken into account that $B=\nabla \times A:$

$$
\partial^2 A/\partial t^2-\nabla(\partial W/\partial t+<\nabla, A>_{\mathbb{E}^3})=\nabla^2 A+J. \tag{6.471}
$$

Thereby, if to impose now the Lorentz condition (6.448), one obtains from (6.471) the corresponding true wave equations in the space-time, taking into account the external charge and current densities conditions (6.469).

6.7.4 *Symplectic reduction*

As the problem of fulfilling the Lorentz constraint (6.448) within the canonical Hamiltonian formalism is still not completely resolved, we are forced to analyzing the structure of the Liouville 1-form (6.456) for the Maxwell equations on a slightly extended functional manifold $M \times L$. For the first step, we rewrite the 1-from (6.456) as

$$
\begin{gathered}
\lambda(\tilde{\alpha}_\rho^{(1)}):=(\tilde{Y}, dA)=(Y+\nabla W, dA)=(Y, dA) \\
+(W, -d<\nabla, A>_{\mathbb{E}^3}):=(Y, dA)+(W, d\eta),
\end{gathered} \tag{6.472}
$$

where

$$
\eta:=-<\nabla, A>_{\mathbb{E}^3}. \tag{6.473}
$$

Considering now the elements $(Y, A; \eta, W) \in T^*(M \times L)$ as new independent canonical variables on the extended cotangent phase space $T^*(M \times L)$, where $L:=C^{(1)}(D;\mathbb{R})$, we can express the symplectic structure (6.457) in the canonical form

$$
\tilde{\omega}_\rho^{(2)}:=d\lambda(\tilde{\alpha}_\rho^{(1)})=(dY, \wedge dA)+(dW, \wedge d\eta). \tag{6.474}
$$

Subject to the Hamiltonian function (6.458), we obtain the expression

$$
H(A, Y; \eta, W)=1/2[(Y-\nabla W, Y-\nabla W)+(\nabla\times A, \nabla\times A)+(\eta, \eta)], \tag{6.475}
$$

with respect to which the corresponding Hamiltonian equations take the form

$$\begin{aligned} \partial A/\partial t &:= \delta H/\delta Y = Y - \nabla W, \quad Y := -E, \\ \partial Y/\partial t &:= -\delta H/\delta A = -\nabla \times (\nabla \times A), \\ \partial \eta/\partial t &:= \delta H/\delta W =< \nabla, Y - \nabla W >_{\mathbb{E}^3}, \\ \partial W/\partial t &:= -\delta H/\delta \eta = -\eta. \end{aligned} \tag{6.476}$$

Thus, from (6.476) we find that

$$\begin{aligned} &\partial B/\partial t + \nabla \times E = 0, \quad \partial^2 W/\partial t^2 - \nabla^2 W =< \nabla, E >_{\mathbb{E}^3}, \\ &\partial E/\partial t - \nabla \times B = 0, \quad \partial^2 A/\partial t^2 - \nabla^2 A = -\nabla(\partial W/\partial t + < \nabla, A >_{\mathbb{E}^3}). \end{aligned} \tag{6.477}$$

These equations describe electromagnetic Maxwell equations in vacuo, not taking into account both the external charge and current density relationships (6.469) and the Lorentz condition (6.448). Hence, as above, we apply to the symplectic structure (6.474) our reduction technique. We then find that under transformations (6.462), the corresponding reduced manifold $\bar{\mathcal{M}}_\rho$ becomes endowed with the symplectic structure

$$\bar{\omega}_\rho^{(2)} := (dB, \wedge dS) + (dW, \wedge d\eta). \tag{6.478}$$

The corresponding expression for the Hamiltonian function (6.475) is

$$H(S, B; \eta, W) = 1/2[(\nabla \times S + F + \nabla W, \nabla \times S + F + \nabla W) + (B, B) + (\eta, \eta)], \tag{6.479}$$

with Hamiltonian equations

$$\begin{aligned} &\partial S/\partial t := \delta H/\delta B = B, \qquad \partial W/\partial t := -\delta H/\delta \eta = -\eta, \\ &\partial B/\partial t := -\delta H/\delta S = -\nabla \times (\nabla \times S + F + \nabla W) = -\nabla \times E, \\ &\partial \eta/\partial t := \delta H/\delta W = - < \nabla, \nabla \times S + F + \nabla W >_{\mathbb{E}^3} = - < \nabla, E >_{\mathbb{E}^3} - \Delta W, \end{aligned} \tag{6.480}$$

coinciding under the constraint (6.462) with the Maxwell equations (6.439), which describe true wave propagation in space-time and include both the imposed external charge and current density relationships (6.469) and the Lorentz condition (6.448). This solves the problem mentioned in [57, 105]. Moreover, it is easy to obtain from (6.480) that

$$\begin{aligned} \partial^2 W/\partial t^2 - \Delta W = \rho, &\qquad \partial W/\partial t + < \nabla, A >_{\mathbb{E}^3} = 0, \\ \nabla \times B = J + \partial E/\partial t, &\qquad \partial B/\partial t = -\nabla \times E. \end{aligned} \tag{6.481}$$

We can now, employing (6.481) and (6.469), easily calculate [63, 321] the magnetic wave equation

$$\partial^2 A/\partial t^2 - \Delta A = J, \tag{6.482}$$

which supplements a suitable wave equation for the scalar potential $W \in L$, thereby completing the calculations. Thus, we have proved the following proposition.

Proposition 6.9. *The electromagnetic Maxwell equations (6.439) together with the Lorentz condition (6.448) are equivalent to the Hamiltonian system (6.480) with respect to the canonical symplectic structure (6.478) and Hamiltonian function (6.479) that, respectively, reduce to the electromagnetic equations (6.481) and (6.482) under external charge and current density relationships (6.469).*

This result, we think, can be used to develop an alternative quantization procedure of Maxwell's equations, which are free of related quantum operator compatibility problems as discussed in [35, 57, 105].

6.8 Symplectic analysis of vortex helicity in magneto-hydrodynamics

6.8.1 *Introduction*

Geometric methods have proven to be very effective in illuminating many interesting physical properties of vortex flows [15]. For example, it was proved using geometric techniques [266, 293] that quantum vortices in superfluid helium can be studied either as open lines with their ends terminating on free surfaces of walls of the container or as closed curves. Nowadays, the closed vortices are treated as topological circles. The existence of structures such as knotted and linked vortex lines in the turbulent phase, which is almost obvious [372] from the geometric perspective, has forced researchers to develop new mathematical tools for their detailed investigation. In this vein, Peradzyński [303] proved a new version of the helicity theorem using differential-geometric methods, which he applied to the description of the vortex dynamics for an incompressible superfluid. The Peradzyński helicity theorem describes in a unique way, both the superfluid equations and the related helicity invariants, which in the conservative case are very important for studying the topological structure of vortices.

In this section we revisit Peradzyński helicity theorem in the context of the modern symplectic theory of differential-geometric structures on manifolds, and present a new unified proof [317, 318] of this fundamental result. In addition, we prove a magneto-hydrodynamic generalization of this theorem for the case of an incompressible superfluid flow. Our analysis also has other useful corollaries: in the conservative case we construct a sequence of nontrivial helicity type conservation laws, which plays a crucial role in determining the stability of superfluids under suitable boundary conditions.

6.8.2 *Symplectic and symmetry analysis*

We consider a quasi-neutral superfluid contained in a domain $M \subset \mathbb{R}^3$, where M is a three-dimensional manifold with a piecewise smooth boundary, interacting with a "frozen" magnetic field $B : M \longrightarrow \mathbb{E}^3$, where $\mathbb{E}^3$ is Euclidean 3-space with the standard inner product $\langle \cdot, \cdot \rangle_{\mathbb{E}^3}$ and vector cross product $\times$. The magnetic field is assumed to be source-less and satisfying the condition $B = \nabla \times A$, where $A : M \longrightarrow \mathbb{E}^3$ is the magnetic field potential. The corresponding electric field $E : M \longrightarrow \mathbb{E}^3$, related to the magnetic potential, satisfies the necessary superconductivity conditions

$$E + u \times B = 0, \qquad \partial E/\partial t = \nabla \times B, \tag{6.483}$$

where $u : M \longrightarrow T(M)$ is the superfluid velocity.

Let ∂M denote the boundary of the domain M. The boundary conditions $\langle n, u \rangle|_{\partial M} = 0$ and $\langle n, B \rangle|_{\partial M} = 0$ are imposed on the superfluid flow, where $n \in T^*(M)$ is the vector normal to the boundary ∂M, considered to be almost everywhere smooth.

Then adiabatic magneto-hydrodynamics (MHD) quasi-neutral superfluid motion can be described, owing to (6.483), by the system of evolution equations

$$\partial u/\partial t = -\langle u, \nabla \rangle u - \rho^{-1} \nabla P + \rho^{-1} (\nabla \times B) \times B,$$

$$\partial \rho/\partial t = -\langle \nabla, \rho u \rangle, \qquad \partial \eta/\partial t = -\langle u, \nabla \eta \rangle, \qquad \partial B/\partial t = \nabla \times (u \times B), \tag{6.484}$$

where $\rho : M \longrightarrow \mathbb{R}_+$ is the superfluid density, $P : M \longrightarrow \mathbb{E}^3$ is the internal pressure and $\eta : M \longrightarrow \mathbb{R}$ is the specific superfluid entropy, which is related to the internal MHD superfluid specific energy function $e = e(\rho, \eta)$ via the First Law of Thermodynamics as

$$T\, d\eta = de(\rho, \eta) - P\rho^{-2} d\rho, \tag{6.485}$$

where $T = T(\rho, \eta)$ is the internal absolute temperature in the superfluid. The system of evolution equations (6.484) conserves the Hamiltonian function (representing the total energy)

$$H := \int_M \left[\frac{1}{2\rho} |\mu|^2 + \rho e(\rho, \eta) + \frac{1}{2} |B|^2 \right] d^3x, \tag{6.486}$$

since the dynamical system (6.484) is a Hamiltonian system on the functional manifold $\mathcal{R} := C^\infty(M; T^*(M) \times \mathbb{R}^2 \times \mathbb{E}^3)$ with respect to the [178] Poisson bracket:

$$\begin{aligned}\{f,g\} := \int_M \Big\{ &\langle \mu, [\tfrac{\delta f}{\delta \mu}, \tfrac{\delta g}{\delta \mu}]_c \rangle + \rho \Big(\langle \tfrac{\delta g}{\delta \mu}, \nabla \tfrac{\delta f}{\delta \rho} \rangle - \langle \tfrac{\delta f}{\delta \mu}, \nabla \tfrac{\delta g}{\delta \rho} \rangle \Big) \\ &+ \eta \langle \nabla, (\tfrac{\delta g}{\delta \mu} \tfrac{\delta f}{\delta \eta} - \tfrac{\delta f}{\delta \mu} \tfrac{\delta g}{\delta \eta}) \rangle + \langle B, [\tfrac{\delta g}{\delta \mu}, \tfrac{\delta f}{\delta B}]_c \rangle \\ &+ \langle \tfrac{\delta f}{\delta B}, \langle B, \nabla \rangle \tfrac{\delta g}{\delta \mu} \rangle - \langle \tfrac{\delta g}{\delta B}, \langle B, \nabla \rangle \tfrac{\delta f}{\delta \mu} \rangle \Big\} dx, \end{aligned} \tag{6.487}$$

where $\mu := \rho u \in T^*(M)$ is the specific momentum of the superfluid motion and $[\cdot,\cdot]_c$ is the canonical Lie bracket of variational gradient vector fields:

$$[\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu}]_c := \langle \frac{\delta f}{\delta \mu}, \nabla \rangle \frac{\delta g}{\delta \mu} - \langle \frac{\delta g}{\delta \mu}, \nabla \rangle \frac{\delta f}{\delta \mu} \tag{6.488}$$

for any smooth functionals $f, g \in \mathcal{D}(M)$ on the functional space $\mathcal{M}$. Moreover, as it was shown in [178], the Poisson bracket (6.487) is actually the canonical Lie–Poisson bracket on the dual space to the Lie algebra $\mathcal{G}$ of the semidirect product of vector fields on M and the direct sum of functions, densities and differential one-forms on M. Namely, the specific momentum $\mu = \rho u \in T^*(M)$ is dual to vector fields, ρ is dual to functions, η is dual to densities and B is dual to the space of two-forms on M. Thus, the set of evolution equations (6.484) can be recast as

$$\begin{aligned} \partial u/\partial t = \{H, u\}, \qquad \partial \rho/\partial t = \{H, \rho\}, \\ \partial \eta/\partial t = \{H, \eta\}, \qquad \partial B/\partial t = \{H, B\}. \end{aligned} \tag{6.489}$$

The Poisson bracket (6.487) can be rewritten for any $f, g \in \mathcal{D}(M)$ as

$$\{f,g\} = (Df, \vartheta\, Dg), \tag{6.490}$$

with $Df := \left(\frac{\delta f}{\delta \mu}, \frac{\delta f}{\delta \rho}, \frac{\delta f}{\delta \eta}, \frac{\delta f}{\delta B} \right)^\intercal \in T^*(\mathcal{M})$, and $\vartheta : T^*(\mathcal{M}) \longrightarrow T(\mathcal{M})$ the corresponding (modulo the Casimir functionals of bracket (6.487)) invertible [176, 177] co-symplectic operator, satisfying the standard [3, 326] properties

$$\vartheta^* = -\vartheta, \qquad \delta(\delta w, \wedge\, \vartheta^{-1} \delta w) = 0, \tag{6.491}$$

where the differential variation complex condition $\delta^2 = 0$ is assumed, the differential variation vector $\delta w := (\delta \mu, \delta \rho, \delta \eta, \delta B)^\intercal \in T^*(\mathcal{M})$ and $*$ denotes the conjugate mapping with respect to the standard bilinear pairing $(\cdot,\cdot)$ of $T^*(\mathcal{M})$ and $T(\mathcal{M})$. Note here that the second condition of (6.491) is equivalent [3, 326] to the fact that the Poisson bracket (6.487) satisfies the

Jacobi commutation condition. Thus, one can define the closed generalized variational differential two-form on $\mathcal{M}$ as

$$\omega^{(2)} := (\delta w, \wedge \vartheta^{-1}\, \delta w), \tag{6.492}$$

which provides a symplectic structure on the functional factor manifold $\mathcal{M}$ (modulo the Casimir functionals of bracket (6.487)).

Let $\mathcal{D}_t(M) = \{\varphi_t : M \to M\}$ be the subgroup of the diffeomorphism group $\mathrm{Diff}_+(M)$, consisting of invertible transformations $\varphi_t : M \to M$, generated by the MHD superfluid evolution equations (6.484). Consequently,

$$d\varphi_t(x)/dt := u(\varphi_t(x)) \tag{6.493}$$

for all $x \in M$ and suitable $t \in \mathbb{R}$, for which solutions to (6.484) exist and are unique. The symplectic structure (6.492) is invariant with respect to the induced mapping of diffeomorphisms $\hat{\varphi}_t : \mathcal{M} \to \mathcal{M}$ on $\mathcal{M}$, that is

$$\hat{\varphi}_{t,*}\omega^{(2)} = \omega^{(2)} \tag{6.494}$$

for suitable $t \in \mathbb{R}$. Then the corresponding diffeomorphism subgroup $\hat{\mathcal{D}}_t(\mathcal{M}) := \{\hat{\varphi}_t : \mathcal{M} \to \mathcal{M}\}$ satisfies the evolution equation

$$d\hat{\varphi}_t(w)/dt := K_H(\hat{\varphi}_t(w)) \tag{6.495}$$

for any $w \in \mathcal{M}$ and the same suitable $t \in \mathbb{R}$, where the vector field $K_H : \mathcal{M} \longrightarrow T(\mathcal{M})$ coincides with the system of MHD evolution equations (6.484). This fact easily follows from the standard [3] differential-geometric considerations related to equation (6.494). Thus, from (6.494) one obtains that

$$0 = \frac{d}{dt}\hat{\varphi}_{t,*}\omega^{(2)} := L_{K_H}\omega^{(2)} = (i_{K_H}\delta + \delta i_{K_H})\omega^{(2)} = \delta i_{K_H}\omega^{(2)} \tag{6.496}$$

for all these suitable $t \in \mathbb{R}$, where L_{K_H} is the standard Lie derivative with respect to the vector field K_H on $\mathcal{M}$ and we used the Cartan formula $L_{K_H} = i_{K_H}\delta + \delta i_{K_H}$. Now, owing to the Hamiltonian equations (6.489), the equality $i_{K_H}\omega^{(2)} = -\delta H$ holds, and since $\delta^2 = 0$, the invariance property (6.494) is proved.

As the properties of equations (6.484) on $\mathcal{M}$ are completely determined by the diffeomorphism subgroup $\mathcal{D}_t(M) \subset Diff_+(M)$, we shall reformulate the set of equations (6.484) making use of the invariant properties on the manifold M. First, observe that the mass conservation law of our superfluid flow is equivalent to

$$\frac{d}{dt}\int_{D_t} \rho\, d^3x = 0 \tag{6.497}$$

for any domain $D_t \subset M$ moving together with the interior fluid. It is an easy calculation to rewrite (6.497) in the following equivalent form:

$$\int_{D_t} (\partial/\partial t + L_u)(\rho\, d^3x) = 0 \tag{6.498}$$

for all domains $D_t \subset M$ and suitable $t \in \mathbb{R}$, where as above, we denoted by $L_u = i_u d + d i_u$ the Lie derivative along the vector field $u : M \longrightarrow T(M)$ on M in the Cartan form.

It follows from (6.498) that we have the following local differential–geometric relationship:

$$(\partial/\partial t + L_u)(\rho\, d^3x) = 0. \tag{6.499}$$

Since the evolution of our superfluid is locally adiabatic, the following equality

$$(\partial/\partial t + L_u)\eta = 0 \tag{6.500}$$

is obvious and implies that $d\eta/dt = 0$ for all suitable $t \in \mathbb{R}$.

Now take the momentum conservation law in the integral Ampere–Newton form

$$\frac{d}{dt}\int_{D_t} \rho u\, d^3x + \int_{S_t = \partial D_t} P\, dS_t - \int_{D_t} (j \times B)\, d^3x = 0, \tag{6.501}$$

where dS_t is the corresponding oriented surface measure on the boundary $S_t := \partial D_t$ of a domain $D_t \subset M$, $P : M \longrightarrow \mathbb{R}$ is the internal pressure and $j : M \longrightarrow \mathbb{E}^3$ is the corresponding induced current density in the MHD superfluid under the superconductivity condition. Hence, owing to neutrality of the superfluid, the induction condition

$$\nabla \times B + j = 0 \tag{6.502}$$

holds. Then from (6.501) and (6.502), one readily computes the infinitesimal form of the evolution for the velocity vector $u : M \longrightarrow T(M)$:

$$(\partial/\partial t + L_{K_H})u = -\rho^{-1}\nabla P + \rho^{-1}(\nabla \times B) \times B, \tag{6.503}$$

which obviously coincides with the first equation of system (6.484).

Consider now at each $t \in \mathbb{R}$ the subgroup of diffeomorphisms $\mathcal{D}_\tau = \{\psi_\tau : M \to M\} \subset \mathrm{Diff}(M)$, generated by the following vector field $v : M \longrightarrow T(M)$ on M:

$$d\psi_\tau(x)/d\tau := v(\psi_\tau(x)) = \rho^{-1}B(\psi_\tau(x)), \tag{6.504}$$

defined for a suitable evolution parameters $\tau \in \mathbb{R}$. Since the subgroup $\mathcal{D}_\tau$ does not depend explicitly on the evolution parameter $t \in \mathbb{R}$, its action can be interpreted as re-arranging the superfluid particles within any chosen domain $D_t \subset M$. Owing now to the commutation property

$$[\partial/\partial t + L_u, L_v] = 0, \tag{6.505}$$

equivalent to the fact that the subgroup $\mathcal{D}_t$ and $\mathcal{D}_\tau$ commute for any suitable $t, \tau \in \mathbb{R}$, from the invariance condition

$$\partial\rho/\partial\tau = 0, \tag{6.506}$$

we see that

$$\gamma_n := L_v^n \gamma \tag{6.507}$$

for all $n \in \mathbb{Z}_+$ are invariants of the MHD superfluid flow (6.484), if the density $\gamma \in \Lambda^3(M)$ is also an invariant on M. In fact, we find that

$$(\partial/\partial t + L_u)\gamma_n = (\partial/\partial t + L_u)L_v^n\gamma = L_v^n(\partial/\partial t + L_u)\gamma = 0, \tag{6.508}$$

since we have

$$(\partial/\partial t + L_u)\gamma = 0. \tag{6.509}$$

Such a density can be found, observing [178] that the superconductivity relations $E + u \times B = 0$, $E = -\partial A/\partial t$ and the last equation of system (6.484) imply the invariance condition

$$(\partial/\partial t + L_u)d\alpha^{(1)} = 0, \tag{6.510}$$

where the one-form $\alpha^{(1)} \in \Lambda^1(M)$ equals

$$\alpha^{(1)} := \langle A, dx\rangle. \tag{6.511}$$

Moreover, since the differential operations $\partial/\partial t + L_u$ and d commute [3], one checks that the stronger cohomological condition

$$(\partial/\partial t + L_u)\alpha^{(1)} = 0 \tag{6.512}$$

holds on M, if the time-dependent gauge mapping $A \longrightarrow A + \nabla\psi$, where $\partial\psi/\partial t + L_u\psi + \langle u, A\rangle = 0$, is applied to the magnetic potential $A : M \longrightarrow \mathbb{E}^3$. Now from conditions (6.510) and (6.512), one easily shows that the density

$$\gamma := \alpha^{(1)} \wedge d\alpha^{(1)} \tag{6.513}$$

satisfies equation (6.509). Thus, it generates, in view of formula (6.507), new conserved quantities, which can be rewritten as

$$\tilde{\gamma}_n := \rho L_v^n(\rho^{-1}\langle B, A\rangle) = \rho L_v^n\langle v, A\rangle \tag{6.514}$$

for all $n \in \mathbb{Z}_+$. Accordingly the following functionals on the functional manifold $\mathcal{M}$

$$\tilde{H}_n := \int_M \tilde{\gamma}_n \, d^3x = \int_M \rho L_v^n(\rho^{-1}\langle B, A\rangle) \, d^3x \tag{6.515}$$

for all $n \in \mathbb{Z}_+$ are invariants of our MHD superfluid dynamical system (6.484). In particular, for $n = 0$ we obtain the well-known [178] magnetic helicity invariant

$$\tilde{H}_0 = \int_M \langle A, \nabla \times A\rangle \, d^3x, \tag{6.516}$$

which exists independently of the boundary conditions imposed on the MHD superfluid flow equations (6.484).

The results obtained prove the following theorem.

Theorem 6.15. *The functionals (6.515), where the Lie derivative L_v is taken along the magnetic vector field $v = \rho^{-1}B$, are global invariants of the system of compressible MHD superfluid and superconductive equations (6.484).*

In the next subsection, we attend to the symmetry analysis of the incompressible superfluid dynamical system and construct the new local and global helicity invariants. Superfluid hydrodynamical flows [346] are of great interest for their many applications featuring complex vorticity structures.

6.8.3 *Incompressible superfluids: Symmetry analysis and conservation laws*

Concerning the helicity theorem result of [303], where the kinematic helicity invariant

$$H_0 := \int_M \langle u, \nabla \times u\rangle \, d^3x \tag{6.517}$$

was derived, making use of differential-geometric tools in Minkowski space in the case of incompressible superfluid at the absent magnetic field $B = 0$, we shall describe its general dynamical symmetries. The governing equations are

$$\partial u/\partial t = -\langle u, \nabla\rangle u + \rho^{-1}\nabla P, \qquad \partial\rho/\partial t + \langle u, \nabla\rho\rangle = 0, \qquad \langle\nabla, u\rangle = 0, \tag{6.518}$$

where the density conservation properties

$$(\partial/\partial t + L_u)\rho = 0, \qquad (\partial/\partial t + L_u)d^3x = 0 \tag{6.519}$$

hold for all suitable $t \in \mathbb{R}$. Define now the vorticity vector $\xi := \nabla \times u$, which from (6.518) satisfies the vorticity flow equation

$$\partial\xi/\partial t = \nabla \times (u \times \xi). \tag{6.520}$$

The first equation of (6.518) can be rewritten as

$$\partial u/\partial t = u \times (\nabla \times u) - \rho^{-1}\nabla P - \frac{1}{2}\nabla|u|^2. \tag{6.521}$$

Then, applying the operation $(\nabla \times \cdot)$ to (6.521), one easily obtains the vorticity equation (6.520). Moreover, equation (6.520) can be recast as

$$\partial\xi/\partial t + \langle u, \nabla\rangle\xi = \langle \xi, \nabla\rangle u, \tag{6.522}$$

which allows a new dynamical symmetry interpretation.

Set

$$\partial x/\partial\tau = v(x,t) := \rho^{-1}\xi, \tag{6.523}$$

which defines for all $\tau \in \mathbb{R}$ the diffeomorphism subgroup $\mathcal{D}_\tau \subset \mathrm{Diff}\,(M)$ of the manifold M. It is easy to check that this subgroup commutes with the subgroup $\mathcal{D}_t \subset \mathrm{Diff}\,(M)$, since the condition

$$(\partial/\partial t + L_u)v = L_v u \tag{6.524}$$

holds for all $t, \tau \in \mathbb{R}$, which is the same as relationship (6.522). The condition (6.524) implies the commutation property

$$[\partial/\partial t + L_u, L_v] = 0, \tag{6.525}$$

which is analogous to (6.505).

Now, we can make use of the above invariant generation technique in the case of the superfluid equations (6.484). For this we need to construct a source density invariant $\gamma \in \Lambda^3(M)$ of equations (6.508) and inductively construct a hierarchy of additional invariants as

$$\gamma_n := L_v^n\gamma \tag{6.526}$$

for all $n \in \mathbb{Z}_+$.

Define $\beta^{(1)} \in \Lambda^1(M)$ to be the one-form

$$\beta^{(1)} := \langle u, dx\rangle \tag{6.527}$$

and observe that

$$(\partial/\partial t + L_u)\beta^{(1)} = -\rho^{-1}dP + \frac{1}{2}d|u|^2 = d(\rho^{-1}P + \frac{1}{2}|u|^2). \tag{6.528}$$

The differential two-form $d\beta^{(1)} \in \Lambda^2(M)$ satisfies

$$(\partial/\partial t + L_u)d\beta^{(1)} = d^2(\rho^{-1}P + \frac{1}{2}|u|^2) = 0 \tag{6.529}$$

owing to the identity $d^2 = 0$. Hence, the differential density three-form $\gamma := \beta^{(1)} \wedge d\beta^{(1)} \in \Lambda^3(M)$ satisfies, owing to (6.528) and (6.529), the condition

$$\begin{aligned}(\partial/\partial t + L_u)\gamma &= (\partial/\partial t + L_u)(\beta^{(1)} \wedge d\beta^{(1)}) \\ &= d\left(\rho^{-1}P + \tfrac{1}{2}|u|^2\right) \wedge d\beta^{(1)} = d\left(\left(\rho^{-1}P + \tfrac{1}{2}|u|^2\right) d\beta^{(1)}\right).\end{aligned} \tag{6.530}$$

Now, integrating of (6.530) over the whole manifold M we obtain, by virtue of Stokes' theorem, the expression

$$\begin{aligned}\tfrac{d}{dt}\int_M \beta^{(1)} \wedge d\beta^{(1)} &= \tfrac{d}{dt}\int_M (u \times (\nabla \times u))\, d^3x = \tfrac{d}{dt}\int_M (u \times \xi)\, d^3x \\ &= \oint_{\partial M} \left(\rho^{-1}P + \tfrac{1}{2}|u|^2\right) \langle du, \wedge dx\rangle = 0,\end{aligned} \tag{6.531}$$

if the boundary conditions $\langle u, n\rangle = 0$ and $\xi|_{\partial M} = 0$ are imposed on the superfluid vorticity flow. The surface measure $\langle du, \wedge dx\rangle$ on ∂M can be equivalently represented as

$$\langle du, \wedge dx\rangle = \langle\langle dx, \nabla\rangle u, \wedge dx\rangle = \langle \nabla \times u, dS\rangle = \langle \xi, dS\rangle, \tag{6.532}$$

where dS is the standard oriented Euclidean surface measure on ∂M. Since the vorticity vector $\xi|_{\partial M} = 0$, the result (6.531) follows immediately.

Assume now that the vorticity vector $\xi = \nabla \times u$ satisfies the additional constraints $L_v^n \xi|_{\partial M} = 0$ for $n \in \mathbb{Z}_+$. Then, we compute from (6.532) and (6.505) that

$$\begin{aligned}\tfrac{d}{dt}\int_M L_v^n\gamma &= \tfrac{d}{dt}\int_M L_v^n \gamma\, d^3x = \int_M L_v^n (\partial/\partial t + L_u)\gamma \\ &= \int_M L_v^n d\left(\rho^{-1}P + \tfrac{1}{2}|u|^2\right) d\beta^{(1)} = \int_M dL_v^n\left(\left(\rho^{-1}P + \tfrac{1}{2}|u|^2\right) d\beta^{(1)}\right) \\ &= \int_M dL_v^n\left(\left(\rho^{-1}P + \tfrac{1}{2}|u|^2\right)\langle du, \wedge dx\rangle\right) = \int_M dL_v^n\left(\left(\rho^{-1}P + \tfrac{1}{2}|u|^2\right)\langle \xi, dS\rangle\right) \\ &= \int_{\partial M} L_v^n\left(\left(\rho^{-1}P + \tfrac{1}{2}|u|^2\right)\langle \xi, dS\rangle\right) \\ &= \int_{\partial M} \sum_{k=0}^n C_n^k \langle L_v^k \xi, L_v^{n-k}\left(\left(\rho^{-1}P + \tfrac{1}{2}|u|^2\right) dS\right)\rangle = 0,\end{aligned} \tag{6.533}$$

which generate the generalized helicity invariants

$$H_n := \int_M \rho L_v^n (u \times \xi)\, d^3x \tag{6.534}$$

for all $n \in \mathbb{Z}_+$. Observe that all of the constraints imposed above on the vorticity vector $\xi = \nabla \times u$ are automatically satisfied if the condition $\operatorname{supp} \xi \cap \partial M = \varnothing$ holds, where supp denotes the usual *support* of a function. The results obtained can be summarized as the following theorem.

Theorem 6.16. *Assume that an incompressible superfluid, governed by the set of equations (6.518) in a domain $M \subset \mathbb{E}^3$, possesses the vorticity vector $\xi = \nabla \times u$, which satisfies the boundary constraints $L^n_{\rho^{-1}\xi}\xi|_{\partial M}$ for all $n \in \mathbb{Z}_+$. Then all of the functionals (6.534) are generalized helicity invariants of (6.518).*

The results above have some interesting modifications. To present them in detail, observe that equality (6.528) can be rewritten as

$$(\partial/\partial t + L_u)\beta^{(1)} - dh = (\partial/\partial t + L_u)\tilde{\beta}^{(1)} = 0, \tag{6.535}$$

where

$$h := \rho^{-1}P + \frac{1}{2}|u|^2, \qquad \tilde{\beta}^{(1)} := \langle u - \nabla\varphi, dx\rangle, \tag{6.536}$$

and the scalar function $\varphi : M \longrightarrow \mathbb{R}$ is chosen such that

$$(\partial/\partial t + L_u)\varphi = \nabla h. \tag{6.537}$$

Then it is clear that

$$(\partial/\partial t + L_u)d\tilde{\beta}^{(1)} = 0, \tag{6.538}$$

which follows from the commutation property $[d, \partial/\partial t + L_u] = 0$. Whence, we see that the density $\tilde{\mu} := \tilde{\beta}^{(1)} \wedge d\tilde{\beta}^{(1)} \in \Lambda^3(M)$ satisfies the condition

$$(\partial/\partial t + L_u)\tilde{\mu} = 0, \tag{6.539}$$

for all $t \in \mathbb{R}$. Similar equalities hold for densities $\tilde{\mu}_n := L^n_v\tilde{\mu} \in \Lambda^3(M)$, $n \in \mathbb{Z}_+$:

$$(\partial/\partial t + L_u)\tilde{\mu}_n = 0, \tag{6.540}$$

owing to the commutation property (6.505). Thus, the following functionals on the corresponding functional manifold $\mathcal{M}$ are invariants of the superfluid flow (6.508):

$$\mathfrak{M}_n := \int_{D_t} \tilde{\mu}_n = \int_{D_t} \rho L^n_{\rho^{-1}\xi}\langle(u - \nabla\varphi), \xi\rangle\ d^3x \tag{6.541}$$

for all $n \in \mathbb{Z}_+$ and an arbitrary domain $D_t \subset M$, independent of boundary conditions imposed on the vorticity vector $\xi = \nabla \times u$ on ∂M. Notice here that only the invariants (6.541) strongly depend on the function $\varphi : M \longrightarrow \mathbb{R}$, which implicitly depends on the velocity vector $u \in T(M)$. We mention here that the practical importance of the constructed invariants (6.541) is still not entirely clear.

6.8.4 *Conclusions*

The symplectic and symmetry analysis of the incompressible MHD superfluid, as shown above, appears to be effective for constructing the related helicity type conservation laws, which are usually important for practical applications. In particular, these conserved quantities play a decisive role [178, 15], when studying the stability of MHD superfluid flows under special boundary conditions.

Here we should also comment that the differential-geometric reformulation of the MHD equations (6.484) suggested in [178] is slightly misleading. Namely, the equality $(\partial/\partial t + L_u)\langle \rho^{-1}B, dx\rangle = 0$ is not equivalent to the magnetic field equation $\partial B/\partial t - \nabla \times (\nabla \times B) = 0$ as one can verify by easy calculations. Nonetheless, the commutator relation $[\partial/\partial t + L_u, L_{\rho^{-1}B}] = 0$ devised and all the Casimir invariants found in [178] are valid. However, there are still some open problems related to the construction of non-Casimir type MHD superfluid flows using their Hamiltonian structure, which warrant further research. Some of the results in this direction may eventually be obtained using group-theoretical and topological tools developed in [15, 335, 388], where the importance of the basic group of diffeomorphisms $\mathrm{Diff}(M)$ of a manifold $M \subset \mathbb{R}^3$ and its differential-geometric characteristics was demonstrated.

6.9 Algebraic-analytic structure of integrability by quadratures of Abel–Riccati equations

6.9.1 *Introduction*

Over a century and a half ago, Liouville posed the problem of determining when the Riccati equations $dy/dx = y^2 + a(x)y + b(x)$ is integrable by quadratures. Since then, many research efforts have been made to solve this problem, and although considerable progress has been made - especially during the last three decades - there still seems to be no completely satisfactory resolution. Within the last two decades, some promising progress has been made on the Liouville problem by studying it in the context of Lax integrability, and this is the general line we take in this section.

In the sequel, we shall further develop an approach to determining the integrability by quadratures of generalized Riccati–Abel equations, which has been applied with some success to a special case, namely the basic Riccati equation. We shall reduce a given Riccati–Abel equation to an equivalent nonlinear evolution partial differential equations with natural Cauchy–Goursat initial data, and determine their Lax type integrability, which is directly related (via Liouville) to integrability by quadratures. The foundation for this approach lies both in modern differential-geometric and Lie-algebraic techniques, and leads to a partial solution of the Liouville problem that essentially resolves the matter for the Riccati equation.

Our purpose is to describe the class of functions a and $b \in C^{\infty}(\mathbb{R};\mathbb{R})$ in the generalized Riccati–Abel equation

$$dy/dx = y^{n} + a\,(x)\,y + b\,(x)\,, \tag{6.542}$$

where $n \in \mathbb{Z}_{+}$, for which this equation is integrable by quadratures (that is, solvable in terms of elementary and algebraic functions and their integrals [14, 358]). The roots of the integrability theory that we employ - in particular the differential-algebraic Picard–Vessiot approach - trace back to the classical articles [189, 199] and were further developed in [208, 347]. First we consider the Cauchy problem for equation (6.542) with fixed functions $a, b \in C^{\infty}(\mathbb{R};\mathbb{R})$:

$$dy/dx = y^{n} + a(x)y + b(x), \quad y(x_0) = y_0, \tag{6.543}$$

with $y_0 \in \mathbb{R}$ being an arbitrary Cauchy datum at a point $x_0 \in \mathbb{R}$. From the regularity and uniqueness Theorem [14] it follows that there exists a unique solution of (6.543), differentiable with respect to x_0 and $y_0 \in \mathbb{R}$ satisfying the conditions

$$\left.\frac{\partial y}{\partial x_0}\right|_{x=x_0} = -\left(y_0^{n} + a\,(x_0)\,y_0 + b\,(x_0)\right), \qquad \left.\frac{\partial y}{\partial y_0}\right|_{x=x_0} = 1. \tag{6.544}$$

Thus, differentiating (6.543) with respect to x_0 and $y_0 \in \mathbb{R}$, we obtain a system of nonlinear evolution equations in the form:

$$dy/dt = v, \quad du/dt = n(n-1)y^{n-2}v \tag{6.545}$$

on the jet submanifold $M_0 = \left\{(u,v)^{\intercal} \in J(\mathbb{R}\times\mathbb{R}^{2};\mathbb{R}^{3}) : v_x = uv\right\}$, where $t \in \{x_0, y_0\} \subset \mathbb{R}$ is an evolution vector parameter, with the following Cauchy–Goursat data:

$$\begin{gathered}\left.\frac{\partial y}{\partial x_0}\right|_{x=x_0} = -\left(y_0^{n} + a\,(x_0)\,y_0 + b\,(x_0)\right), \qquad \left.\frac{\partial y}{\partial y_0}\right|_{x=x_0} = 1, \\ u|_{x=x_0} = ny_0^{n-1} + a\,(x_0)\,, \qquad v|_{x=x_0} = \left.\frac{\partial y}{\partial t}\right|_{x=x_0}.\end{gathered} \tag{6.546}$$

Solutions of (6.543) and (6.545) are characterized by the following simple but important lemma (see also [90]).

Lemma 6.8. *All solutions of equations* (6.545) *with conditions* (6.546) *that reduce to quadratures are also solutions of equation* (6.543), *which is also reducible to quadratures.*

The analysis that follows is focused on proving integrability by quadratures of the Cauchy–Goursat problem (6.546) for the system of partial differential equations (6.545) on the jet submanifold M_0^∞. First, the detailed analysis will be carried out for the system (6.545).

6.9.2 *General differential-geometric analysis*

We begin by proving that the evolution equations (6.545) are integrable by quadratures, inasmuch as they can be linearized via a Lax representation. Making use of the gradient-holonomic algorithm system (6.545) on the jet-submanifold $M_0^\infty \subset J_{top}(\mathbb{R} \times \mathbb{R}^2; \mathbb{R}^3)$ can be recast as a set of 2-forms $\{\alpha\} \subset \Lambda^2(J^0_{top}(\mathbb{R}\times\mathbb{R}^2;\mathbb{R}^3))$ on the adjoint jet-manifold $J^0_{top}(\mathbb{R}\times\mathbb{R}^2;\mathbb{R}^3)$ as follows:

$$\{\alpha\} = \Big\{ \alpha_1 := dy^{(0)} \wedge dx + v^{(0)} dx \wedge dt; \quad \alpha_2 := dv^{(0)} \wedge dt - u^{(0)} v^{(0)} dx \wedge dt; \tag{6.547}$$

$$\alpha_3 := du^{(0)} \wedge dx + n\,(n-1)\,(y^{(0)})^{n-2} v^{(0)} dx \wedge dt : \left(t, x, u^{(0)}, v^{(0)}, y^{(0)}\right)^{\mathsf{T}} \in \tilde{M} \Big\},$$

where $\tilde{M}$ is a finite-dimensional submanifold in $J^0_{top}\left(\mathbb{R}\times\mathbb{R}^2;\mathbb{R}^3\right)$ with local coordinates $(x, t, u^{(0)} = u, v^{(0)} = v, y^{(0)} = y)$, $t \in \{x_0, y_0\} \subset \mathbb{R}$.

The set of 2-forms (6.547) generates a closed ideal $I(\alpha) \subset \Lambda^2(\tilde{M}) \otimes \mathfrak{g}$, that is $dI(\alpha) \subset I(\alpha)$, since

$$d\alpha_1 = -dx \wedge \alpha_2, \quad d\alpha_2 = -v^{(0)} dt \wedge \alpha_3 + u^{(0)} dx \wedge \alpha_2,$$

$$d\alpha_3 = n(n-1)(n-2) v^{(0)} (y^{(0)})^{n-3} dt \wedge \alpha_1 - n(n-1)(y^{(0)})^{n-2} dx \wedge \alpha_2. \tag{6.548}$$

Therefore, the ideal $I(\alpha) \subset \Lambda^2(\tilde{M}) \otimes \mathfrak{g}$, where $\mathfrak{g}$ is a Lie algebra, and is Cartan–Frobenius integrable (due to the Cartan theorem) with the three-dimensional integral submanifold $N^3 = \left\{(x, x_0, y_0) \in \mathbb{R}^3\right\} \subset \tilde{M}$ defined locally by the condition $I(\alpha)|_N = 0$.

Integrability by quadratures of system (6.545) is equivalent [90, 308] to the vanishing on the integral submanifold $N^3 \subset \tilde{M}$ of the curvature

$\Omega \in \Lambda^2(\tilde{M}) \otimes \mathfrak{g}$ of the corresponding connection form Γ_λ, $\lambda \in \mathbb{R}$, on the principal fiber space $P(\tilde{M}, G)$:

$$\Omega_\lambda = d\Gamma_\lambda + \Gamma_\lambda \wedge \Gamma\lambda \in I(\alpha) \otimes \mathfrak{g}, \tag{6.549}$$

where $\mathfrak{g}$ is the Lie algebra of a structural group G of the principal fiber bundle $P(\tilde{M}, G)$.

Now, we seek this connection form $\Gamma\lambda \in \Lambda^1(\tilde{M}) \otimes \mathfrak{g}$ belonging to some, as yet undetermined, Lie algebra $\mathfrak{g}$ of a structure group G. This 1-form can be represented using (6.547), as follows:

$$\Gamma_\lambda := \Gamma_\lambda^{(x)} \left(u^{(0)}, v^{(0)}, y^{(0)}\right) dx + \Gamma_\lambda^{(t)} \left(u^{(0)}, v^{(0)}, y^{(0)}\right) dt, \tag{6.550}$$

where the elements $\Gamma_\lambda^{(x)}, \Gamma_\lambda^{(t)} \in \mathfrak{g}$, $t \in \{x_0, y_0\}$, satisfy the determining equations

$$\begin{aligned}
\Omega_\lambda = {} & \frac{\partial \Gamma_\lambda^{(x)}}{\partial u^{(0)}} du^{(0)} \wedge dx + \frac{\partial \Gamma_\lambda^{(x)}}{\partial v^{(0)}} dv^{(0)} \wedge dx + \frac{\partial \Gamma_\lambda^{(x)}}{\partial y^{(0)}} dy^{(0)} \wedge dx + \frac{\partial \Gamma_\lambda^{(t)}}{\partial u^{(0)}} du^{(0)} \wedge dt \\
& + \frac{\partial \Gamma_\lambda^{(t)}}{\partial v^{(0)}} dv^{(0)} \wedge dt + \frac{\partial \Gamma_\lambda^{(t)}}{\partial y^{(0)}} dy^{(0)} \wedge dt + [\Gamma_\lambda^{(x)}, \Gamma_\lambda^{(t)}] dx \wedge dt \\
= {} & g_1(dy^{(0)} \wedge dx + v^{(0)} dx \wedge dt) + g_2(dv^{(0)} \wedge dt - u^{(0)} v^{(0)} dx \wedge dt) \\
& + g_3(du^{(0)} \wedge dx + n(n-1)(y^{(0)})^{n-2} v^{(0)} dx \wedge dt) \in I(\alpha) \otimes \mathfrak{g}
\end{aligned} \tag{6.551}$$

for some $\mathfrak{g}$-valued functions $g_1, g_2, g_3 : M \to \mathfrak{g}$.

From (6.551) one easily finds that

$$[c]l \frac{\partial \Gamma_\lambda^{(x)}}{\partial u^{(0)}} = g_3, \quad \frac{\partial \Gamma_\lambda^{(x)}}{\partial v^{(0)}} = 0, \quad \frac{\partial \Gamma_\lambda^{(x)}}{\partial y^{(0)}} = g_1, \quad \frac{\partial \Gamma_\lambda^{(t)}}{\partial u^{(0)}} = 0, \quad \frac{\partial \Gamma_\lambda^{(t)}}{\partial v^{(0)}} = g_2, \tag{6.552}$$

$$\frac{\partial \Gamma_\lambda^{(t)}}{\partial y^{(0)}} = 0, \quad \left[\Gamma_\lambda^{(x)}, \Gamma_\lambda^{(t)}\right] = g_1 v^{(0)} - g_2 u^{(0)} v^{(0)} + g_3 n(n-1)(y^{(0)})^{n-2} v^{(0)}.$$

The set (6.552) has the following unique global solution:

$$\Gamma_\lambda^{(x)} = X_1 u^{(0)} + \sum_{m=2}^{n} X_m (y^{(0)})^{n-m}, \quad \Gamma_\lambda^{(t)} = X_0 v^{(0)}, \tag{6.553}$$

where $X_j \in \mathfrak{g}$, $0 \leq j \leq n$, are some constant elements on M of the Lie algebra $\mathfrak{g}$ that we seek, which satisfying the structural equations

$$[X_1, X_0] = -X_0, \ [X_2, X_0] = n\,(n-1)\,X_1, \ [X_{m+1}, X_0] = (n-m)\,X_m, \tag{6.554}$$

where $n \geq m \geq 2$.

6.9.3 Lie-algebraic analysis of the case $n = 2$

We can use (6.549) to determine the Lie algebra structure of $\mathfrak{g}$, by taking into account the holonomy Lie group reduction theorem of Ambrose, Singer and Loos [12, 173, 238]. Namely, the holonomy Lie algebra $\mathfrak{g}(h) \subset \mathfrak{g}$ is generated by the covariant derivatives composition of the $\mathfrak{g}$-valued curvature form $\Omega_\lambda \in \Lambda^2(M) \otimes \mathfrak{g}$:

$$\mathfrak{g} := \mathrm{span}_{\mathbb{C}}\{\nabla_1^{j_1}\nabla_2^{j_2}\dots\nabla_n^{j_n}\Omega_\lambda \in \mathfrak{g} : j_k \in \mathbb{Z}_+\}, 1 \le k \le n\}, \tag{6.555}$$

where the covariant derivatives $\nabla_j : \Lambda(M) \longrightarrow \Lambda(M)$ are given as $\nabla_j := \partial/\partial z_j + \Gamma_\lambda^{(j)}(z)$, $z_j \in M$, $j = 1, \dots, n$. Therefore, reducing the associated principal fibered frame space $P(M, G)$ to the principal fiber bundle $P(M, G(h))$ using the Ambrose–Singer theorem, where $G(h) \subset G$ is the corresponding holonomy Lie group of the connection Γ_λ on P, we must verify that $\mathfrak{g}(h) \subset \mathfrak{g}$ is a subalgebra in $\mathfrak{g}$: $\nabla_x^m \nabla_t^n \Omega_\lambda \in \mathfrak{g}(h)$ for all $m, n \in \mathbb{Z}_+$.

To do this, we shall wrap up the above transfinite procedure. One can easily verify that the simplest equality

$$\mathfrak{g}(h) = \mathfrak{g}(h)_1 := \mathrm{span}_{\mathbb{C}}\left\{\nabla_x^m \nabla_t^n \Omega \in \mathfrak{g} : m + n = 0, 1\right\}$$

meets all of the conditions mentioned above. This means that one can put

$$\begin{aligned}
\mathfrak{g}(h) = \mathfrak{g}(h)_1 &:= \mathrm{span}_{\mathbb{C}}\left\{\nabla_x^m \nabla_t^n g_j \in \mathfrak{g} : j = 1, 2, 3, m + n = 0, 1\right\} \\
&= \mathrm{span}_{\mathbb{C}}\left\{g_j \in \mathfrak{g}; \frac{\partial g_j}{\partial x} + [g_j, \Gamma^{(x)}], \frac{\partial g_j}{\partial t} + [g_j, \Gamma^{(t)}] \in \mathfrak{g} : j = 1, 2, 3\right\} \\
&= \mathrm{span}_{\mathbb{C}}\left\{X_0, X_1, [X_1, X_2]\right\} = \mathrm{span}_{\mathbb{C}}\left\{X_j \in \mathfrak{g} : j = 0, 1, 3\right\},
\end{aligned} \tag{6.556}$$

where $[X_1, X_2] = X_3 \in \mathfrak{g}$. To satisfy the set of relations (6.554) we need to use expansions over the basis of the external element $X_2 \in \mathfrak{g}(h)$:

$$X_2 := \sum_{j=0, j\neq 2}^{3} q_j X_j. \tag{6.557}$$

Substituting the expansion (6.557) into (6.554), we compute that $q_0 = -\lambda$, $q_1 = 0$, $q_3 = 1$, for an arbitrary complex parameter $\lambda \in \mathbb{C}$, that is

$$\mathfrak{g}(h) = \mathrm{span}_{\mathbb{C}}\left\{X_0, X_1, X_3\right\},$$

where

$$[X_0, X_3] = -2X_1, \quad [X_1, X_3] = -\lambda X_0 + X_3, \quad X_2 = -\lambda X_0 + X_3. \tag{6.558}$$

We can now prove that this finite-dimensional holonomy Lie algebra $\mathfrak{g}(h)$, which is generated by commutator relationships (6.554) and (6.558), has the general solution

$$X_0 = L_{-1} - 2L_0 + L_1, \quad X_1 = L_{-1} - L_0,$$
$$X_2 = \left(-\frac{\lambda}{2} - 1\right) L_{-1} + \lambda L_0 - \frac{\lambda}{2} L_1, \tag{6.559}$$

with $L_{-1}, L_0, L_1 \in \mathfrak{g}(h)$ satisfying the canonical sl(2)-commutation relations with parameter $\lambda \in \mathbb{C}$.

It is straightforward to find a (2×2)-matrix representation of L-elements of $\mathfrak{g}$; namely,

$$L_{-1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad L_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}. \tag{6.560}$$

Therefore, from (6.553), (6.559) and (6.560) we obtain the following expressions for $\Gamma_\lambda^{(x)}$, $\Gamma_\lambda^{(t)}$, $t \in \{x_0, y_0\}$:

$$\Gamma_\lambda^{(x)} = \begin{pmatrix} -\frac{u}{2} + \frac{\lambda}{2} & \frac{\lambda}{2} \\ u - \frac{\lambda}{2} - 1 & \frac{u}{2} - \frac{\lambda}{2} \end{pmatrix}, \quad \Gamma_\lambda^{(t)} = \begin{pmatrix} -v & -v \\ v & v \end{pmatrix}, \tag{6.561}$$

and the 1-form $\Gamma_\lambda \in \Lambda^1(M) \otimes \mathfrak{g}$

$$\Gamma_\lambda = (X_1 u + X_2)dx + X_0 v dt, \tag{6.562}$$

generating parallel transport of vectors $f \in \mathcal{H}$ from a linear representation space $\mathcal{H}$ of the holonomy Lie algebra $\mathfrak{g}(h)$:

$$df + \Gamma_\lambda f = 0 \tag{6.563}$$

along the integral submanifold $N^3 \subset M$ of the ideal $I(\alpha) \subset \Lambda^2(\tilde{M}) \times \mathfrak{g}$, which is generated by the set of 2-forms (6.547). The result (6.563) means also that the dynamical system (6.545) is endowed with the standard Lax representation, containing the spectral parameter $\lambda \in \mathbb{C}$, which is a necessary condition for its integrability by quadratures [14, 227, 262, 326, 406].

6.9.4 *Generalized spectral problem*

Now we proceed to consider the following generalized spectral problem for the system (6.545) on the 2π-periodic manifold $\tilde{M}$:

$$Lf := \left(\partial/\partial x + \Gamma_\lambda^{(x)}[u, v, \lambda]\right) f = 0, \tag{6.564}$$

$f \in \mathcal{H} := L^\infty(\mathbb{R};\mathbb{C}^2)$. Still, we need to analyze the spectral properties of problem (6.564) in more detail.

Let $Y(x, x_0; \lambda)$ be the fundamental solution of equation (6.564), which is normalized to the identity matrix at $x = x_0 \in \mathbb{R}$, that is $Y(x_0, x_0; \lambda) = \mathbf{1}$ for all $x_0 \in \mathbb{R}$, $\lambda \in \mathbb{C}$. Any solution of (6.564) can obviously be represented as

$$f(x, x_0; \lambda) = Y(x, x_0; \lambda) f_0(\lambda), \tag{6.565}$$

where $f_0(\lambda) \in \mathbb{C}^2$ is some initial Cauchy data at $x = x_0 \in \mathbb{R}$.

Consider the value of $f(x, x_0; \lambda) \in \mathcal{H}$ at $x = x_0 + 2\pi N$, where $N \in \mathbb{Z}_+$. Owing to the periodicity of the manifold $\tilde{M}$ in the independent variable $x \in \mathbb{R}$, one readily infers from (6.565) that

$$f(x_0 + 2\pi N, x_0; \lambda) = S^N(x_0; \lambda) f_0(\lambda),$$

where $S(x_0; \lambda) := Y(x_0 + 2\pi, x_0; \lambda)$ is the monodromy (transfer) matrix of the periodic differential equation (6.564).

The monodromy matrix $S(x_0; \lambda)$, $\lambda \in \mathbb{C}$, possesses the following useful properties [232, 233, 262, 406]: 1°) the matrix $S(x_0; \lambda)$, $\lambda \in \mathbb{C}$, as a function of the parameter $x_0 \in \mathbb{R}$ satisfies the Novikov–Marchenko equation

$$dS/dx_0 = [-\Gamma_\lambda^{(x)}, S] \tag{6.566}$$

for all $x_0 \in \mathbb{R}$, where $[\cdot, \cdot]$ is the usual matrix commutator; 2°) the eigenvalue $\varrho(\lambda)$ of the matrix $S(x_0; \lambda)$, $\lambda \in \mathbb{C}$, does not depend on the variable $x_0 \in \mathbb{R}$; and 3°) the eigenvalue $\varrho(\lambda)$ of the matrix $S(x_0; \lambda)$, $\lambda \in \mathbb{C}$, as a functional on manifold M_0^∞, is independent of the evolution parameter $t \in \{x_), y_)\} \subset \mathbb{R}$. Since $d\varrho(\lambda)/dt = 0$ for all $t \in \{x_0, y_0\} \subset \mathbb{R}$, we claim and shall show that as $|\lambda| \to \infty$, $\varrho(\lambda) \in \mathcal{D}(\tilde{M})$ is a generating functional of conservation laws of system (6.545). Hence, one can find an infinite hierarchy of conservation laws of system (6.545), making use of (6.564). We introduce the function $\sigma(x; \lambda) := \dfrac{\partial}{\partial x} \log \bar{f}_1(x, x_0; \lambda)$, $\lambda \in \mathbb{C}$, where $\bar{f}_1(x, x_0; \lambda)$ is the first coordinate of the vector-eigenfunction $\bar{f} \in \mathcal{H}$ of the monodromy matrix $S(x; \lambda)$:

$$S(x; \lambda)\bar{f}(x, x_0; \lambda) = \varrho(\lambda)\bar{f}(x, x_0; \lambda),$$

which is normalized at $x = x_0$, that is $\bar{f}_1(x = x_0, x_0; \lambda) = 1$. Substituting the function $\bar{f} \in \mathcal{H}$ into (6.564) one readily finds the following differential Riccati equation for the function $\sigma(x; \lambda)$:

$$\begin{aligned} \sigma_x &= -\sigma^2 + \frac{1}{2}u_x - \left(-u + \frac{\lambda}{2} + 1\right)\frac{\lambda}{2} + \left(\frac{1}{2}u - \frac{\lambda}{2}\right)^2, \\ u &= v_x/v, \end{aligned} \tag{6.567}$$

where $(\cdot)_{nx} := d^n(\cdot)/dx^n$, $x \in \mathbb{R}$, $n \in \mathbb{Z}_+$.

Assuming that $\sigma(x;\lambda)$ has an asymptotic solution of the form

$$\sigma(x;\lambda) \sim \sum_{j\in\mathbb{Z}_+} \sigma_j[u,v]\varrho^{-j+\tilde{\sigma}}(\lambda), \tag{6.568}$$

as $|\lambda| \to +\infty$, with respect to the parameter $\varrho(\lambda) \in \mathbb{C}$ which is an analytic function, and $\tilde{\sigma} \in \mathbb{Z}_+$ is a fixed number, one easily derives from (6.567) the recursion formulae for $\sigma_j[u,v]$, $j \in \mathbb{Z}_+$:

$$-\sigma_0^2 = \frac{1}{2}, \quad \sigma_{0,x} = -(\sigma_0\sigma_1)^2, \quad \sigma_{1,x} = -(2\sigma_0\sigma_2 + \sigma_1^2) + \frac{u_x}{2} + \frac{u^2}{4},$$

$$\sigma_{2,x} = -(2\sigma_0\sigma_3 + 2\sigma_1\sigma_2), \quad \sigma_{3,x} = -(2\sigma_0\sigma_4 + 2\sigma_1\sigma_3 + \sigma_2^2),$$

$$\sigma_{4,x} = -(2\sigma_0\sigma_5 + 2\sigma_1\sigma_4 + 2\sigma_2\sigma_3), \dots ,$$

and from them the following polynomial functionals $\sigma_j[u,v]$, $j \in \mathbb{Z}_+$:

$$\sigma_0 = \frac{\sqrt{2}}{2}i,\ \sigma_1 = 0,\ \sigma_2 = -\frac{\sqrt{2}i}{4}\left(\frac{1}{2}u^2 + u_x\right),\ \sigma_3 = \frac{1}{4}(u_{xx} + uu_x),$$

$$\sigma_4 = \frac{i\sqrt{2}}{16}\left(2uu_{xx} + 2u_{xxx} + u_x^2 - \frac{u^4}{4} - u_xu^2\right),$$

$$\sigma_5 = -\frac{1}{8}(u_{xxxx} + uu_{xxx} + u_xu_{xx} - u^3u_x - u^2u_{xx} - 2u_x^2u),$$

$$\sigma_6 = \frac{\sqrt{2}}{32}i\left(-2u_{xxxxx} - 2u_xu_{xxx} - u_{xx}^2 - 2uu_{xxxx} + \frac{13}{2}u^2u_x^2 + 3u_{xx}u^3\right.$$

$$\left.+3u^2u_{xxx} + 16uu_xu_{xx} + 5u_x^3 - \frac{3}{4}u^4u_x - \frac{u^6}{8}\right),$$

and so on.

Since

$$\varrho(\lambda) = \exp\left[\int_0^{2\pi} \sigma(x;\lambda)dx\right],$$

$\lambda \in \mathbb{C}$, (6.568) and 3) implies that all the functionals

$$\gamma_j = \int_0^{2\pi} dx\sigma_j\left[u,v\right],$$

$j \in \mathbb{Z}_+$, are conservation laws for (6.545). Thus, we have explicit solutions for γ_j, $j \in \mathbb{Z}_+$; namely,

$$\gamma_0 = \pi i\sqrt{2}, \quad \gamma_1 = 0, \quad \gamma_2 = \frac{i\sqrt{2}}{8}\int\limits_0^{2\pi} u^2 dx,$$

$$\gamma_3 = 0, \quad \gamma_4 = \frac{i\sqrt{2}}{16}\int\limits_0^{2\pi}\left(uu_{xx} - \frac{1}{4}u^4\right)dx,$$

$$\gamma_5 = 0, \quad \gamma_6 = -\frac{i\sqrt{2}}{32}\int\limits_0^{2\pi}\left(-u_x u_{xxx} + \frac{5}{2}u^2 u_x^2 + \frac{u^6}{8}\right)dx, \quad \gamma_7 = 0, \tag{6.569}$$

$$\gamma_8 = \frac{i\sqrt{2}}{64}\int\limits_0^{2\pi}\left(-u_{xxxxx}u_x + \frac{7}{2}u^2 u_x u_{xxx} + \frac{7}{4}uu_x^2 u_{xx} + \frac{7}{8}u^5 u_{xx} - \frac{5}{64}u^8\right)dx, \ \ldots,$$

and so on.

6.9.5 *Novikov–Marchenko commutator equation*

We shall now make use of the Novikov–Marchenko equation (6.566) for a deeper analysis of the system (6.545). Let us denote $\Delta(\lambda) := \operatorname{tr} S(x;\lambda)$, $\lambda \in \mathbb{C}$, as a normalized trace of the monodromy matrix $S(x;\lambda) : \mathbb{C}^2 \to \mathbb{C}^2$ at $x \in \mathbb{R}$. By virtue of the results stated above, we know that the functional $\Delta(\lambda) \in \mathcal{D}(M_0^\infty)$ is a generating function of conservation laws for (6.545). Clearly, the same is true for all functionals $\operatorname{tr} S^k(x;\lambda)$, $k \in \mathbb{Z}$, $\lambda \in \mathbb{C}$, at $x \in \mathbb{R}$, but they may not all be functionally independent. It follows from the Novikov–Marchenko equation (6.566) and [247, 406] that the following relation holds

$$\eta \operatorname{grad}\Delta(\lambda) = \lambda\vartheta \operatorname{grad}\Delta(\lambda), \tag{6.570}$$

for all $\lambda \in \mathbb{C}$, where (η, ϑ) is a pair of implectic operators [137, 262, 286, 288, 290, 291, 326] acting from $T^*(M_0^\infty)$ into $T(M_0^\infty)$. Since the system (6.545) is considered on the singular submanifold M_0, we are forced to introduce new frame-regularizing coordinates $(\widetilde{x}, \widetilde{t}) \in \mathbb{R}\times\mathbb{R}^2$ defined as

$$\widetilde{x} = x, \quad \widetilde{t} = x + t, \tag{6.571}$$

where $t \in \{x_0, y_0\} \subset \mathbb{R}$. In these coordinates, the system (6.545) takes the form of a nondegenerate evolution system on $J(\mathbb{R};\mathbb{R}^2)$:

$$\frac{d\widetilde{u}}{d\widetilde{t}} = 2\widetilde{v}, \quad \frac{d\widetilde{v}}{d\widetilde{t}} = \widetilde{v}_{\widetilde{x}} - \widetilde{u}\widetilde{v}. \tag{6.572}$$

The connection matrices $\Gamma_\lambda^{(\widetilde{x})}$, $\Gamma_\lambda^{(\widetilde{t})} \in \mathfrak{g}$ corresponding to (6.561) are given as follows:

$$\Gamma_\lambda^{(\widetilde{x})} = \begin{pmatrix} -\frac{\widetilde{u}}{2} - \widetilde{v} + \frac{\lambda}{2} & \frac{\lambda}{2} - \widetilde{v} \\ \widetilde{u} + \widetilde{v} - \frac{\lambda}{2} - 1 & \frac{u}{2} + \widetilde{v} - \frac{\lambda}{2} \end{pmatrix}, \quad \Gamma_\lambda^{(\widetilde{t})} = \begin{pmatrix} -\widetilde{v} & -\widetilde{v} \\ \widetilde{v} & \widetilde{v} \end{pmatrix}. \tag{6.573}$$

Whence, the Novikov–Marchenko equation (6.566) takes the form

$$\begin{aligned} s_{\widetilde{x}} &= \left(\widetilde{v} - \frac{\lambda}{2}\right) s_{21} + \left(\widetilde{v} + \widetilde{u} - \frac{\lambda}{2} - 1\right) s_{12}, \\ s_{12,\widetilde{x}} &= -2\left(\widetilde{v} - \frac{\lambda}{2}\right) s + (2\widetilde{v} + \widetilde{u} - \lambda) s_{12}, \\ s_{21,\widetilde{x}} &= -2\left(\widetilde{v} + \widetilde{u} - \frac{\lambda}{2} - 1\right) s + (-\widetilde{u} - 2\widetilde{v} + \lambda) s_{21}, \end{aligned} \tag{6.574}$$

for the components $s_{ij}(\widetilde{x};\lambda)$, $i,j = 1,2$, of the monodromy matrix $S(\widetilde{x};\lambda) : \mathbb{C}^2 \to \mathbb{C}^2$ at $\widetilde{x} \in \mathbb{R}$ for all $\lambda \in \mathbb{C}$, where $s := \frac{1}{2}(s_{11} - s_{22})$. For the operator $L := \partial/\partial\widetilde{x} + \Gamma_\lambda^{(\widetilde{x})}$, with $\Gamma_\lambda^{(\widetilde{x})}$ given by (6.573), one readily finds that

$$\operatorname{grad} \Delta(\lambda) = \frac{1}{2}\begin{pmatrix} s - s_{12} \\ 2s - s_{12} + s_{21} \end{pmatrix}, \tag{6.575}$$

for all $\lambda \in \mathbb{C}$. Now, it follows directly from expression (6.575) and equations (6.574) that we have the spectral gradient identity

$$\widetilde{\eta} \operatorname{grad}\Delta(\lambda) = \lambda\widetilde{\vartheta} \operatorname{grad}\Delta(\lambda), \tag{6.576}$$

valid for all $\lambda \in \mathbb{C}$, where

$$\widetilde{\eta} = \begin{pmatrix} -2\partial & \partial^2 + \partial\widetilde{u} \\ -\partial^2 + \widetilde{u}\partial & \widetilde{v}\partial + \partial\widetilde{v} \end{pmatrix}, \quad \widetilde{\vartheta} = \begin{pmatrix} 0 & 0 \\ 0 & \partial \end{pmatrix}, \tag{6.577}$$

with $\partial := \partial/\partial\widetilde{x}$. Using (6.575) and (6.576), one can further make the Moser reduction [270, 271, 326] of problem (6.572) on a finite-dimensional nonlocal invariant submanifold carrying a natural symplectic structure, which gives rise to a new reduction of the problem to that of Liouville–Arnold type.

6.9.6 *Representation of the holonomy Lie algebra sl(2)*

We now analyze, using some results of [12, 173, 197, 198, 238], the following useful representation of the holonomy Lie algebra sl(2) by means of the following vector fields on the circle $\mathbb{S}^1$:

$$L_{-1} = \frac{\partial}{\partial \xi}, \quad L_0 = \xi \frac{\partial}{\partial \xi}, \quad L_1 = \xi^2 \frac{\partial}{\partial \xi},$$

$\xi \in \mathbb{S}^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$. The corresponding connection operators $\Gamma_\lambda^{(x)}$ and $\Gamma_\lambda^{(t)} \in \mathfrak{g}$ then take the form

$$\Gamma_\lambda^{(x)} = \left[-\frac{1}{2}\lambda\xi^2 - (u-\lambda)\xi + u - \frac{\lambda}{2} - 1 \right] \frac{\partial}{\partial \xi}, \tag{6.578}$$

$$\Gamma^{(t)} = (v - 2v\xi - v\xi^2)\frac{\partial}{\partial \xi}.$$

Consider now the system

$$\frac{\partial f}{\partial x} = A\frac{\partial f}{\partial \xi}, \quad \frac{\partial f}{\partial t} = B\frac{\partial f}{\partial \xi}, \tag{6.579}$$

where $f \in \mathcal{H}(\mathbb{S}^1)$ is a complex analytic continuation of f from the circle $\mathbb{S}^1$ to all $\mathbb{C}$, and $A\frac{\partial}{\partial \xi} = -\Gamma_\lambda^{(x)}$, $B\frac{\partial}{\partial \xi} = -\Gamma_\lambda^{(t)}$, as defined by (6.578).

The following lemmas [66, 67, 65, 228] can be readily verified. A proof of the first result below (in particular, (6.580)) follows simply from the definition of the associative product in the space of reduced pseudo-differential expressions, which can be verified by a straightforward calculation.

Lemma 6.9. *The dynamical system* (6.545) *is equivalent to the vector field Lax representation*

$$\frac{d\widetilde{L}}{dt} = \left[\widetilde{L}, p(\widetilde{L})\right], \tag{6.580}$$

where

$$\widetilde{L} = \frac{\partial l}{\partial x}\frac{\partial}{\partial \xi} - \frac{\partial l}{\partial \xi}\frac{\partial}{\partial x}, \quad p(\widetilde{L})\,(l) = \frac{\partial p}{\partial x}\frac{\partial}{\partial \xi} - \frac{\partial p}{\partial \xi}\frac{\partial}{\partial x}$$

and $l := l\,(x;\xi) \sim \sum_{j\geq 0} l_j\,[u,v]\,\xi^{-(j-s)} \in \mathcal{H}(\mathbb{C})$ *is an element of the Lie algebra,* $\xi \in \mathbb{C}$ *is symbol expression and* $s \in \mathbb{Z}_+$ *is a nonnegative integer.*

Lemma 6.10. *The following equality holds:* $l = f$*, where* $f \in \mathcal{H}(\mathbb{C})$ *is an analytic solution of* (6.579).

Proof. Since $\widetilde{L}f = 0$ with the operator $\widetilde{L}$ defined in Lemma 6.10, using the usual method of characteristics we find that

$$\frac{d\xi}{dx} = -\frac{\partial l}{\partial x} \Big/ \frac{\partial l}{\partial \xi}.$$

On the other hand, from the characteristic equations of the first equation in (6.579), one finds that $dx/1 = -d\xi/A$. Whence, $A\dfrac{\partial l}{\partial \xi} = \dfrac{\partial l}{\partial x}$, so $l \in \mathcal{H}(\mathbb{C})$ is a solution of (6.579), and the proof is complete. $\square$

Taking into account Lemmas 6.8 and 6.10, one can find the following asymptotic solution of (6.579)

$$l\,(x;\xi) \sim \sum_{j\geq 1} l_j\,[u,v]\,\xi^{-(j-1)}.$$

After simple calculations, we find for functionals $l_j \in \mathcal{H}(\mathbb{C})$, $j \in \mathbb{Z}_+$ the following recurrent chain of differential equations:

$$l_{1,x} = -\frac{\lambda}{2} l_2, \quad l_{2,x} = (\lambda - u)l_2 - \lambda l_3,$$

$$l_{3,x} = \left(u - \frac{\lambda}{2} - 1\right) l_2 + 2(\lambda - u)l_3 - \frac{3}{2}\lambda l_4, \ldots,$$

$$l_{k,x} = \left(u - \frac{\lambda}{2} - 1\right)(k-2)l_{k-1} + (k-1)(\lambda - u)l_k - \frac{k}{2}\lambda l_{k+1}, \ldots\ ,$$

and so on, which yield

$$\begin{aligned}
&l_1 = \varphi, \quad l_2 = -\frac{2}{\lambda}\varphi_x, \quad l_3 = \frac{2}{\lambda^2}((\lambda - u)\varphi_x + \varphi_{xx}),\\
&l_4 = \frac{4}{3\lambda^3}\left[\left(-2u^2 - u_x + 3\lambda u - \frac{3}{2}\lambda^2 + \lambda\right)\varphi_x + 3(\lambda - u)\varphi_{xx} - \varphi_{xxx}\right],\\
&l_5 = \frac{2}{3\lambda^4}\left[\left(-4\lambda^3 + (12u + 4)\lambda^2 - (14u^2 + 4u + 6u_x)\lambda + 6u^3 + 7uu_x + u_{xx}\right)\varphi_x\right.\\
&\quad \left. +(10\lambda^2 - 2\lambda(10u + 1) + 11u^2 + 4u_x)\varphi_{xx} + 6(u - \lambda)\varphi_{xxx} + \varphi_{xxxx}\right], \ldots,
\end{aligned}$$

and so on, with $\varphi = \varphi\,(x;\lambda)$, $\lambda \in \mathbb{C}$, a t-independent function. Thus, we have accumulated the desired information about the integrable by quadratures Cauchy–Goursat conditions (6.546). A more detailed analysis of the structure obtained requires new tools that are better suited to this task.

6.9.7 Algebraic-geometric properties of the integrable Riccati equations: The case $n = 2$

A proof of the following result can readily be fashioned directly from our work in subsection 6.9.2.

Lemma 6.11. *The system* (6.545) *on* $\tilde{M}$ *possesses the commutator Lax matrix representation*

$$[X_\lambda, T_\lambda] = 0, \tag{6.581}$$

where $X_\lambda := \partial/\partial x + \Gamma_\lambda^{(x)}$, $T_\lambda := \partial/\partial t + \Gamma_\lambda^{(t)}$, $t \in \{x_0, y_0\}$, *that is*

$$X_\lambda = \frac{\partial}{\partial x} - \begin{pmatrix} \frac{u}{2} - \frac{\lambda}{2} & -\frac{\lambda}{2} \\ -u + \frac{\lambda}{2} + 1 & -\frac{1}{2}u + \frac{\lambda}{2} \end{pmatrix}, \quad T_\lambda = \frac{\partial}{\partial t} - \begin{pmatrix} v & v \\ -v & -v \end{pmatrix}, \tag{6.582}$$

are linear matrix differential operators in the space of complex vector-valued functions with $\lambda \in \mathbb{C}$ *an arbitrary complex parameter.*

The notation (6.581) implies that the commutator of the operators (6.582) on the solutions of (6.545) is identically zero for all $\lambda \in \mathbb{C}$. The Lax representation (6.581) enables us to find the solutions to equations (6.545) by means of the classical algebraic-geometric methods [111, 262, 358, 406], via reduction to the Jacobi problem of inversion of Abelian integrals on hyperelliptic Riemann surfaces, which employs multi-dimensional Riemann ϑ-functions [406, 409]. Following [247, 262, 406], we consider the following compatible linear equations for the vector-valued complex function $g_\lambda = (g_1, g_2)^\intercal \in \mathcal{H} = L_\infty(\mathbb{R}^3; \mathbb{C}^2)$:

$$X_\lambda g_\lambda(x, x_0, y_0) = 0, \quad T_\lambda g_\lambda(x, x_0, y_0) = 0, \tag{6.583}$$

where $\lambda \in \mathbb{C}$ is an arbitrary parameter. By a direct computation from (6.583), we obtain the equations for $\zeta = g_1 g_2$, $\psi = -g_1^2$, $\chi = g_2^2$ in the form

$$\frac{\partial \zeta}{\partial x} = -\frac{\lambda}{2}\chi + \psi\left(u - \frac{\lambda}{2} - 1\right), \quad \frac{\partial \zeta}{\partial t} = (\chi + \psi)v, \quad \frac{\partial \psi}{\partial x} = \psi(u - \lambda) + \zeta\lambda,$$
$$\frac{\partial \psi}{\partial t} = 2v(\psi - \zeta), \quad \frac{\partial \chi}{\partial x} = -\chi(u - \lambda) + 2\zeta\left(-u + \frac{\lambda}{2} + 1\right),$$
$$\frac{\partial \chi}{\partial t} = -2v(\chi + \zeta), \tag{6.584}$$

which are compatible on the solution submanifold of the system (6.545). The solutions of (6.584) are characterized by the following lemma [247, 262]:

Lemma 6.12. *The system of differential equations* (6.584) *possesses a polynomial* (*in* $\lambda \in \mathbb{C}$) *solution*

$$\zeta = \sum_{k=0}^{N} \zeta_k(x, x_0, y_0)\lambda^k, \quad \psi = \sum_{k=0}^{N} \psi_k(x, x_0, y_0)\lambda^k, \quad \chi = \sum_{k=0}^{N} \chi_k(x, x_0, y_0)\lambda^k \tag{6.585}$$

with $N \in \mathbb{Z}_+$ *fixed, if and only if the coefficients* ζ_k, ψ_k, χ_k, $0 \leq k \leq N$, *satisfy certain compatible, autonomous systems of nonlinear nonlocal ordinary differential equations, and*

$$u(x, x_0, y_0) = \frac{\psi_{N-1} - \zeta_{N-1}}{\zeta_N}, \quad v_x(x, x_0, y_0) = \frac{\psi_{N-1} - \zeta_{N-1}}{\zeta_N}\, v(x, x_0, y_0). \tag{6.586}$$

If, furthermore, the relations

$$\begin{aligned}
&\zeta(x', x_0', y_0', \lambda^\star) = \zeta^\star(x', x_0', y_0', \lambda), \quad \psi(x', x_0', y_0', \lambda^\star) = \psi^\star(x', x_0', y_0', \lambda),\\
&\psi_N(x', x_0', y_0') = \zeta_N(x', x_0', y_0') = -\chi_N(x', x_0', y_0'),\\
&\chi(x', x_0', y_0', \lambda^\star) = \chi^\star(x', x_0', y_0', \lambda),\\
&\zeta_{N-1}(x', x_0', y_0') - \psi_{N-1}(x', x_0', y_0') = \zeta^\star_{N-1}(x', x_0', y_0') - \psi^\star_{N-1}(x', x_0', y_0')
\end{aligned} \tag{6.587}$$

obtain at some point $(x', x_0', y_0') \in \mathbb{R} \times \mathbb{R}^2$, *then the autonomous systems of nonlinear ordinary differential equations have a solution for all* $x \in \mathbb{R}$, $t \in \{x_0, y_0\} \subset \mathbb{R}$, *and the functions* $u(x, x_0, y_0)$ *and* $v(x, x_0, y_0)$ *found from* (6.586) *are real infinitely differentiable solutions of* (6.545).

Proof. First, we substitute solution (6.585) into system (6.584) and equate the coefficients of like powers of $\lambda \in \mathbb{C}$. As a result, we obtain systems both of differential and algebraic equations. Using these algebraic equations, we obtain two autonomous nonlocal systems of nonlinear ordinary differential equations of the form

$$\frac{\partial z_i}{\partial x} = F_{1i}(z_0, \ldots, z_{3N+2}), \quad \frac{\partial z_i}{\partial t} = F_{2i}(z_0, \ldots, z_{3N+2}), \tag{6.588}$$

where $z_i = \zeta_i$, $z_{N+1+i} = \psi_i$, $z_{2N+2+i} = \chi_i$, $0 \leq i \leq N$, and F_{ki}, $k = 1, 2$, $0 \leq i \leq 3N+2$, are polynomials in $z \in \mathbb{R}^{3N+3}$. The compatibility condition of system (6.588) can be written as

$$\sum_{j=0}^{3N+2} \left(\frac{\partial F_{1i}}{\partial z_j} F_{2j} - \frac{\partial F_{2i}}{\partial z_j} F_{1j} \right) = 0, \quad 0 \leq i \leq 3N+2,$$

which can be verified by a direct calculation. On the other hand, from these autonomous systems one easily obtains the relationship for the function u:

$$u = \frac{\psi_{N-1} - \zeta_{N-1}}{\zeta_N},$$

which together with the condition $v_x = \dfrac{\psi_{N-1} - \zeta_{N-1}}{\zeta_N} v$ coincides with system (6.545). Thus, the proof is complete. $\square$

As a consequence of system (6.584), one readily deduces that $\dfrac{\partial}{\partial t}(\zeta^2 + \chi\psi) = 0$, $t \in \{x_0, y_0\} \subset \mathbb{R}$, and $\dfrac{\partial}{\partial x}(\zeta^2 + \chi\psi) = 0$, determining in the natural way from initial data for ζ, ψ, χ a certain polynomial $P_{2N-1}(\lambda)$, $\lambda \in \mathbb{C}$, with constant real coefficients of the form

$$P_{2N-1}(\lambda) = \zeta^2 + \chi\psi = \sum_{k=0}^{2N-1} p_k \lambda^k = \prod_{j=0}^{2N-2} (\lambda - E_j), \tag{6.589}$$

with the conditions $P(0) \neq 0$ and $E_i \neq E_j \neq 0$ for $i \neq j = 0, \ldots, 2N - 2$. Expanding the polynomial solution $\psi(x, x_0, y_0, \lambda)$ with respect to zeros $\mu_j(x, x_0, y_0)$, $0 \leq j \leq N$, as

$$\psi = \psi_N \prod_{j=1}^{N} (\lambda - \mu_j), \tag{6.590}$$

from (6.584), passing to the limit as $\lambda \to \mu_j(x, x_0, y_0)$, we obtain the following system of nonlinear equations for the zeros μ_j, $1 \leq j \leq N$:

$$\frac{\partial \mu_j}{\partial x} = \frac{-\mu_j \sqrt{P_{2N-1}(\mu_j)}}{\prod\limits_{n \neq j} (\mu_j - \mu_n)}, \qquad \frac{\partial \mu_j}{\partial t} = u_t \frac{\sqrt{P_{2N-1}(\mu_j)}}{\prod\limits_{n \neq j} (\mu_j - \mu_n)}, \tag{6.591}$$

where $u_t := du/dt$, $t \in \{x_0, y_0\} \subset \mathbb{R}$. Note that since ψ_N is a constant, we have assumed for convenience that $\psi_N = 1$.

6.9.8 *Jacobi inversion and Abel transformation*

Equations (6.591) belong to the class of equations integrable by means of the Abel transformation on the hyperelliptic Riemann surface $\mathcal{R}$ of the function $w = \sqrt{P_{2N-1}(\lambda)}$, $\lambda, w \in \mathbb{C}$. So we shall consider systems (6.591) as defined on the hyperelliptic Riemann surface $\mathcal{R}$ of genus $N-1$ of the function $\sqrt{P_{2N-1}(\lambda)}$, which can be realized as a two-sheeted covering surface of the extended complex plane $\mathbb{C}$ with cuts along the intervals $[E_{2j}, E_{2j+1}]$, $0 \leq j \leq N-2$, $[E_{2N-2}, \infty]$. Let $\omega_j(\lambda)$, $1 \leq j \leq N$, be the following Abelian integrals:

$$\omega_j(\lambda) = \int_{\lambda_0}^{\lambda} \frac{q_j(\xi) d\xi}{\xi \sqrt{P_{2N-1}(\xi)}}, \quad q_j(\xi) = \sum_{k=1}^{N} C_{jk} \xi^{N-k}, \tag{6.592}$$

normalized by the conditions:

$$\oint_{a_k} d\omega_j(\lambda) = \delta_{kj}, \tag{6.593}$$

where a_k, $1 \le k \le N-1$, are a-cycles of the Riemann surface $\mathcal{R}$ and a_N is the outline on the upper sheet of $\mathcal{R}$ surrounding $0 \in \mathcal{R}$. It follows directly from (6.593) [262, 406, 409] that the coefficients C_{jk}, $1 \le j,k \le N$, of (6.592) are unique. The zeros E_j, $0 \le j \le 2N-2$, of the polynomial $P_{2N-1}(\lambda)$ are not singular, so the integrals $\omega_j(\lambda)$ have singularities only at the points 0^+ and $0^- \in \mathcal{R}$ (zeros on the upper and lower sheet of $\mathcal{R}$). The differentials $d\omega_j$, $1 \le j \le N$, in neighborhoods of $0^\pm \in \mathcal{R}$ have the form

$$d\omega_j(\lambda) = \left(\pm \frac{C_{jN}}{\lambda\sqrt{P_{2N-1}(0)}} + \mathrm{reg}\,(\lambda) \right) d\lambda,$$

with logarithmic residues at 0^+ and $0^- \in \mathcal{R}$ equal to

$$\pm \frac{C_{jN}}{\sqrt{P_{2N-1}(0)}} = \pm \frac{\delta_{jN}}{2\pi i},$$

respectively. Thus $d\omega_j$, $1 \le j \le N-1$, are Abelian differentials of the first kind on $\mathcal{R}$, and $d\omega_N$ is an Abelian differential of the third kind on $\mathcal{R}$ with logarithmic singularities at 0^+ and $0^- \in \mathcal{R}$ and residues equal to 1 and -1, respectively. The differential $d\omega_N$ is normalized, since its a-periods are equal to zero. The standard [247, 262, 406, 409] substitution

$$\nu_j(x, x_0, y_0) = \sum_{k=1}^{N} \omega_j(\mu_k(x, x_0, y_0)), \tag{6.594}$$

transforms (6.591) into

$$\nu_j(x, x_0, y_0) = C_{j1}x + \frac{C_{jN}}{(-1)^N}\gamma(x_0, y_0) + \nu_j(0,0), \tag{6.595}$$

where $1 \le j \le N$ and the function $\gamma(x_0, y_0)$ satisfies the equations

$$\frac{\partial \gamma}{\partial t} = -u_t \prod_{j=1}^{N} \mu_j^{-1}, \quad \frac{\partial \gamma}{\partial x} = 0, \tag{6.596}$$

where $t \in \{x_0, y_0\} \subset \mathbb{R}$ and $\gamma(0,0) = 0$. Considering (6.594) as expressions on $\mathcal{R}$, we have the following inversion problem: given $\nu_j(x, x_0, y_0)$, find $\mu_j(x, x_0, y_0)$, $1 \le j \le N$, making use of equations (6.596). This Jacobi inversion problem is clearly nonstandard, as it is considered on the Riemann surface $\mathcal{R}$ of genus $N-1$ with the normalized base of Abelian integrals of the first kind ω_j, $1 \leq j \leq N-1$, and one normalized Abelian integral of the third kind, ω_N. To solve this problem we consider the ε-deformed hyperelliptic Riemann surface $\mathcal{R}_\varepsilon$, $\varepsilon \in \mathbb{C}$, of the function $\sqrt{(\lambda^2+\varepsilon^2)P_{2N-1}(\lambda)}$, having the genus N, and the following augmented equations on this surface:

$$\frac{\partial \mu_{j,\varepsilon}}{\partial x} = -\frac{\sqrt{(\mu_{j,\varepsilon}^2+\varepsilon^2)P_{2N-1}(\mu_{j,\varepsilon})}}{\prod_{n\neq j}(\mu_{j,\varepsilon}-\mu_{n,\varepsilon})},$$

$$\frac{\partial \mu_{j,\varepsilon}}{\partial t} = \frac{u_t\sqrt{(\mu_{j,\varepsilon}^2+\varepsilon^2)P_{2N-1}(\mu_{j,\varepsilon})}}{\mu_{j,\varepsilon}\prod_{n\neq j}(\mu_{j,\varepsilon}-\mu_{n,\varepsilon})}, \tag{6.597}$$

where $t \in \{x_0, y_0\} \subset \mathbb{R}$ and $\mu_{j,\varepsilon} \in \mathcal{R}_\varepsilon$, $1 \leq j \leq N$. The following result obtains.

Lemma 6.13. *Let the initial data for equations* (6.591) *and* (6.597) *satisfy the inequalities*

$$\begin{aligned} &\max_{1\leq j\leq N} |\mu_j(0,0,0)| < M, \quad \min_{i\neq j} |\mu_i(0,0,0)-\mu_j(0,0,0)| > m_2, \\ &\min_{1\leq j\leq N} |\mu_j(0,0,0)| > m_1, \;\; \mu_j(0,0,0) = \mu_{j,\varepsilon}(0,0,0), \end{aligned} \tag{6.598}$$

for all $\varepsilon \in \mathbb{C}$, where $M, m_1, m_2 \in \mathbb{R}_+$ are positive. Then, there exists a sequence $\{\varepsilon_k : k \in \mathbb{Z}_+\}$ with $\lim_{k\to\infty}\varepsilon_k = 0$, such that $\mu_{j,\varepsilon_k}(x, x_0, y_0)$ converge uniformly to $\mu_j(x, x_0, y_0)$, $1 \leq j \leq N$, as $k \to \infty$ for sufficiently small $x \in \mathbb{R}$.

Proof. It follows from (6.597) and (6.598) that for small enough $x \in \mathbb{R}$ and $t \in \{x_0, y_0\} \subset \mathbb{R}$, there exist positive constants c_1 and c_2 that are independent of $\varepsilon \in \mathbb{C}$, and $(x, x_0, y_0) \in \mathbb{R}\times\mathbb{R}^2$, such that

$$\left|\frac{\partial}{\partial x}\mu_{j,\varepsilon}(x,x_0,y_0)\right| \leq c_1, \quad \left|\frac{\partial}{\partial t}\mu_{j,\varepsilon}(x,x_0,y_0)\right| \leq c_2,$$

for all $1 \leq j \leq N$. Therefore, the set of functions $\{\mu_{j,\varepsilon_k}(x, x_0, y_0) : 1 \leq j \leq N\}$ is compact in the uniform metric. Whence, there exists a sequence $\{\varepsilon_k : k \in \mathbb{Z}_+\}$, with $\lim_{k\to\infty}\varepsilon_k = 0$, such that the sequence of functions $\{\mu_{j,\varepsilon_k}(x, x_0, y_0) : k \in \mathbb{Z}_+\}$ converges uniformly (with respect to $x \in \mathbb{R}$ and $x_0, y_0 \in \mathbb{R}$) to $\mu_j(x, x_0, y_0)$, $1 \leq j \leq N$, satisfying equations (6.591) and the chosen initial data. Now, by virtue of (6.598), the solution of (6.591) is unique, so for all subsequences of $\{\varepsilon_k : k \in \mathbb{Z}_+\}$ one obtains the same $\lim_{k\to\infty}\mu_{j,\varepsilon_k}(x, x_0, y_0)$, $(x, x_0, y_0) \in \mathbb{R}\times\mathbb{R}^2$, which completes the proof. $\square$

Let $\omega_{j,\varepsilon}(z)$, $1 \le j \le N$, be a normalized base of Abelian integrals of the first kind in $\mathcal{R}_\varepsilon$

$$\omega_{j,\varepsilon}(\lambda) = \int\limits_{\lambda_0}^{\lambda} \frac{q_{j,\varepsilon}(\xi)d\xi}{\sqrt{(\xi^2+\varepsilon^2)P_{2N-1}(\xi)}}, \quad q_{j,\varepsilon}(\xi) = \sum_{k=1}^{N} C_{jk,\varepsilon}\xi^{N-k}. \tag{6.599}$$

Then, the Jacobi inversion problem

$$\nu_{j,\varepsilon}(x,x_0,y_0) = \sum_{k=1}^{N} \omega_{j,\varepsilon}(\mu_{k,\varepsilon}(x,x_0,y_0)) \tag{6.600}$$

recasts (6.597) into

$$\nu_{j,\varepsilon}(x,x_0,y_0) = C_{j1,\varepsilon}x + (-1)^N C_{jN,\varepsilon}\,\gamma_\varepsilon(x_0,y_0) + \nu_{j,\varepsilon}(0,0,0), \tag{6.601}$$

where

$$\frac{\partial\gamma_\varepsilon}{\partial t} = -u_t \prod_{j=1}^{N} \mu_{j,\varepsilon}^{-1}, \quad \frac{\partial\gamma_\varepsilon}{\partial x} = 0, \quad \gamma_\varepsilon(0) = 0$$

for all $x \in \mathbb{R}$, $t \in \{x_0, y_0\} \subset \mathbb{R}^2$, due to the linearity of (6.601) in $x \in \mathbb{R}$ and $\gamma_\varepsilon(x_0,y_0) \in \mathbb{C}$.

6.9.9 *Convergence of Abelian integrals*

To study the convergence of the Abelian integrals (6.599), we make canonical cuts of the Riemann surface $\mathcal{R}_\varepsilon$ by a- and b-cycles. An arbitrary branch of each integral $\omega_{j,\varepsilon}(\lambda)$ is regular on the cut region $\mathcal{R}_\varepsilon^*$, and tends continuously as $\varepsilon \to 0$ (away from the points 0^+ and 0^-), to the respective branch of the integrals $\omega_j(\lambda)$, $1 \le j \le N$, on $\mathcal{R}^*$, the region obtained by cutting $\mathcal{R}$. Since the set of regular Abelian integrals $\{\omega_{j,\varepsilon}(\lambda) : 1 \le j \le N\}$ is uniformly continuous and bounded with respect to sufficiently small $\varepsilon \in \mathbb{C}$ on an arbitrary compact region $\mathcal{K} \subset \mathcal{R}^*$ located at a positive distance from the points 0^+ and $0^- \in \mathcal{R}^*$, the Abelian integrals $\omega_{j,\varepsilon}(\lambda)$, $1 \le j \le N$, converge uniformly to $\omega_j(\lambda)$ on $\mathcal{K} \subset \mathcal{R}^*$, as $\varepsilon \to 0$. Whence, $\lim\limits_{\varepsilon\to 0} \nu_{j,\varepsilon}(x,x_0,y_0) = \nu_j(x,x_0,y_0)$ if and only if the initial data $\mu_j(0,0,0)$, $1 \le j \le N$, do not belong to small neighborhoods of 0^+ and 0^-. Therefore, the problem (6.600) approximates the problem (6.594) as $\varepsilon \to 0$. The problem (6.594) is obviously the standard Jacobi inversion problem on the Riemann surface $\mathcal{R}_\varepsilon$, and we have proved the following important lemma.

Lemma 6.14. *If* $\varepsilon \to 0$*, then the Jacobi inversion problem* (6.600) *tends to the Jacobi problem* (6.594)*, where* $\gamma(x_0,y_0) := \lim_{\varepsilon\to 0}\gamma_\varepsilon(x_0,y_0)$*,* $x_0, y_0 \in \mathbb{R}$*, and* $\omega_j(\lambda) = \lim\limits_{\varepsilon\to 0}\omega_{j,\varepsilon}(\lambda)$*,* $1 \le j \le N$*.*

Assuming that $\mu_k(0,0,0) \in \mathcal{R}$, $1 \le k \le N$, are pairwise distinct, it follows from a basic theorem of Riemann that there exists a nontrivial Riemann ϑ-function $\vartheta_\varepsilon(\lambda) = \vartheta_\varepsilon(\omega_\varepsilon(\lambda) - e_\varepsilon)$ defined as

$$\vartheta_\varepsilon(u) = \sum_{m \in \mathbb{Z}^N} \exp[\pi i < B_\varepsilon m, m > +2\pi i < u, m >],$$

where B_ε is the matrix of B-periods of the base $\{\omega_{j,\varepsilon}(\lambda) : 1 \le j \le N\}$ on $\mathcal{R}_\varepsilon$, $u \in \mathbb{C}^N$, $< . , . >$ is the usual inner product in $\mathbb{C}^N$, $e_\varepsilon = (\nu_{j,\varepsilon}(x, x_0, y_0) + k_{j,\varepsilon} : 1 \le j \le N)$, and $k_{j,\varepsilon} = \frac{1}{2}\sum_{r=1}^{N} B_{rj,\varepsilon} - \frac{j}{2}$, $1 \le j \le N$. The ϑ-function is regular on the (a,b)-cycles cut surface $\mathcal{R}^\star_\varepsilon$ and has just N zeros $\mu_{j,\varepsilon}(x, x_0, y_0)$, $1 \le j \le N$; namely, the solution of the Jacobi inversion problem. It follows from a simple computation that the following limit exists:

$$\begin{aligned} T(\lambda) := \lim_{\varepsilon \to 0} \vartheta_\varepsilon(\lambda) &= \vartheta_{N-1}(\omega(\lambda) - e^+) \\ &+ \exp[2\pi i(\omega_N(\lambda) - e_N)]\vartheta_{N-1}(\omega(\lambda) - e^-) \end{aligned} \tag{6.602}$$

where $e^\pm = (\nu_j(x, x_0, y_0) + \widehat{k}_j \pm \frac{1}{2} B_{Nj} : 1 \le j \le N-1)$, $e_N = \nu_N(x, x_0, y_0) + \widehat{k}_N$, $\widehat{k}_j = \frac{1}{2}\sum_{k=1}^{N-1} B_{kj} - \frac{j}{2}$, $\omega(\lambda) = (\omega_j(\lambda),\ 1 \le j \le N-1)$, and $\vartheta_{N-1}(u)$ is the usual Riemann ϑ-function of $u \in C^{N-1}$, and B_{Nj} are B-periods of the Abelian integral $\omega_N(\lambda)$ on the surface $\mathcal{R}$. Let us also consider the following τ-function on $\mathbb{C}^N$ associated with (6.602):

$$\begin{aligned} \tau(u) = \tau(u_1, \ldots, u_N) &= \vartheta_{N-1}\left(u_1 - \frac{B_{N1}}{2}, \ldots, u_{N-1} - \frac{B_{N,N-1}}{2}\right) \\ &+ \exp(2\pi i u_N)\vartheta_{N-1}\left(u_1 + \frac{B_{N1}}{2}, \ldots, u_{N-1} + \frac{B_{N,N-1}}{2}\right), \end{aligned} \tag{6.603}$$

where $u \in \mathbb{C}^N$, and B_{Nj}, $1 \le j \le N-1$, are defined as above. It is easy to see that

$$T(\lambda) = \tau(\omega(\lambda) - \nu(x, x_0, y_0) - e),$$

with $\nu(x, x_0, y_0) = (\nu_j(x, x_0, y_0) : 1 \le j \le N-1)$, and ω, $e \in \mathbb{C}^{N-1}$ are defined as above. Additional properties of the function $T(\lambda)$ can be found in [112, 113, 409].

Lemma 6.15. *The zeros of the T-function* (6.602) *are the solution to the Jacobi inversion problem* (6.600).

Proof. It suffices to show that $\mu_j(x, x_0, y_0) = \lim_{\varepsilon\to 0} \mu_{j,\varepsilon}(x, x_0, y_0)$, $1 \le j \le N$. But this follows immediately from the fact that the functions $\vartheta_\varepsilon(\lambda)$ tend continuously to the function $T(\lambda)$, $\lambda \in \mathcal{K}$, which is bounded on any compact $\mathcal{K}$ in $\mathcal{R}^*$ that is a positive distance from 0^-. Thus, the proof is complete. □

6.9.10 *Analytical expressions for exact solutions*

Following [247, 262, 406, 409], one can find explicit expressions for the symmetric functions $\sum \ln \mu_j$ and $\sum \mu_j$. To find the first sum, we consider the region $\mathcal{R}^*$ with one more intersection along the curve σ starting at the point $0^+ \in \mathcal{R}^*$ and ending at $\infty \in \mathcal{R}^*$, and the function $\beta(\lambda) = \log\lambda + 2\pi i\omega_N(\lambda)$, $\lambda \in \mathcal{R}$, which is an Abelian integral of the third kind with logarithmic residues at 0^+ and $\infty \in \mathcal{R}$ equal to 2 and -2, respectively. Consider now the integral

$$\mathcal{I} = \frac{1}{2\pi i} \int\limits_{\partial\mathcal{R}^*\cup L^+\cup L^-} \beta(\lambda)\ d\log T(\lambda).$$

It follows from the classical residue theorem that

$$\mathcal{I} = \sum_{k=1}^{N} \beta(\mu_k) - \beta(0^-) = \sum_{k=1}^{N} \log\mu_k + 2\pi i \sum_{k=1}^{N} \omega_N(\mu_k) - \beta(0^-)$$

$$= \sum_{k=1}^{N} \log\mu_k + 2\pi i\nu_N(x,t) - \beta(0^-).$$

On the other hand, since $\beta(\lambda)$ and $T(\lambda)$ are continuous on b-cycles, we have

$$\beta^+(\lambda) - \beta^-(\lambda) = -B_j, \quad B_j = \oint\limits_{b_j} d\log\lambda + 2\pi i \oint\limits_{b_j} d\omega_N(\lambda),$$

$\lambda \in a_j$, $1 \le j \le N-1$, and $\beta^+(\lambda) - \beta^-(\lambda) = -2\pi i$, $\lambda \in \sigma$, so

$$\mathcal{I} = \frac{1}{2\pi i} \sum_{k=1}^{N-1} \left(\oint_{a_k^+} \beta^+(\lambda) d\log T^+(\lambda) - \oint_{a_k^-} \beta^-(\lambda) d\log T^-(\lambda) \right)$$

$$+ \frac{1}{2\pi i} \left(\int_{\sigma^+} \beta^+(\lambda) d\log T^+(\lambda) - \int_{\sigma^-} \beta^-(\lambda) d\log T^-(\lambda) \right)$$

$$= \sum_{k=1}^{N-1} \left(\oint_{a_k} \beta^-(\lambda) d\log \frac{T^+(\lambda)}{T^-(\lambda)} \right) - \frac{1}{2\pi i} \sum_{k=1}^{N-1} B_k \oint_{a_k} d\log T^-(\lambda) - \int_{L^+} d\log T(\lambda).$$

Therefore,

$$\sum_{k=1}^{N} \log \mu_k(x, x_0, y_0) = -2\pi i \nu_N(x, x_0, y_0) + \beta(0^-)$$

$$+ \sum_{j=1}^{N-1} \oint_{a_j} \beta^-(\lambda) d\omega_j(\lambda) - \sum_{j=1}^{N-1} B_j n_j + 2 \ln \frac{T(\infty)}{T(0^+)}, \tag{6.604}$$

where $n_j = \oint_{a_j} d \log T^-(\lambda)$, $1 \leq j \leq N$. Following [247, 262], we find similarly that

$$\sum_{k=1}^{N} \mu_k(x, x_0, y_0) = \sum_{j=1}^{N-1} \oint_{a_j} \lambda d\omega_j(\lambda) - \mathrm{res}_{\lambda=\infty}(\lambda d \ln T(\lambda)). \tag{6.605}$$

First we compute the residue on the right-hand side of (6.605), making use of the following expansion of the function $\frac{d}{d\tau} \log T(\lambda)$ in $\tau = \lambda^{-1/2}$:

$$\frac{d}{d\tau} \log T(\lambda) = -2\lambda^{3/2} \frac{d}{d\lambda} \log T(\lambda) = -2\lambda^{3/2} \sum_{j=1}^{N-1} \frac{\partial \log T(\lambda)}{\partial \overline{\omega}_j} \frac{d\omega_j(\lambda)}{d\lambda} =$$

$$-2\lambda^{3/2} \sum_{j=1}^{N-1} \frac{q_j(\lambda)}{\lambda \sqrt{P_{2N-1}(\lambda)}} \frac{\partial \log T(\lambda)}{\partial \overline{\omega}_j}.$$

Since $q_j(\lambda) = \sum_{j=1}^{N} C_{jk} \lambda^{N-k}$, $1 \leq j \leq N-1$, (see (6.592)) one has

$$\frac{q_j(\lambda)}{\lambda \sqrt{P_{2N-1}}(\lambda)} = C_{j1} \lambda^{-3/2} \left(1 - \frac{d_0}{2} \lambda^{-1} - \frac{d_1}{2} \lambda^{-2} + \cdots \right)$$

$$+ C_{j2} \lambda^{-5/2} \left(1 - \frac{d_0}{2} \lambda^{-1} - \frac{d_1}{2} \lambda^{-2} + \cdots \right) + \cdots,$$

where d_0, d_1 are constants. Whence,

$$\lambda \frac{d}{d\tau} \log T(\lambda) = -2 \sum_{j=1}^{N-1} \left[\frac{C_{j1}}{\tau^2} + \ldots \right] \frac{\partial \log T(\lambda)}{\partial \overline{\omega}_j}$$

and

$$\mathrm{res}_{\lambda=\infty} \left\{ \lambda d \log T(\lambda) \right\} = -2 \sum_{j=1}^{N-1} C_{j1} \frac{\partial}{\partial \tau} \left(\frac{\partial \log T(\lambda)}{\partial \overline{\omega}_j} \right) \Big|_{\tau=0}$$

$$= 4 \sum_{i,j=1}^{N-1} C_{j1} C_{i1} \frac{\partial^2 \log T(\lambda)}{\partial \overline{\omega}_j \partial \overline{\omega}_i}.$$

Moreover, it follows from (6.595) that

$$\frac{\partial}{\partial x^2}\log T(\lambda) = \sum_{i,j=1}^{N-1} C_{j1}C_{i1}\frac{\partial^2 \log T(\lambda)}{\partial\overline{\omega}_j\partial\overline{\omega}_i}.$$

Thus, from these results we find that

$$\sum_{k=1}^{N}\mu_k(x,x_0,y_0) = \sum_{j=1}^{N-1}\oint_{a_j}\lambda d\omega_j(\lambda) - 4\frac{\partial^2}{\partial x^2}\log T(\infty). \tag{6.606}$$

From (6.586) and (6.545) we easily find an exact expression for the solution $y(x,x_0,y_0)$, $(x,x_0,y_0)\in\mathbb{R}\times\mathbb{R}^2$:

$$y(x,x_0,y_0) = -\frac{a(x)}{2} + \frac{1}{2}\frac{\partial}{\partial x}\log\prod_{k=1}^{N}\mu_k. \tag{6.607}$$

Furthermore, owing to (6.546), the following relations are valid for $\gamma(x_0,y_0)$:

$$\frac{\partial\gamma}{\partial x_0} = 2\frac{y_0^2 + a(x_0)y_0 + b(x_0)}{N(x_0;\gamma)}, \qquad \frac{\partial\gamma}{\partial y_0} = -\frac{2}{N(x_0;\gamma)}, \tag{6.608}$$

where $N(x;\gamma(x_0,y_0)) = \prod_{j=1}^{N}\mu_j(x;x_0,y_0)$, $x_0,y_0\in\mathbb{R}$, is determined as

$$N(x;\gamma(x_0,y_0)) = \frac{T^2(\infty)}{T^2(0^+)}\exp\Bigg\{-2\pi i\nu_N(x,t) + \beta(0^-) + \sum_{j=1}^{N-1}\oint_{a_j}\beta^-(\lambda)d\omega_j(\lambda) - \sum_{j=1}^{N-1}B_jn_j\Bigg\}. \tag{6.609}$$

Lemma 6.16. *The identity*

$$\gamma(x,y) = \gamma(x_0,y_0) \tag{6.610}$$

holds for any solution to the Cauchy–Goursat problem (6.545), (6.546).

Proof. It is straightforward to show that the function $\gamma:\mathbb{R}^2\to\mathbb{R}$ satisfies the equation

$$\frac{d}{dx}\gamma(x,y(x;x_0,y_0)) = \frac{\partial\gamma}{\partial x} + \frac{\partial\gamma}{\partial y}\frac{d}{dx}y(x;x_0,y_0)$$

$$= 2\frac{y^2 + a(x)y + b(x)}{N(x;\gamma)} - \frac{2}{N(x;\gamma)}\frac{dy(x)}{dx} = 0,$$

for the solutions to (6.604), where $t \in \{x_0, y_0\} \subset \mathbb{R}$, that is $\gamma(x, y) = \gamma(x_0, y_0)$ for all $x_0, y_0 \in \mathbb{R}$. The equality of the mixed derivatives of $\gamma(x_0, y_0)$ yields the expression

$$y_0 = \frac{1}{2}\frac{\partial}{\partial x}\log N(x = x_0; \gamma(x_0, y_0)) - \frac{a(x_0)}{2}, \tag{6.611}$$

determining the function $\gamma(x_0, y_0)$ implicitly. Now from (6.545) and (6.611) we also compute that

$$u = \frac{\partial}{\partial x}\log N(x;\gamma), \quad \mathcal{L}_N(x;\gamma(x_0, y_0)) := \frac{u_x}{2} - \frac{1}{4}u^2$$

$$= -\frac{a^2(x)}{4} + \frac{a'(x)}{2} + b(x) := \mathcal{L}(x), \tag{6.612}$$

that is

$$\mathcal{L}_N(x;\gamma(x_0, y_0)) = \mathcal{L}(x), \tag{6.613}$$

for all $x_0, y_0 \in \mathbb{R}$. Hence, it follows from (6.545), (6.584), and (6.590) that $v = \psi_{N-1,t}/2(\psi_{N-1} - \zeta_{N-1})$, thus $u_t = -\sum \mu_{j,t}/u$, $t \in \{x_0, y_0\} \subset \mathbb{R}$, so that using the condition (6.613) one can easily reduce it to the characteristic expression

$$\frac{\partial}{\partial \gamma}\left(\sum_{j=1}^{N}(\mu_j + \frac{\partial}{\partial x}\log \mu_j)\right) = 0, \tag{6.614}$$

which directly yields the desired result. $\square$

The above equation and the exact formulas (6.604) and (6.606), for detecting the Riemann surface parameters generating integrable Riccati equations, and this leads directly to a proof of the next result.

Theorem 6.17. *The Riemann surface $\mathcal{R}$ of the function $w^2 = P_{2N-1}(\lambda)$, $(\lambda, w) \in \mathcal{R}$, satisfying the condition* (6.614)*, generates solutions to* (6.543) *representable by quadratures via* (6.607) *for given differentiable functions $a, b : \mathbb{R} \to \mathbb{R}$ if the function $z := \log N(\cdot;\gamma) :\to \mathbb{R}$, with $N(\cdot;\gamma)$ given by* (6.609) *and defining via* (6.607) *and* (6.612) *the desired functions $a, b \in C^\infty(\mathbb{R};\mathbb{R})$, satisfies the well-known Liouville equation*

$$\partial^2 z/\partial x \partial \gamma + e^z = 0, \tag{6.615}$$

$x \in \mathbb{R}$, which is integrable by quadratures and equivalent to (6.614)*.*

At the same time equation (6.608) makes it possible to find a still unknown function $\gamma : \mathbb{R}^2 \to \mathbb{R}$ incorporated implicitly into the solution (6.607) to the Riccati equation (6.543).

6.9.11 *Abel equation integrability analysis for $n = 3$*

For the case $n = 3$, the holonomy Lie algebra $\mathfrak{g}(h)$ is strictly infinite-dimensional. This means that there exists no finite-dimensional representation of the holonomy Lie algebra. The simplest infinite-dimensional Lie algebra containing sl(2) as a subalgebra is the Lie algebra of the group of diffeomorphisms of the circle $\mathbb{S}^1$ or the *Virasoro algebra*:

$$L_j = \xi^{j+1}\frac{\partial}{\partial \xi}, \quad [L_i, L_j] = (j-i)\, L_{i+j}, \tag{6.616}$$

where $\xi \in \mathbb{S}^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$ and $j \in \mathbb{Z}_+$. The associated Lie algebra $\mathfrak{g}(h)$ is generated by the system (6.554) and has the general representation

$$\begin{aligned} X_0 &= -L_{-1}, \quad X_1 = L_{-1} + L_0, \quad X_2 = \left(3 + \frac{\lambda}{2}\right) L_{-1} + 6L_0 + 3L_1, \\ X_3 &= \left(2 + \frac{\lambda}{6}\right) L_{-1} + \left(3 + \frac{\lambda}{2}\right) L_0 + 3L_1 + L_2, \end{aligned} \tag{6.617}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter. Upon substituting the representation (6.616) into (6.617), we readily find the following (A, B) expressions of the induced Cartan–Ehresmann connection Γ_λ :

$$A = \left(-\frac{\lambda}{2}y - \frac{1}{3}\right) - \left(u + \frac{\lambda}{2}\right)\xi - 3y\xi^2 - \xi^3, \quad B = v, \tag{6.618}$$

where $A\dfrac{\partial}{\partial \xi} = -\Gamma_\lambda^{(x)}$, $B\dfrac{\partial}{\partial \xi} = -\Gamma_\lambda^{(t)}$, and $t \in \{x_0, y_0\} \subset \mathbb{R}$. The corresponding symbol expression $l\,(x;\xi) \sim \sum_{j\geq 1} l_j \xi^{-(j-1)}$ solving equation (6.578) is described by the series of coefficients

$$\begin{aligned} l_1 &= \varphi, \quad l_2 = 0, \quad l_3 = \frac{\varphi_x}{2}, \quad l_4 = -y\varphi_x, \\ l_5 &= \frac{1}{4}\left(\frac{1}{2}\varphi_{xx} - \left(u + \frac{\lambda}{2}\right)\varphi_x + 9u^2\varphi_x\right), \ldots, \end{aligned}$$

where $\varphi = \varphi\,(\cdot;\lambda) : \mathbb{R} \to \mathbb{R}$ is also a $(x_0, y_0) \in \mathbb{R}^2$-independent function (as in the case of $n = 2$) containing information about the integrable by quadratures Cauchy–Goursat conditions (6.546). It is easy to verify that for $n = 3$ one has obvious analogs of Lemma 6.9 and Lemma 6.10.

6.9.12 *A final remark*

Important problems concerning the integrability of the Riccati–Abel equation (6.542) still remain open in the area of analyzing the Lie-algebraic properties of the corresponding solution manifolds for the $n = 3$ case. Naturally, these problems are related to the Cauchy–Goursat conditions (6.546), and their corresponding finite-dimensional Moser reductions via the momentum mapping approach and methods from modern symplectic and differential-algebraic theories.

Chapter 7

Versal Deformations of a Dirac Operator on a Sphere and Related Dynamical Systems

7.1 Introduction: Diff $(\mathbb{S}^1)$-actions

If we are given a smooth differential operator in the variable $x \in \mathbb{R}/2\pi\mathbb{Z} \simeq \mathbb{S}^1$, its normal form, as is well known, is the simplest form obtainable by means of the Diff$(\mathbb{S}^1)$-group action on the space of all such operators. A versal deformation of this operator is a normal form for some parametric infinitesimal family including the operator. Here we shall undertake the analysis of versal deformations of a Dirac differential operator using the theory of induced Diff$(\mathbb{S}^1)$-actions endowed with centrally extended Lie–Poisson brackets. After constructing a general expression for transversal deformations of a Dirac differential operator, we interpret it via the Lie-algebraic theory of induced Diff$(\mathbb{S}^1)$-actions on a special Poisson manifold and determine its generic moment mapping. Using the standard Marsden–Weinstein reduction with respect to certain Casimir generated distributions, we describe an extensive class of versally deformed Dirac differential operators that depend on complex parameters. Suppose we are given the linear 2-vector first order Dirac differential operator on the real axis $\mathbb{R}$:

$$L_\lambda f := -\frac{df}{dx} + l_\lambda[u, v; z]f, \quad l_\lambda[u, v; z] := \begin{pmatrix} z - \lambda & u \\ v & \lambda - z \end{pmatrix}, \tag{7.1}$$

acting on the Sobolev space $L^\infty(\mathbb{R}; \mathbb{R}^2)$ and depending on 2π-periodic coefficients $u, v, z \in C^\infty(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{R})$ and a complex parameter $\lambda \in \mathbb{C}$. The variety of all operators (7.1), parametrized by $\lambda \in \mathbb{C}$, will be denoted by $\mathcal{L}_\lambda$. Let $\mathcal{A} := \mathrm{Diff}(\mathbb{S}^1)$ be the group of orientation preserving diffeomorphisms of the circle $\mathbb{S}^1$. A group action of $\mathcal{A}$ on $\mathcal{L}_\lambda$ can be defined as follows: Fixing a parametrization of $\mathbb{S}^1$, i.e., a C^∞-covering $p : \mathbb{R} \to \mathbb{S}^1$ such that the mapping $p : [a, a+2\pi) \rightleftharpoons \mathbb{S}^1$ is injective for every real $a \in \mathbb{R}$ and $p(x+2\pi) = p(x)$ for all $x \in \mathbb{R}$, each $\varphi \in \mathcal{A}$ can obviously be represented by a smooth mapping $\varphi : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}$ such that

$$\varphi(\xi + 2\pi) = \varphi(\xi) + 2\pi \quad \text{and} \quad \varphi'(\xi) > 0 \tag{7.2}$$

for all $\xi \in \mathbb{R}$. Upon making the change of variables

$$x = \varphi(\xi), \quad f\left(\varphi(\xi)\right) = \varphi(\xi) f(\xi), \tag{7.3}$$

with $\varphi \in \mathcal{A}$, $\varphi \in G := C^\infty\left(\mathbb{R}/2\pi\mathbb{Z}; SL(2;\mathbb{R})\right)$ and $x, \xi \in \mathbb{R}$, in (7.1), it is easy to see that the differential operator L_λ transforms into $L_\lambda^{(\varphi,\Phi)} : L^\infty \to W^\infty$ defined as

$$L_\lambda^{(\varphi,\Phi)} f(\xi) := -\frac{df}{d\xi} + l_\lambda^{(\varphi,\Phi)}[u, v; z] f, \tag{7.4}$$

where

$$l_\lambda^{(\varphi,\Phi)}[u, v; z] := -\Phi^{-1}(\xi)\frac{d\Phi(\xi)}{d\xi} + \varphi'(\xi)\Phi^{-1}(\xi) l_\lambda[u, v; z]\Phi(\xi). \tag{7.5}$$

We assume now that the matrix $\Phi(\xi)$, $\xi \in \mathbb{S}^1$, is chosen so that $l_\lambda^{(\varphi,\Phi)}[u, v; z] = l_\lambda[\tilde{u}, \tilde{v}; \tilde{z}]$ for all $\lambda \in \mathbb{R}$ and some mapping $(\tilde{u}, \tilde{v}; \tilde{z})^\intercal \in C^\infty(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{R}^3)$.

Whence, we obtain an induced nonlinear transformation $A^*(\varphi, \Phi) : \mathcal{L}_\lambda \to \mathcal{L}_\lambda$, $(\varphi, \Phi) \in \mathcal{A} \times G$, where

$$A^*(\varphi, \Phi) l_\lambda[u, v; z] := l_\lambda^{(\varphi,\Phi)}[u, v; z] \tag{7.6}$$

for all mappings in $C^\infty(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{R}^3)$.

This together with expression (7.5) determines an automorphism A^* of $\mathcal{A}$, for a fixed φ, that we shall study in detail. We are primarily interested in describing normal forms and versal deformations of (7.1) with respect to the automorphism A^*.

It is well known [14] that a normal form of the operator (7.1) is the simplest (in some sense) representative of its orbit under the group action of $\mathcal{A}$ on the space $\mathcal{L}_\lambda$. A versal deformation of (7.1) is a normal form for a stable parametric infinitesimal family including (7.1). As will be shown in the remainder of this chapter, all such deformations can be described by means of Lie-algebraic analysis of this group action on $\mathcal{L}_\lambda$ and an associated momentum mapping reduced on certain invariant subspaces.

7.2 Lie-algebraic structure of the $\mathcal{A}$-action

Let us consider the loop group $G := G_{\mathbb{S}^1}(SL(2;\mathbb{R}))$ of all smooth mappings $\mathbb{S}^1 \to SL(2;\mathbb{R})$ and its corresponding group $\mathcal{A}$-action on a functional manifold $M \subset C^\infty(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^3)$, which is assumed to be equivariant; that is, the diagram

$$\begin{array}{ccc} M & \xrightarrow{l} & \mathfrak{g}^* \\ A_\Phi\downarrow & & \downarrow Ad^*_{\Phi^{-1}} \\ M & \xrightarrow{l} & \mathfrak{g}^* \end{array} \tag{7.7}$$

commutes for all $l \in \mathfrak{g}^*$ of the loop Lie algebra and $\Phi \in G$. Consequently, we can define on M a natural Poisson structure that induces the following canonical Lie–Poisson structure on $\mathfrak{g}^*$: for any $\gamma, \mu \in \mathcal{D}(\mathfrak{g}^*)$,

$$\{\gamma, \mu\} := (l, [\nabla\gamma(l), \nabla\mu(l)]). \tag{7.8}$$

Here $(\cdot,\cdot)$ is the usual Killing type nondegenerate, symmetric, invariant scalar product on the loop Lie algebra $\mathfrak{g} = C_{\mathbb{S}^1}(sl(2;\mathbb{R}))$, i.e. for any $a, b \in \mathfrak{g}$,

$$(a, b) := \int_0^{2\pi} dx \, \mathrm{tr}\,(ab) \tag{7.9}$$

and $\nabla : \mathcal{D}(\mathfrak{g}^*) \to \mathfrak{g}$ is defined as $(\nabla\gamma(l), \delta l) := \frac{d}{d\epsilon}\gamma(l + \epsilon\delta l) \mid_{\epsilon=0}$ for any $\delta l \in \mathfrak{g}^*, \gamma \in \mathcal{D}(\mathfrak{g}^*)$.

In order to address the problems to be studied, we need to centrally extend the group action $A_\Phi : M \to M$, $\Phi \in G$, as follows: for $\hat{\Phi} := (\Phi, c) \in \hat{G} := G \times \mathbb{R}$ the corresponding action $A_{\hat{\Phi}} : M \to M$ is defined so that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\hat{l}} & \hat{\mathfrak{g}}^* \\ A_{\hat{\Phi}}\downarrow & & \downarrow Ad^*_{\hat{\Phi}^{-1}} \\ M & \xrightarrow{\hat{l}} & \hat{\mathfrak{g}}^* \end{array} \tag{7.10}$$

commutes for all $\hat{\Phi} \in \hat{G}$ and $\hat{l} = (l, c) \in \hat{\mathfrak{g}}^*$. This leads to the following (unique) choice of the extended Ad^*-action in (7.10):

$$Ad^*_{\hat{\Phi}^{-1}} : (l, c) \in \mathfrak{g}^* \to \left(\varphi'(\xi) Ad_{\Phi^{-1}} l(x) - c\Phi^{-1}\frac{d\Phi}{d\xi}, c\right) \tag{7.11}$$

for all $\hat{\Phi} \in \hat{G}$, $l \in \mathfrak{g}^*$ at $\xi \in \mathbb{R}$, $x = \varphi(\xi)$ and $c \in \mathbb{R}$.

This expression follows from the fact that the loop Lie algebra $\mathfrak{g}$ admits only the central extension $\hat{\mathfrak{g}} \oplus \mathbb{R}$. As the homology groups $H_1(\mathfrak{g}) = 0$ and $H_2(\mathfrak{g}) = 1$, it is represented as

$$[(a, \alpha), (b, \beta)] := ([(a, b)], (a, db/dx)) \tag{7.12}$$

for any $a, b \in \mathfrak{g}$ and $\alpha, \beta \in \mathbb{R}$.

Taking c to be unity and defining an appropriate diffeomorphism $x \to \varphi(x) = \xi$, it is easy to see that $Ad^*_{\hat{\Phi}^{-1}}$ has the same structure element as that of the action $A^*(\varphi, \Phi)$ on $\mathcal{L}_\lambda$ defined above. Accordingly it is clear that our Lie-algebraic analysis is intimately connected with the structure of the G-orbits induced by the diffeomorphism group $\mathcal{A} = \mathrm{Diff}(\mathbb{S}^1)$. We define a natural Lie–Poisson bracket on the adjoint space $\hat{\mathfrak{g}}^*$ as follows: for any $\gamma, \mu \in \mathcal{D}(\hat{\mathfrak{g}}) \subset \hat{\mathfrak{g}}^*$,

$$\{\gamma, \mu\}_0 := (l, [\nabla\gamma(l), \nabla\mu(l)]) + \left(\nabla\gamma(l), \frac{d\nabla\mu(l)}{dx}\right), \tag{7.13}$$

and deform it into a pencil of brackets using a constant parameter $\lambda \in \mathbb{R}$ via

$$\{\gamma, \mu\}_0 \xrightarrow{\lambda} \{\gamma, \mu\}_\lambda := (\nabla\gamma(l), \frac{d}{dx}\nabla\mu(l)) + (l + \lambda J, [\nabla\gamma(l), \nabla\mu(l)]), \tag{7.14}$$

where $J \in sl^*(2; \mathbb{R})$ is chosen here to be the constant matrix

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7.15}$$

The following compatibility condition is almost obvious [242, 243].

Lemma 7.1. *A pencil of brackets* (7.14) *is a Poisson brackets pencil for each $\lambda \in \mathbb{C}$ and $J \in sl^*(2; \mathbb{R})$, i.e. it is compatible.*

Proof. It is well known that the Lie derivative of a Poisson bracket is also a Poisson bracket if and only if

$$\{\gamma, \mu\}_1 := L_K\{\gamma, \mu\}_0 - \{L_K\gamma, \mu\}_0 - \{\gamma, L_K\mu\}_0 \tag{7.16}$$

satisfies the Jacobi identity for all $\gamma, \mu \in \mathcal{D}(\mathfrak{g}^*)$, where L_K is the Lie derivative with respect to a vector field $K : \mathfrak{g}^* \to T(\mathfrak{g}^*)$. Choosing $K(l) := J$, it is easy to verify that the bracket (7.16) satisfies the Jacobi identity and is the usual Poisson bracket on $\mathfrak{g}^*$. Consequently, the Poisson bracket (7.16) is also a Poisson bracket along a generic orbit of the vector field $dl/d\lambda = J$. Hence, the deformation (7.14) is also Poisson, as was to be proved. $\square$

7.3 Casimir functionals and reduction problem

A Casimir functional $h \in I_\lambda(\hat{\mathfrak{g}}^*)$ is defined, as usual, as a functional $h \in \mathcal{D}(\hat{\mathfrak{g}}^*)$ that is invariant with respect to the following λ-deformed $Ad^*_{\hat{\varphi}^{-1}}$-action:

$$Ad^*_{\hat{\varphi}^{-1}} : (l, 1) \in \hat{\mathfrak{g}}^* \to \left(Ad^*_{\varphi^{-1}}(l + \lambda J) - \varphi^{-1}\frac{d\varphi}{dx}, 1\right) \tag{7.17}$$

for any $\varphi \in G$, $l \in \mathfrak{g}^*$ and $\lambda \in \mathbb{R}$. It is easy to see from this definition that $h \in I_\lambda(\hat{\mathfrak{g}}^*)$ if the equation

$$\frac{d\nabla h(l)}{dx} = [l + \lambda J, \nabla h(l)] \tag{7.18}$$

is satisfied for all $\lambda \in \mathbb{R}$. Assuming further that there exists an asymptotic expansion of the form

$$h(\lambda) \sim \sum_{j \in_+} h_j \lambda^{-j} \tag{7.19}$$

as $|\lambda| \to \infty$, one can readily verify that $h_0 \in I_1(\hat{\mathfrak{g}}^*)$ and that for all $j, k \in \mathbb{Z}_+$

$$\{h_j, h_k\}_0 = 0 = \{h_j, h_k\}_1, \qquad \{\gamma, h_j\}_0 = \{\gamma, h_{j+1}\}_1, \tag{7.20}$$

where $\gamma \in \mathcal{D}(\hat{\mathfrak{g}}^*)$ is arbitrary. Let us now consider the action (7.7) at a fixed $l = l[u, v; z] \in \hat{\mathfrak{g}}^*$. It is easy to see that this action does not necessarily preserve the form of the element $l \in \hat{\mathfrak{g}}^*$. Thus, we must reduce the initial $\hat{G}$-action on $\hat{\mathfrak{g}}^*$ to an appropriate subgroup; for this we develop the reduction procedure employed in [243, 250, 326]. Define the distribution

$$D_1 := \{K \in T(\hat{\mathfrak{g}}^*) : K(l) = [J, \nabla\gamma(l)], l \in \hat{\mathfrak{g}}^*, \gamma \in \mathcal{D}(\hat{\mathfrak{g}}^*)\}. \tag{7.21}$$

Distribution D_1 is integrable, that is $[D_1, D_1] \subset D_1$, since the bracket $\{\cdot,\cdot\}_1$ is Poisson. Now define another distribution

$$D_0 := \{K \in T(\hat{\mathfrak{g}}^*) : K(l) = [l - d/dx, \nabla h_0], h_0 \in I_1(\hat{\mathfrak{g}}^*)\}, \tag{7.22}$$

which is clearly also integrable on $\hat{\mathfrak{g}}^*$, since $[D_0, D_0] \subset D_0$. The set of maximal integral submanifolds of (7.22) generates the foliation $\hat{\mathfrak{g}}^*_J \backslash D_0$ whose leaves are the intersections of fixed integral submanifolds $\hat{\mathfrak{g}}^*_J \subset \hat{\mathfrak{g}}^*$ of the distribution D_1 passing through an element $l[u, v; z] \in \hat{\mathfrak{g}}^*$. If the foliation $\hat{\mathfrak{g}}^*_J \backslash D_0$ is sufficiently smooth, one can define the quotient manifold $\hat{\mathfrak{g}}^*_{\mathrm{red}} := \hat{\mathfrak{g}}^*_J / (\hat{\mathfrak{g}}^*_J \backslash D_0)$ with its associated projection mapping $\hat{\mathfrak{g}}^*_J \to \hat{\mathfrak{g}}^*_{\mathrm{red}}$. To continue this line of reasoning, we shall obtain explicit constructions of the objects introduced. The distribution D_1 is obviously generated by the vector fields

$$\frac{dl}{dt} = \begin{pmatrix} 0 & 2b \\ -2c & 0 \end{pmatrix}, \quad \nabla\gamma(l) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \tag{7.23}$$

where $t \in \mathbb{R}$ is a complex evolution parameter and $l \in \hat{\mathfrak{g}}^*_J$, where $\hat{\mathfrak{g}}^*_J \subset \hat{\mathfrak{g}}^*$ is the isotropy Lie subalgebra of the element $J \in \hat{\mathfrak{g}}^*$. Hence, the integral submanifold $\hat{\mathfrak{g}}^*_J$ consists of orbits of an element $l = l[u, v; z] \in \hat{\mathfrak{g}}^*$, with

$z \in \mathbb{R}$, with respect to the vector fields (7.23). The distribution D_0 on $T(\hat{\mathfrak{g}}^*)$ is generated by the vector fields

$$\frac{dl}{d\tau} = \begin{pmatrix} -\chi_x & -2u\chi \\ 2v\chi & \chi_x \end{pmatrix}, \quad \nabla h_0(l) = \begin{pmatrix} \chi & 0 \\ 0 & -\chi \end{pmatrix}, \tag{7.24}$$

where $\tau \in \mathbb{R}$ is an evolution parameter and $l = l[u, v; z] \in \hat{\mathfrak{g}}^*$. It follows immediately from (7.24) that

$$\frac{dz}{d\tau} = -\chi_x, \quad \frac{du}{d\tau} = -2u\chi \quad \frac{dv}{d\tau} = 2v\chi \tag{7.25}$$

for all $\tau \in \mathbb{R}$ along D_0. Eliminating the variable χ from (7.25), we obtain

$$\frac{d}{d\tau}\left[\frac{d}{dx}(\log u) - 2z\right] = 0 = \frac{d}{d\tau}\left[\frac{d}{dx}(\log v) + 2z\right]; \tag{7.26}$$

that is, the map

$$\hat{\mathfrak{g}}^* \ni l = \begin{pmatrix} z & u \\ v & -z \end{pmatrix} : \xrightarrow{\nu} \begin{pmatrix} 0 & \exp(\partial^{-1}\alpha) \\ \exp(\partial^{-1}\beta) & 0 \end{pmatrix} \to \hat{\mathfrak{g}}^*_{\mathrm{red}}, \tag{7.27}$$

where

$$\alpha := u_x u^{-1} - 2z, \quad \beta := v_x v^{-1} + 2z, \tag{7.28}$$

explicitly determines the reduction $\nu : \hat{\mathfrak{g}}^* \to \hat{\mathfrak{g}}^*_{\mathrm{red}}$ discussed above. We are now in a position to compute the bracket (7.14) reduced upon the submanifold $\hat{\mathfrak{g}}^*_{\mathrm{red}}$ by defining the functionals $\lambda, \mu \in \mathcal{D}(\hat{\mathfrak{g}}^*)$ to be constant along the distribution D_0, that is

$$\gamma := \tilde{\gamma} \circ \nu, \quad \mu := \tilde{\mu} \circ \nu, \tag{7.29}$$

for any $\tilde{\gamma}, \tilde{\mu} \in \mathcal{D}(\hat{\mathfrak{g}}^*_{\mathrm{red}})$. From (7.28) one readily obtains the expressions

$$\nabla\gamma(l)|_{l\in\hat{\mathfrak{g}}^*_{\mathrm{red}}} = \begin{pmatrix} \frac{\delta\tilde{\gamma}}{\delta\beta} - \frac{\delta\tilde{\gamma}}{\delta\alpha} & -\frac{1}{v}\left(\frac{\delta\tilde{\gamma}}{\delta\beta}\right)_x \\ -\frac{1}{u}\left(\frac{\delta\tilde{\gamma}}{\delta\alpha}\right)_x & \frac{\delta\tilde{\gamma}}{\delta\alpha} - \frac{\delta\tilde{\gamma}}{\delta\beta} \end{pmatrix},$$

$$\nabla\mu(l)|_{l\in\hat{\mathfrak{g}}^*_{\mathrm{red}}} = \begin{pmatrix} \frac{\delta\tilde{\mu}}{\delta\beta} - \frac{\delta\tilde{\mu}}{\delta\alpha} & -\frac{1}{v}\left(\frac{\delta\tilde{\mu}}{\delta\beta}\right)_x \\ -\frac{1}{u}\left(\frac{\delta\tilde{\mu}}{\delta\alpha}\right)_x & \frac{\delta\tilde{\mu}}{\delta\alpha} - \frac{\delta\tilde{\mu}}{\delta\beta} \end{pmatrix}, \tag{7.30}$$

which satisfy the desired identities

$$(\nabla\gamma(l), dl/d\tau) = 0 = (\nabla\mu(l), dl/d\tau) \tag{7.31}$$

for all $l \in \hat{\mathfrak{g}}^*_{\mathrm{red}} \subset \hat{\mathfrak{g}}^*$. Substituting now (7.30) into (7.14), we obtain

$$\{\tilde{\gamma}, \tilde{\mu}\}_\lambda := \{\gamma, \mu\}_\lambda \,|_{l\in\hat{\mathfrak{g}}^*_{\mathrm{red}}} = (\nabla\tilde{\gamma}, (\eta + \lambda\vartheta)\nabla\tilde{\mu}), \tag{7.32}$$

where we have used the obvious relationship

$$\{\tilde{\gamma}, \tilde{\mu}\}_\lambda \circ \nu = \{\tilde{\gamma} \circ \nu, \tilde{\mu} \circ \nu\}_\lambda, \tag{7.33}$$

and where

$$\eta := \begin{pmatrix} 2\partial \\ -\partial \exp[-\partial^{-1}(\alpha+\beta)]\partial^2 \\ -2\partial - \partial \cdot \alpha \exp[-\partial^{-1}(\alpha+\beta)] \cdot \partial - \partial \exp[-\partial^{-1}(\alpha+\beta)]\partial^2 \\ -2\partial - \partial \cdot \beta \exp[-\partial^{-1}(\alpha+\beta)] \cdot \partial \\ 2\partial \end{pmatrix}, \tag{7.34}$$

$$\vartheta := \begin{pmatrix} 0 & 2\partial \exp[-\partial^{-1}(\alpha+\beta)]\partial \\ -2\partial \exp[-\partial^{-1}(\alpha\beta)]\partial & 0 \end{pmatrix}. \tag{7.35}$$

It is straightforward to verify that these integro-differential, implectic (co-symplectic or Poisson) operators are compatible [137, 173, 242, 326] (see also [4, 6–8, 10] for a general theory of iso-symplectic structures on functional manifolds) on the reduced submanifold $\hat{\mathfrak{g}}^*_{\mathrm{red}}$ on which it defines a bi-Hamiltonian structure.

7.4 Associated momentum map and versal deformations of the Diff($\mathbb{S}^1$) action

Let us introduce some additional notation for versal deformations [14, 42]. By a *deformation* of (7.1), we shall mean an operator of the same form with a matrix $l_\lambda(\epsilon)$ whose entries are analytic in ϵ in a neighborhood of $\epsilon = 0$ in $\mathbb{R}^n$ and satisfies $l_\lambda(0) = l_\lambda$ for all $\lambda \in \mathbb{R}$. The coordinates $\epsilon_i \in \mathbb{R}$, $1 \le i \le n$, of ϵ are called the deformation parameters and the space of these parameters is called the base of the deformation. Two deformations $l'_\lambda(\epsilon)$ and $l''_\lambda(\epsilon)$ of a matrix l_λ are *equivalent* if there exists a deformation $A^*(\varphi_\epsilon) : l'_\lambda(\epsilon) \to l''_\lambda(\epsilon)$ generated by a diffeomorphism $\varphi_\epsilon \in \mathrm{Diff}(\mathbb{S}^1)$ satisfying $\varphi_\epsilon \mid_{\epsilon=0} = id$. From a given deformation $l_\lambda(\epsilon)$ one can obtain a new deformation $\tilde{l}_\lambda(\tilde{\epsilon})$ by setting $\tilde{l}_\lambda(\tilde{\epsilon}) := l_\lambda(\epsilon(\tilde{\epsilon}))$, where $\epsilon : \mathbb{R}^m \to \mathbb{R}^n$ is an analytic mapping in a neighborhood of $\tilde{\epsilon} = 0$ in $\mathbb{R}^m$ such that $\epsilon(0) = 0$. The deformation $\tilde{l}_\lambda(\tilde{\epsilon})$ is said to be *induced* from $l_\lambda(\epsilon)$ by the map $\epsilon : \mathbb{R}^m \to \mathbb{R}^n$. A deformation $l_\lambda(\epsilon)$, $\epsilon \in \mathbb{R}^n$, is called *versal* if every one of its deformations $l_\lambda(\tilde{\epsilon})$, $\tilde{\epsilon} \in \mathbb{R}^m$, is equivalent to one of its induced deformations. A versal deformation is said to be *universal* if the induced deformation is unique. Before we give a definition of a transversal deformation for the induced group $\hat{\mathfrak{g}}_{\mathrm{red}}$ orbits, let us consider a family of smooth induced transformations $\varphi_\sigma(x) \in \hat{G}_{\mathrm{red}}$, $\sigma \in \mathbb{R}$, where $\varphi_\sigma(x) = 1 + O(\sigma)$ as $\sigma \to 0$. Each such transformation generates (via formula (7.5)) a new matrix $l_\lambda(\sigma)$, $\sigma \to 0$, that obviously belongs to the orbit space associated to the $\hat{G}_{\mathrm{red}}$ action. The set of matrices

$$\left.\frac{dl_\lambda(\sigma)}{d\sigma}\right|_{\sigma=0} \in \hat{\mathfrak{g}}^*_{\mathrm{red}} \tag{7.36}$$

spans a linear subspace $\hat{V}_\lambda \subset \hat{\mathfrak{g}}^*_{\mathrm{red}}$ of finite codimension. Consider an arbitrary deformation $l_\lambda(\epsilon)$, $\epsilon \in \mathbb{R}^n$, of a given matrix $l_\lambda \in \hat{\mathfrak{g}}^*_{\mathrm{red}}$ and denote by $\hat{E}_\lambda$ the linear span in $\hat{\mathfrak{g}}^*_{\mathrm{red}}$ over the matrices $\partial l_\lambda(\epsilon)/\partial\epsilon_i\mid_{\epsilon=0}$, $1 \le i \le n$. The above deformation is said to be *transverse* to the induced $\hat{G}_{\mathrm{red}}$ orbit if the subspaces $\hat{E}_\lambda$ and $\hat{V}_\lambda$ together span their ambient space, that is

$$\hat{E}_\lambda + \hat{V}_\lambda = \hat{\mathfrak{g}}^*_{\mathrm{red}}. \tag{7.37}$$

The following general theorem [42] holds for versal deformations of the Dirac operator (7.1) and can be proved by applying standard perturbation theory techniques.

Theorem 7.1. *A deformation $l_\lambda(\epsilon)$, $\epsilon \in \mathbb{R}^n$, is versal if and only if it is transverse to the induced group $\hat{G}$ orbit.*

We are now ready to make use of the above results to describe the spaces $\hat{E}_\lambda$ and $\hat{V}_\lambda$ analytically. Let $\tilde{\gamma} \in \mathcal{D}(\hat{\mathfrak{g}}^*_{\mathrm{red}})$ be any smooth functional on $\hat{\mathfrak{g}}^*_{\mathrm{red}}$; it generates a flow on the loop group $\hat{G}_{\mathrm{red}}$ orbit via the (σ, x)-evolutions

$$\frac{dl}{d\sigma} := \{\tilde{\gamma}, l\}_\lambda, \quad \frac{dl}{dx} := \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} l \tag{7.38}$$

with respect to the Poisson bracket (7.32). In view of (7.38), formula (7.32) implies that the subspace $\hat{V}_\lambda$ is isomorphic to the following subspace of vector functions in $T^*(M)$:

$$V_\lambda := \{\vartheta_\lambda\psi := (\eta + \lambda\vartheta)\psi : \nabla\tilde{\gamma} = \psi \in T^*(M)\}. \tag{7.39}$$

Theorem 7.1 suggests the following construction of versal deformations for the Dirac type operator (7.1): As ϑ_λ is skew-symmetric, the operator $i\vartheta_\lambda$ is formally self-adjoint in the space $L^2(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^2)$. Therefore, the orthogonal complement to the subspace V_λ with respect to the natural scalar product in $L^2(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^2)$ consists of 2π-periodic solutions to the equation

$$\vartheta_\lambda\psi = 0. \tag{7.40}$$

Whence, we have the following characterization of versal deformations of the operator (7.1).

Theorem 7.2. *The prolongation of the matrix $l_\lambda \in \hat{\mathfrak{g}}^*_{red}$ defined as*

$$\bar{l}_\lambda(\epsilon) := \begin{pmatrix} \lambda & \exp(\partial^{-1}\beta) \\ \exp(\partial^{-1}\alpha) & -\lambda \end{pmatrix} + \sum_{i,j=1}^{2} \epsilon_{ij} \bar{f}_i \otimes \bar{f}_j \tag{7.41}$$

generates a versal deformation of Dirac operator (7.1)*. Here $\otimes$ is the usual Kronecker tensor product in $\mathbb{R}^2$, $\epsilon_{ij} \in \mathbb{R}$, $i,j = 1,2$, $\epsilon_{12} = -\epsilon_{21}$ are any deformation constants, and $f_i \in L^\infty(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^2)$, $i = 1,2$, are two linearly independent, normalized solutions to the Dirac equations*

$$\frac{d\bar{f}_i}{dx} + \bar{l}_\lambda \bar{f}_i = 0, \qquad \{\bar{f}_i, \bar{f}_j\}\big|_{x=0} = 1, \tag{7.42}$$

with spectral parameter $\lambda \in \mathbb{R}$.

Proof. It is easy to verify that the set of solutions to equation (7.40) is isomorphic to the set of functions

$$\hat{\psi} = \sum_{i,j=1}^{2} \epsilon_{ij} \bar{f}_i \otimes \bar{f}_j,$$

and these functions satisfy the canonical Casimir equation

$$\left[l_\lambda, \hat{\psi}\right] - \frac{d\hat{\psi}}{dx} = 0, \tag{7.43}$$

which is equivalent to equation (7.40). Owing to the fact that any matrix $l_\lambda \in \hat{\mathfrak{g}}^*$ in (7.1) can be transformed into the expression $\bar{l}_\lambda(0) \in \hat{\mathfrak{g}}^*$ with functional parameters α, β given by (7.28), this leads to the general form (7.41) for versal deformations of operator (7.1), which finishes the proof. $\square$

The techniques developed here can be used for constructing classes of the Dirac operators with a prescribed parametric dependence, which are important for several applications in mathematical physics.

This page intentionally left blank

Chapter 8

Integrable Spatially Three-dimensional Coupled Dynamical Systems

8.1 Short introduction

The Lax representations [173, 227, 326, 406] for integrable (1+1)-dimensional nonlinear dynamical system hierarchies [54, 173, 326] on functional manifolds were first interpreted as Hamiltonian flows on the dual space to the Lie algebra of integro-differential operators in [7]. An algebraic method for constructing Lax integrable (2+1)-dimensional nonlinear dynamical systems by means of two commuting flows from the hierarchy on a suitable co-adjoint action orbit of an integro-differential operator with an infinite integral part was proposed in [101, 369]. The relationship among some Lax integrable (1+1)- and (2+1)-dimensional systems with the corresponding hierarchies of Hamiltonian flows on dual spaces to centrally extended (via the standard Maurer–Cartan two-cocycle) Lie algebras has been intensively investigated, in particular, in [121, 333, 357, 354].

Every such Hamiltonian flow on the dual space, either to the operator Lie algebra or to its central extension, can be written as the compatibility condition of the spectral relationship for the corresponding integro-differential operator and a suitable eigenfunction evolution. If the spectral relationship admits a finite set of eigenvalues, there is the important problem of finding the Hamiltonian representation for the Lax hierarchy coupled with the evolution of eigenfunctions and appropriate adjoint eigenfunctions. This problem was partially solved in [288, 325, 337, 365] for the Lie algebra of integro-differential operators and its super-generalization by means of a property of variational Casimir functionals under a certain Lie–Bäcklund transformation.

In what follows, we shall first develop a general Lie-algebraic scheme for constructing a hierarchy of Lax integrable flows from Hamiltonian systems on the dual space to the centrally extended Lie algebra of integro-differential operators with matrix-valued coefficients.

Next, we study the Hamiltonian structure for the related coupled Lax hierarchy that is obtained by means of the Bäcklund transformation technique developed in [337, 365].

Then, the corresponding hierarchies of additional or ghost symmetries for the coupled Lax flows are also shown to be Hamiltonian. It is established that an additional hierarchy of Hamiltonian flows is generated by the Poisson structure that is equivalent to the tensor product of the $\mathcal{R}$-deformed canonical Lie–Poisson bracket [54, 325, 337, 365] with the standard Poisson bracket on related eigenfunctions and adjoint eigenfunction spaces [14, 325, 337, 365], and the corresponding natural powers of suitable eigenvalue are their Hamiltonian functions. The method for introducing one more variable into $(2+1)$-dimensional nonlinear dynamical systems using additional symmetries, which preserve Lax integrability, is developed and an integrable $(3+1)$-dimensional analog of the Davey–Stewartson system [368, 403] is constructed.

8.2 Lie-algebra of Lax integrable (2+1)-dimensional dynamical systems

Let $\tilde{\mathcal{G}} := C^\infty(\mathbb{S}\times\mathbb{S};\mathfrak{g})$ be a current Lie algebra of mappings taking values in a semisimple matrix Lie algebra $\mathfrak{g}$. Using $\tilde{\mathcal{G}}$, one constructs the Lie algebra $\hat{\mathfrak{g}}$ of matrix integro-differential operators

$$a := \mathbf{I}\xi^m + \sum_{j<m} a_j \xi^j,$$

where $a_j \in \hat{\mathfrak{g}}$, $j < m$, $j \in \mathbb{Z}$, $m \in \mathbb{N}$, and the symbol $\xi := \partial/\partial x$ denotes differentiation with respect to the independent variable $x \in \mathbb{R}/2\pi\mathbb{Z} \simeq \mathbb{S}^1$. The Lie structure in $\hat{\mathfrak{g}}$ is defined as

$$[a,b] := a\circ b - b\circ a$$

for all $a,\ b \in \hat{\mathfrak{g}}$, where "$\circ$" is the composition operator taking the form:

$$a\circ b := \sum_{\alpha\in\mathbb{Z}_+} \frac{1}{\alpha!}\frac{\partial^\alpha a}{\partial \xi^\alpha}\frac{\partial^\alpha b}{\partial x^\alpha}.$$

On the Lie algebra $\hat{\mathfrak{g}}$ there exists the ad-invariant nondegenerate symmetric bilinear form:

$$(a,b) := \int_0^{2\pi}\int_0^{2\pi} \mathrm{Tr}\,(a\circ b)\ dxdy\,, \tag{8.1}$$

where Tr-operation for all $a \in \tilde{\mathcal{G}}$ is defined as

$$\operatorname{Tr} a := \operatorname{res}_{\xi} \operatorname{tr} a = \operatorname{tr} a_{-1},$$

and tr and res are the usual matrix trace and residue operators, respectively. Employing the scalar product (8.1) one transforms the Lie algebra $\hat{\mathfrak{g}}$ into a metrizable Lie algebra. Accordingly its dual linear space of matrix integro-differential operators $\hat{\mathfrak{g}}^*$ is identified with the Lie algebra, that is $\hat{\mathfrak{g}}^* \simeq \hat{\mathfrak{g}}$.

The linear subspaces $\hat{\mathfrak{g}}^*_+ \subset \hat{\mathfrak{g}}^*$ and $\hat{\mathfrak{g}}^*_- \subset \hat{\mathfrak{g}}$ defined as

$$\hat{\mathfrak{g}}_+ := \left\{ a := \xi^{n(\hat{a})} + \sum_{j=0}^{n(\hat{a})-1} a_j \xi^j : \ a_j \in \hat{\mathfrak{g}}, \quad j = 0, \ldots, n(\hat{a}) \right\} ,$$

$$\hat{\mathfrak{g}}_- := \left\{ b := \sum_{j=0}^{\infty} \xi^{-(j+1)} b_j : \ b_j \in \hat{\mathfrak{g}}, \quad j \in \mathbb{Z}_+ \right\} , \tag{8.2}$$

are Lie subalgebras in $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_+ \oplus \hat{\mathfrak{g}}_-$. Owing to the splitting $\hat{\mathfrak{g}}$ into the direct sum of its Lie subalgebras (8.2) one can construct a Lie–Poisson structure on $\hat{\mathfrak{g}}^*$ by employing the special linear endomorphism $\mathcal{R}$ of $\hat{\mathfrak{g}}$ [54, 121, 354, 357]:

$$\mathcal{R} := (P_+ - P_-)/2, \quad P_\pm \hat{\mathfrak{g}} := \hat{\mathcal{G}}_\pm, \quad P_+ P_- = 0.$$

The centrally extended Lie commutator on $\hat{\mathfrak{g}}_c := \hat{\mathfrak{g}} \oplus \mathbb{C}$ is given as [121, 333, 357]:

$$[(a, \alpha), (b, \beta)] := ([a, b], \omega(\hat{a}, \hat{b})) , \tag{8.3}$$

where $\alpha, \beta \in \mathbb{C}$, is generated by means of the standard Maurer–Cartan two-cocycle on $\hat{\mathfrak{g}}$:

$$\omega(a, b) := (a, [\partial/\partial y, b]) ,$$

$\partial/\partial y$ is differentiation with respect to the independent variable $y \in S$ and $[\partial/\partial y, b] := \partial b/\partial y$. The commutator (8.3) can be deformed using the above endomorphism $\mathcal{R}$ of $\hat{\mathfrak{g}}$:

$$[(a, \alpha), (b, \beta)]_{\mathcal{R}} := ([a, b]_{\mathcal{R}}, \omega_{\mathcal{R}}(a, b)) , \tag{8.4}$$

where the $\mathcal{R}$-commutator is

$$[a, b]_{\mathcal{R}} := [\mathcal{R}a, b] + [a, \mathcal{R}b] ,$$

and the $\mathcal{R}$-deformed two-cocycle is determined as

$$\omega(a, b)_{\mathcal{R}} := \omega(\mathcal{R}a, b) + \omega(a, \mathcal{R}b) .$$

For any Fréchet smooth functionals $\gamma, \mu \in \mathcal{D}(\hat{\mathfrak{g}}_c^*)$, the Lie–Poisson bracket on $\hat{\mathfrak{g}}_c^*$ related to the commutator (8.4) and the extended scalar product:

$$((a,\alpha),(b,\beta)) := (a,b) + \alpha\beta \ ,$$

where $a, b \in \hat{\mathfrak{g}}$ and $\alpha, \beta \in \mathbb{C}$, is given as

$$\{\gamma,\mu\}_{\mathcal{R}}(\ell) = (\ell, [\nabla\gamma(\ell), \nabla\mu(\ell)]_{\mathcal{R}}) + c\omega_{\mathcal{R}}(\nabla\gamma(\ell), \nabla\mu(\ell)) \ , \tag{8.5}$$

where $\ell \in \hat{\mathfrak{g}}^*$ and $c \in \mathbb{C}$. Owing to the scalar product (8.1), the gradient $\nabla\gamma(\ell) \in \hat{\mathcal{G}}$ of a functional $\gamma \in \mathcal{D}(\hat{\mathfrak{g}}_c^*)$ at the point $\ell \in \hat{\mathfrak{g}}^*$ is naturally defined as

$$\delta\gamma(\ell) := (\nabla\gamma(\ell), \delta\ell) \ .$$

Consider the Casimir functionals $\gamma_n \in I(\hat{\mathfrak{g}}_c^*)$, $n \in N$, given as

$$\gamma_n(\ell) := \int_0^{2\pi}\int_0^{2\pi} Tr\,(\xi^n \ell_0) dx dy \ , \tag{8.6}$$

which are invariant with respect to the Ad^*-action of the abstract Lie group $\hat{\mathfrak{g}}_c$ corresponding to $\hat{\mathfrak{g}}_c^*$ and satisfy the following condition [354]

$$(\ell - c\partial/\partial y) \circ \Phi = \Phi \circ (\ell_0 - c\partial/\partial y) \tag{8.7}$$

at a point $\ell \in \hat{\mathfrak{g}}^*$. In (8.7)

$$\hat{\ell}_0 := \xi^m + \sum_{j<m} c_j \xi^j \in \hat{\mathfrak{g}}^*,$$

where $c_j \in \tilde{\mathcal{G}}$, $[\xi, c_j] = 0$, $j < m$, $j \in \mathbb{Z}$ and $m \in \mathbb{N}$,

$$\Phi = 1 + \sum_{r>0} \Phi_r \xi^{-r} \in \hat{\mathfrak{g}}_-,$$

where $\Phi_r \in \tilde{\mathfrak{g}}$, $r \in \mathbb{N}$, and $\hat{\mathfrak{g}}_-$ denotes the appropriate abstract Lie group [101, 121, 354] generated by the Lie subalgebra $\hat{\mathfrak{g}}_-$. As with [354], one can show that the condition (8.7) is equivalent to

$$[\ell - c\partial/\partial y, \nabla\gamma_n(\ell)] = 0 \ , \tag{8.8}$$

for all $n \in \mathbb{N}$. When $c = 0$, the Casimir functionals take the usual Adler form [7, 325].

The Lie–Poisson bracket (8.5) generates a hierarchy of Hamiltonian dynamical systems on $\hat{\mathfrak{g}}_c^*$ with the Casimir functionals $\gamma_n \in I(\hat{\mathfrak{g}}_c^*)$, $n \in N$, as the corresponding Hamiltonian functions, which take the form:

$$d\hat{\ell}/dt_n := [\mathcal{R}\nabla\gamma_n(\ell),\ \ell - c\partial/\partial y] = [(\nabla\gamma_n(\ell))_+,\ \ell - c\partial/\partial y], \tag{8.9}$$

where the lower index "+" denotes the differential part of the corresponding integro-differential operator. This equation is equivalent to the usual commutator Lax representation. It is easy to verify that for every $n \in \mathbb{Z}_+$ the relationship above is the compatibility condition of a system of linear integro-differential equations of the form:

$$(\ell - c\partial/\partial y)f = \lambda f \tag{8.10}$$

and

$$df/dt_n = (\nabla \gamma_n(\ell))_+ f \ , \tag{8.11}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter, $f \in W := W(\mathbb{S} \times \mathbb{S}; H)$ and H is a matrix representation space of the Lie algebra $\mathfrak{g}$. The dynamical system related to (8.11) on the adjoint function space $W^* := W^*(\mathbb{S} \times \mathbb{S}; H)$ is

$$df^*/dt_n = -(\nabla \gamma_n(\ell))_+^* f^*, \tag{8.12}$$

where $f^* \in W^*$ is a solution of the adjoint spectral equation

$$(\ell^* + c\partial/\partial y)f^* = \nu f^*, \tag{8.13}$$

with spectral parameter $\nu \in \mathbb{C}$.

In the sequel, it shall be further assumed that the spectral equation (8.10) admits $N \in \mathbb{N}$ distinct eigenvalues $\lambda_i \in \mathbb{C}$, $1 \leq i \leq N$, in which case one studies the algebraic properties of equation (8.9) combined with $N \in \mathbb{N}$ copies of (8.11):

$$df_i/dt_n = (\nabla \gamma_n(\hat{\ell}))_+ f_i \tag{8.14}$$

for the corresponding eigenfunctions $f_i \in W(\mathbb{S} \times \mathbb{S}; H)$, $1 \leq i \leq N$, and the same number of copies of (8.12):

$$df_i^*/dt_n = -(\nabla \gamma_n(\hat{\ell}))_+^* f_i^* \ , \tag{8.15}$$

for the suitable adjoint eigenfunctions $f_i^* \in W^*(\mathbb{S} \times \mathbb{S}; H)$ for $N \in \mathbb{N}$ different eigenvalues $\nu_i \in \mathbb{C}$, $1 \leq i \leq N$ of (8.13), being considered as a coupled evolution system on the space $\hat{\mathfrak{g}}_c^* \oplus W^N \oplus W^{*N}$. The same problem for $c = 0$ and $N = 1$ has been studied in [287, 288, 325, 326].

8.2.1 *The Poisson bracket on the extended phase space*

To make the exposition here more concise, the following notation shall be used for the gradient vector:

$$\nabla \gamma(\tilde{\ell}, \tilde{f}, \tilde{f}^*) := (\delta\gamma/\delta\tilde{\ell},\ \delta\gamma/\delta\tilde{f},\ \delta\gamma/\delta\tilde{f}^*)^\top,$$

where $\tilde{f} := (\tilde{f}_1, \ldots, \tilde{f}_N)$, $\tilde{f}^* := (\tilde{f}_1^*, \ldots, \tilde{f}_N^*)$ and $\delta\gamma/\delta\tilde{f} := (\delta\gamma/\delta\tilde{f}_1, \ldots, \delta\gamma/\delta\tilde{f}_N)$, $\delta\gamma/\delta\tilde{f}^* := (\delta\gamma/\delta f_1^*, \ldots, \delta\gamma/\delta\tilde{f}_N^*)$, at a point $(\tilde{\ell}, \tilde{f}, \tilde{f}^*)^\top \in \hat{\mathfrak{g}}^* \oplus W^N \oplus W^{*N}$ for any smooth functional $\gamma \in \mathcal{D}(\hat{\mathfrak{g}}_c^* \oplus W^N \oplus W^{*N})$.

On the spaces $\hat{\mathfrak{g}}_c^*$ and $W^N \oplus W^{*N}$ there exist canonical Poisson structures such as

$$\delta\gamma/\delta\tilde{\ell} :\xrightarrow{\tilde{\theta}} [\tilde{\ell} - c\partial/\partial y, (\delta\gamma/\delta\tilde{\ell})_+] - [\tilde{\ell} - c\partial/\partial y, \delta\gamma/\delta\tilde{\ell}]_+ \ , \tag{8.16}$$

where $\tilde{\theta} : T^*(\hat{\mathfrak{g}}_c^*) \to T(\hat{\mathfrak{g}}_c^*)$ is an implectic operator corresponding to (8.5) at a point $\tilde{\ell} \in \hat{\mathfrak{g}}^*$ and

$$(\delta\gamma/\delta\tilde{f}, \, \delta\gamma/\delta\tilde{f}^*)^\top :\xrightarrow{\tilde{J}} (-\delta\gamma/\delta\tilde{f}^*, \, \delta\gamma/\delta\tilde{f})^\top \ , \tag{8.17}$$

where $\tilde{J} : T^*(W^N \oplus W^{*N}) \to T(W^N \oplus W^{*N})$ is an implectic operator corresponding to the symplectic form $\omega^{(2)} = \sum_{i=1}^N d\tilde{f}_i^* \wedge d\tilde{f}_i$ at a point $(\tilde{f}, \tilde{f}^*) \in W^N \oplus W^{*N}$. It should be noted here that the Poisson structure (8.16) generates equation (8.9) for any Casimir functional $\gamma \in I(\hat{\mathfrak{g}}_c^*)$. Thus, on the extended phase space $\hat{\mathfrak{g}}_c^* \oplus W^N \oplus W^{*N}$, one can obtain a Poisson structure as the tensor product $\tilde{\Theta} := \tilde{\theta} \otimes \tilde{J}$ of (8.16) and (8.17).

Consider the Bäcklund transformation

$$(\tilde{\ell}, \tilde{f}, \tilde{f}^*)^\top :\xrightarrow{B} (\ell(\tilde{\ell}, \tilde{f}, \tilde{f}^*), \, f = \tilde{f}, \, f^* = \tilde{f}^*)^\top \ , \tag{8.18}$$

generating on $\hat{\mathfrak{g}}_c^* \oplus W^N \oplus W^{*N}$ a Poisson structure $\Theta : T^*(\hat{\mathfrak{g}}_c^* \oplus W^N \oplus W^{*N}) \to T(\hat{\mathfrak{g}}_c^* \oplus W^N \oplus W^{*N})$. The main condition imposed on the mapping (8.18) is that the resulting dynamical system

$$(d\ell/dt_n, \, df/dt_n, \, df^*/dt_n)^\top := -\Theta \, \nabla\overline{\gamma}_n(\ell, f, f^*) \tag{8.19}$$

coincide with equations (8.9), (8.14) and (8.15) in the case when functionals $\overline{\gamma}_n \in I(\hat{\mathfrak{g}}_c^*)$, $n \in \mathbb{N}$, do not depend on the variables $(f, f^*) \in W^N \oplus W^{*N}$.

To satisfy that condition, a variation of a Casimir functional $\overline{\gamma}_n := \gamma_n|_{\ell=\ell(\tilde{\ell}, f, f^*)} \in \mathcal{D}(\hat{\mathfrak{g}}_c^* \oplus W^N \oplus W^{*N})$, $n \in \mathbb{N}$, will be found, under the constraint $\delta\tilde{\ell} = 0$, taking into account the evolutions (8.14), (8.15) and the

Bäcklund transformation (8.18). One has

$$\begin{aligned}
\delta\overline{\gamma}_n(\tilde{\ell},\tilde{f},\tilde{f}^*)\Big|_{\delta\tilde{\ell}=0} &= \sum_{i=1}^{N}\Big(< \delta\overline{\gamma}_n/\delta\tilde{f}_i, \delta\tilde{f}_i > + < \delta\overline{\gamma}_n/\delta\tilde{f}_i^*, \delta\tilde{f}_i^* >\Big) \\
&= \sum_{i=1}^{N}\Big(< -d\tilde{f}_i^*/dt_n, \delta\tilde{f}_i > + < d\tilde{f}_i/dt_n, \delta\tilde{f}_i^* >\Big)\Big|_{\tilde{\mathrm{f}}=f,\,\tilde{\mathrm{f}}^*=f^*} \\
&= \sum_{i=1}^{N}\big(< (\delta\gamma_n/\delta\ell)_+^* f_i^*, \delta f_i > + < (\delta\gamma_n/\delta\ell)_+ f_i, \delta f_i^* >\big) \\
&= \sum_{i=1}^{N}(< f_i^*, (\delta\gamma_n/\delta\ell)_+ \delta f_i > + < (\delta\gamma_n/\delta\ell)_+ f_i, \delta f_i^* >) \\
&= \sum_{i=1}^{N}\big((\delta\gamma_n/\delta\ell, (\delta f_i)\xi^{-1}\otimes f_i^*) + (\delta\gamma_n/\delta\ell, f_i\xi^{-1}\otimes \delta f_i^*)\big) \\
&= \left(\delta\gamma_n/\delta\ell, \delta\sum_{i=1}^{N} f_i\xi^{-1}\otimes f_i^*\right) := (\delta\gamma_n/\delta\ell, \delta\ell)\ , \quad (8.20)
\end{aligned}$$

where $\gamma_n \in I(\hat{\mathfrak{g}}_c^*)$, $n \in \mathbb{N}$, and the bracket $< \cdot,\cdot >$ denotes a pairing of the spaces W^* and W.

As a result of expression (8.20), one obtains

$$\delta\ell|_{\delta\tilde{\ell}=0} = \sum_{i=1}^{N}\delta(f_i\xi^{-1}\otimes f_i^*)\ . \tag{8.21}$$

As ℓ and $\tilde{\ell} \in \hat{\mathfrak{g}}^*$ are linearly dependent, one finds immediately from (8.21) that

$$\ell = \tilde{\ell} + \sum_{i=1}^{N} f_i\xi^{-1}\otimes f_i^*. \tag{8.22}$$

Thus, the Bäcklund transformation (8.18) can be written as

$$(\tilde{\ell},\tilde{f},\tilde{f}^*)^\top :\xrightarrow{B} (\ell = \tilde{\ell} + \sum_{i=1}^{N} f_i\xi^{-1}\otimes f_i^*, f, f^*)^\top. \tag{8.23}$$

Expression (8.23) generalizes results obtained both for the scalar form of the Lie algebra of integro-differential operators in [337] and for the matrix form in [325]. The existence of the Bäcklund transformation (8.23) leads directly to the following result.

Theorem 8.1. *The dynamical system (8.19) on $\hat{\mathfrak{g}}_c^* \oplus W^N \oplus W^{*N}$ is equivalent to the following system of evolution equations:*

$$d\tilde{\ell}/dt_n = [(\nabla\overline{\gamma}_n(\tilde{\ell}))_+, \tilde{\ell}] - [\nabla\overline{\gamma}_n(\tilde{\ell}), \tilde{\ell}]_+ \ ,$$
$$d\tilde{f}/dt_n = \delta\overline{\gamma}_n/\delta\tilde{f}^* \ , \quad d\tilde{f}^*/dt_n = -\delta\overline{\gamma}_n/\delta\tilde{f},$$

where $\overline{\gamma}_n := \gamma_n|_{\ell=\ell(\tilde{\ell},f,f^*)} \in \mathcal{D}(\hat{\mathfrak{g}}_c^* \oplus W^N \oplus W^{*N})$ and $\gamma_n \in I(\hat{\mathfrak{g}}_c^*)$ is a Casimir functional at a point $\ell \in \mathfrak{g}^*$ for every $n \in \mathbb{Z}_+$ under the Bäcklund transformation (8.23).

Now, simple calculations using

$$\Theta = B^{'}\tilde{\Theta}B^{'*},$$

where $B^{'} : T(\hat{\mathfrak{g}}_c^* \oplus W^N \oplus W^{*N}) \to T(\hat{\mathfrak{g}}_c^* \oplus W^N \oplus W^{*N})$ is a Fréchet derivative of (8.23), readily produce the following form of the Bäcklund transformed Poisson structure Θ on $\hat{\mathfrak{g}}_c^* \oplus W^N \oplus W^{*N}$:

$$\nabla\gamma(\ell, f, f^*) :\xrightarrow{\Theta} \begin{pmatrix} [\ell - c\partial/\partial y, (\delta\gamma/\delta\ell)_+] - [\ell - c\partial/\partial y, \delta\gamma/\delta\ell]_+ + \\ \sum_{i=1}^{N}(f_i\xi^{-1} \otimes (\delta\gamma/\delta f_i) - (\delta\gamma/\delta f_i^*)\xi^{-1} \otimes f_i^*) \\ -\delta\gamma/\delta f^* - (\delta\gamma/\delta\ell)_+ f \\ \delta\gamma/\delta f + (\delta\gamma/\delta\ell)_+^* f^* \end{pmatrix} , \tag{8.24}$$

where $\gamma \in \mathcal{D}(\hat{\mathfrak{g}}_c^* \oplus W^N \oplus W^{*N})$ is an arbitrary smooth functional. Whence, one has the following theorem.

Theorem 8.2. *The hierarchy of dynamical systems (8.9), (8.14) and (8.15) is Hamiltonian with respect to the Poisson structure Θ in the form (8.24) that has the functionals $\overline{\gamma}_n := \gamma_n \in I(\hat{\mathfrak{g}}_c^*)$, $n \in \mathbb{N}$, as Casimir invariants on $\hat{\mathfrak{g}}_c^*$.*

Using expression (8.19), one can construct a new hierarchy of Hamiltonian evolution equations describing commutative flows generated by Casimir invariants $\gamma_n \in I(\hat{\mathfrak{g}}_c^*)$, $n \in \mathbb{N}$, which are in involution with respect to the Poisson bracket (8.5) on the extended phase space $\hat{\mathfrak{g}}_c^* \oplus W^N \oplus W^{*N}$.

8.2.2 *Hierarchies of additional symmetries*

The hierarchy (8.9), (8.14) and (8.15) of evolution equations possesses another natural set of invariants including all higher powers of the eigenvalues λ_k, $k = 1, \ldots, N$. They can be considered as Fréchet smooth functionals on the extended phase space $\hat{\mathfrak{g}}_c^* \oplus W^N \oplus W^{*N}$ owing to the obvious representation:

$$\lambda_k^s =< f_k^*, (\ell - c\partial/\partial y)^s f_k > \ , \tag{8.25}$$

where $s \in N$, holding under the normalizing constraints

$$< f_k^*, f_k >= 1 \ .$$

In the case of the Bäcklund transformation (8.22), where

$$\ell := \ell_+ + \sum_{i=1}^{N} f_i \xi^{-1} \otimes f_i^* \tag{8.26}$$

formula (8.25) gives rise to the following variation of the functionals $\lambda_k^s \in \mathcal{D}(\hat{\mathfrak{g}}_c^* \oplus W^N \oplus W^{*N})$, $1 \leq k \leq N$:

$$\begin{aligned}
&\delta\lambda_k^s =< \delta f_k^*, (\ell - c\partial/\partial y)^s f_k > \\
&+ < (f_k^*, \delta(\ell - c\partial/\partial y)^s) f_k > + < f_k^*, (\ell - c\partial/\partial y)^s (\delta f_k) > \\
&= (M_k^s, \delta\ell_+) + \sum_{i=1}^{N} \big(< (-M_k^s + \delta_k^i (\ell - c\partial/\partial y)^s)^* f_i^*, \delta f_i >\big) \\
&+ \sum_{i=1}^{N} \big(< (-M_k^s + \delta_k^i (\ell - c\partial/\partial y)^s) f_i, \delta f_i^* >\big) \ ,
\end{aligned}$$

where δ_k^i is the Kronecker delta and the operators M_k^s, $s \in \mathbb{N}$, are determined as

$$M_k^s := \sum_{p=0}^{s-1} ((\ell - c\partial/\partial y)^p f_k) \xi^{-1} \otimes ((\ell^* + c\partial/\partial y)^{s-1-p} f_k^*) \ .$$

Thus, one obtains the exact forms of gradients for the functionals $\lambda_k^s \in \mathcal{D}(\hat{\mathfrak{g}}_s^* \oplus W^N \oplus W^{*N})$, $1 \leq k \leq N$:

$$\begin{aligned}
&\nabla\lambda_k^s(\ell_+, f, f^*) = (M_k^s, (-M_k^s + \delta_k^i (\ell - c\partial/\partial y)^s)^* f_i^*, \\
&(-M_k^s + \delta_k^i (\ell - c\partial/\partial y)^s) f_i : \ 1 \leq i \leq N)^\top \ .
\end{aligned} \tag{8.27}$$

The expressions (8.27), (8.16) and (8.17) lead one to a new hierarchy of coupled evolution equations on $\hat{\mathfrak{g}}_c^* \oplus W^N \oplus W^{*N}$:

$$d\ell_+/d\tau_{s,k} = -[M_k^s, \ell_+ - c\partial/\partial y]_+ \ , \tag{8.28}$$

$$df_i/d\tau_{s,k} = (-M_k^s + \delta_k^i (\ell - c\partial/\partial y)^s) f_i \ , \tag{8.29}$$

$$df_i^*/d\tau_{s,k} = (M_k^s - \delta_k^i (\ell - c\partial/\partial y)^s)^* f_i^* \ , \tag{8.30}$$

where $i = 1, \ldots, N$ and $\tau_{s,k} \in \mathbb{R}$, $s \in \mathbb{N}$, are evolution parameters. Owing to the Bäcklund transformation (8.26), equation (8.28) can be rewritten in the commutator form:

$$\begin{aligned}
&d\ell/d\tau_{s,k} = -[M_k^s, \ell - c\partial/\partial y] \\
&= -\lambda_k^p \nu_k^{s-1-p} [M_k^1, \ell - c\partial/\partial y] = \lambda_k^p \nu_k^{s-1-p} d\ell/d\tau_{1,k} \ ,
\end{aligned} \tag{8.31}$$

where $p = 0, \ldots .s-1$. Whence, we are led to the following result.

Theorem 8.3. *For $k = 1, \ldots, N$ and $s \in \mathbb{N}$, the dynamical systems (8.31), (8.29) and (8.30) are Hamiltonian with respect to the Poisson structure Θ in the form (8.24), and the invariant functionals $\overline{\gamma}_s := \lambda_k^s \in \mathcal{D}(\hat{\mathfrak{g}}_c^* \oplus W^N \oplus W^{*N})$.*

Proof. It suffices to show that

$$[d/dt_n, d/d\tau_{1,k}] = 0 \ , \quad [d/d\tau_{1,k}, d/d\tau_{1,q}] = 0 \ , \tag{8.32}$$

where $1 \leq k, q \leq N$ and $n \in \mathbb{N}$. The first equality in formula (8.32) follows from the identities

$$d(\nabla\gamma_n(\ell))_+/d\tau_{1,k} = [(\nabla\gamma_n(\ell))_+, M_1^1]_+ \ , \quad dM_1^1/dt_n = [(\nabla\gamma_n(\ell))_+, M_1^1]_- \ ,$$

and the second one is a consequence of the equation

$$dM_k^1/d\tau_{1,q} - dM_q^1/d\tau_{1,k} = [M_k^1, M_q^1],$$

so the proof is complete. □

Thus, for every $k = 1, \ldots, N$ and all $s \in \mathbb{Z}_+$ the dynamical systems (8.31), (8.29) and (8.30) on $\hat{\mathfrak{g}}_c^* \oplus W^N \oplus W^{*N}$ form a hierarchy of additional homogeneous "ghost" symmetries for the Lax flows (8.9), (8.14) and (8.15) on $\hat{\mathfrak{g}}_c^* \oplus W^N \oplus W^{*N}$.

If $N \geq 2$, one can obtain a new class of nontrivial Hamiltonian flows $d/dT_n := d/dt_n + \sum_{k=1}^{N-1} d/d\tau_{n,k}$, $n \in \mathbb{Z}_+$, on $\hat{\mathfrak{g}}_c^* \oplus W^N \oplus W^{*N}$ in Lax form using the invariants considered above for the centrally extended Lie algebra $\hat{\mathfrak{g}}_c^*$ of integro-differential operators. These flows act on the eigenfunctions $(f_i, f_i^*) \in W \oplus W^*$, $1 \leq i \leq N$, and generate integrable $(N+1)$-dimensional nonlinear dynamical systems.

For example, in the case in which $\ell := \partial/\partial x + f_1\xi^{-1} \otimes f_1^* + f_2\xi^{-1} \otimes f_2^* \in \hat{\mathfrak{g}}^*$ with $(f_1, f_2, f_1^*, f_2^*)^\intercal \in W^2(\mathbb{S} \times \mathbb{S}; H) \times W^{*2}(\mathbb{S} \times \mathbb{S}; H)$, the flows $d/d\tau := d/d\tau_{1,1}$ and $d/dT := d/dT_2 = d/dt_2 + d/d\tau_{2,1}$ on $\hat{\mathfrak{g}}_c^* \oplus W^2 \oplus W^{*2}$ acting on the functions f_i, f_i^*, $i = 1, 2$, give rise to dynamical systems such as

$$\begin{aligned} &f_{1,\tau} = f_{1,x} - cf_{1,y} + f_2 u \ , \quad f_{1,\tau}^* = f_{1,x}^* - cf_{1,y}^* + f_2^* \bar{u} \ , \\ &f_{2,\tau} = -f_1\bar{u} \ , \quad f_{2,\tau}^* = -f_1^* u \ , \end{aligned} \tag{8.33}$$

and

$$\begin{aligned} f_{1,T} &= f_{1,xx} + f_{1,\tau\tau} + wf_1 + 2f_1 v_\tau \ , \\ f_{1,T}^* &= -f_{1,xx}^* - f_{1,\tau\tau}^* - wf_1^* - 2f_1^* v_\tau \ , \\ f_{2,T} &= f_{2,xx} + wf_2 - f_{1,\tau}\bar{u} + f_1\bar{u}_\tau \ , \\ f_{2,T}^* &= -f_{2,xx}^* - wf_2^* + f_{1,\tau}^* u - f_1^* u_\tau \ , \\ cw_y &= w_x - 2(f_1 \otimes f_1^* + f_2 \otimes f_2^*)_x \ , \\ u_x &= f_1^T f_2^* \ , \quad \bar{u}_x = f_1^{*T} f_2 \ , \quad v_x = f_1^T f_1^* \ , \end{aligned} \tag{8.34}$$

where one sets $(\nabla\gamma_2(\ell))_+ := \partial^2/\partial x^2 + w$ for a function $w \in \tilde{G}$ depending parametrically on the variables $\tau, T \in \mathbb{R}$. The systems (8.33) and (8.34) represent a Lax integrable (3+1)-dimensional generalization of the (2+1)-dimensional system equivalent to that of Davey–Stewartson [368, 403] with an infinite sequence of conservation laws, which can be found from (8.6) and have the following form

$$\gamma_n(\ell) := tr \int_0^{2\pi}\int_0^{2\pi} (f_1\partial^{n-1}f_1^*/\partial x^{n-1} + f_2\partial^{n-1}f_2^*/\partial x^{n-1})dxdy \ ,$$

where $n \in N$. Its Lax linearization is given by the spectral problem (8.10) extended by the set of evolution equations:

$$f_\tau = -M_1^1 f \ , \tag{8.35}$$

$$f_T = ((\nabla\gamma_2(\ell))_+ - M_1^2)f \ , \tag{8.36}$$

for an arbitrary eigenfunction $f \in W(\mathbb{S}\times\mathbb{S}; H)$. The relationships (8.35) and (8.36) give rise to the additional nonlinear constraint:

$$w_\tau = 2(f_1 \otimes f_1^*)_x. \tag{8.37}$$

When $\dim H = 1$, the Lax representation (8.10), (8.35) and (8.36) for the above (3+1)-dimensional generalization (8.33), (8.34) and (8.37) of the Davey–Stewartson system [368, 403] has the equivalent matrix form

$$\frac{dF}{dx} = \begin{pmatrix} 0 & 0 & f_1^* \\ 0 & 0 & f_2^* \\ -f_1 & -f_2 & \lambda + c\partial/\partial y \end{pmatrix} F \ ,$$

$$\frac{dF}{d\tau} = \begin{pmatrix} -(\lambda + c\partial/\partial y) & \bar{u} & f_1^* \\ -u & 0 & 0 \\ -f_1 & 0 & 0 \end{pmatrix} F \ ,$$

$$\frac{dF}{dT} = CF \ ,$$

where $F = (F^1, F^2, F^3 = f)^\top \in W(\mathbb{S} \times \mathbb{S}; \mathbb{C}^3)$, $C := \{C_{mn} \in g\ell(3; \mathbb{C}) : 1 \leq m, n \leq 3\}$, and

$$
\begin{aligned}
C_{11} &= -(\lambda + c\partial/\partial y)^2 - u\bar{u} - 2f_1 f_1^* \ , \\
C_{12} &= -f_1 f_2^* - (\lambda + c\partial/\partial y)\bar{u} - \bar{u}_\tau \ , \\
C_{13} &= 2((\lambda + c\partial/\partial y) f_1^* - f_{1,x}^*) - \bar{u} f_2^* \ , \\
C_{21} &= -(\lambda + c\partial/\partial y)u - u_\tau - f_1 f_2^* \ , \\
C_{22} &= -f_2 f_2^* + u\bar{u} \ , \\
C_{23} &= (\lambda + c\partial/\partial y) f_2^* - f_{2,x}^* + u f_1^* \ , \\
C_{31} &= -(\lambda + c\partial/\partial y) f_1 - f_{1,x} - f_{1,\tau} \ , \\
C_{32} &= -(\lambda + c\partial/\partial y) f_2 - f_{2,x} + \bar{u} f_1 \ , \\
C_{33} &= (\lambda + c\partial/\partial y)^2 + w - f_2 f_2^*,
\end{aligned}
$$

to which one can effectively apply the standard inverse spectral transform method [111, 244, 262, 406].

The results obtained above can also be used to construct a wide class of integrable (3+1)-dimensional nonlinear dynamical systems with triple Lax type linearizations [325].

8.2.3 *Closing remarks*

Several regular Lie-algebraic approaches [325, 333, 354, 369] for constructing Lax type integrable multi-dimensional (mainly (2+1)) nonlinear dynamical systems on functional manifolds and their super-symmetric generalizations are well known. In this chapter a new method has been developed for introducing one more variable into Lax integrable (2+1)-dimensional dynamical systems arising as flows on dual spaces to a centrally extended matrix Lie-algebra of integro-differential operators. This approach is based on the recently discovered natural hierarchy of additional invariants [337]. The integrable (3+1)-dimensional dynamical systems obtained using this method possess infinite sequences of conservation laws and related triple Lax type linearizations. Owing to the latter property, their soliton type solutions can be found by means of either the standard inverse spectral transform method [244, 406] or Darboux–Bäcklund transformations [253, 343, 366].

The structure of the constructed Lie–Bäcklund transformation (8.23), which is a key ingredient in the method devised, is strongly dependent on the *ad*-invariant scalar product chosen for an operator Lie algebra $\hat{\mathcal{G}}$

and on a suitable Lie algebra decomposition (see [54, 326]). Since there exist other possibilities for choosing the corresponding *ad*-invariant scalar products on $\hat{\mathfrak{g}}$, such decompositions give rise naturally to other Bäcklund transformations.

Chapter 9

Hamiltonian Analysis and Integrability of Tensor Poisson Structures and Factorized Operator Dynamical Systems

9.1 Problem setting

Consider a Tr-metrizable associative operator algebra $\mathfrak{g}$ endowed with the standard commutator Lie structure and admitting a decomposition into two Lie subalgebras $\mathfrak{g}_+ \oplus \mathfrak{g}_- = \mathfrak{g}$. Then it follows from the basics [121, 326] of the Lie-algebraic theory of dynamical systems that one can construct on $\mathfrak{g}^*$ a so-called Lax flow as follows:

$$\frac{dl}{dt} = [\nabla\gamma_+(l), l] \tag{9.1}$$

here $l \in \mathfrak{g}^*$, $\gamma \in \mathrm{I}(\mathfrak{g}^*)$ is a Casimir function on $\mathfrak{g}^*$, that is $[\nabla\gamma(l), l] = 0$, with the associated gradient decomposition $\nabla\gamma(l) := \nabla\gamma_+(l) \oplus \nabla\gamma_-(l)$ for all $l \in \mathfrak{g}^*$, where $t \in \mathbb{R}$ is an evolution parameter.

The flow (9.1) is Lax integrable since its Casimir functions on $\mathfrak{g}^*$ generate a complete set of mutually commuting invariants in view of the well-known [7, 121, 206, 326] Adler–Kostant–Symes theorem. In general, a Casimir function $\gamma \in \mathrm{I}(\mathfrak{g}^*)$ can be constructed as an analytic functional on $\mathfrak{g}^*$ of the form

$$\gamma := \mathrm{Tr}\,\gamma[l], \tag{9.2}$$

where $\mathrm{Tr}\,(ab) := (a, b)$ is the ad-invariant nondegenerate symmetric Tr-metric on $\mathfrak{g} \simeq \mathfrak{g}^*$.

The expression (9.1) clearly defines the Hamiltonian vector field d/dt with respect to the usual Lie–Poisson bracket on $\mathfrak{g}^*$, modified to conform to the Lie-bracket $[\cdot,\cdot]_{\mathcal{R}} := [P_+(\cdot), P_+(\cdot)] - [P_-(\cdot), P_-(\cdot)]$ on $\mathfrak{g}$, where $P_\pm\mathfrak{g} := \mathfrak{g}_\pm$ are the corresponding projectors. Let us take another element $\widetilde{l} \in \mathfrak{g}^*$, and construct the flow d/dt on $\mathfrak{g}^*$:

$$d\tilde{l}/dt = \left[\nabla\widetilde{\gamma}_+(\widetilde{l}), \widetilde{l}\right] \tag{9.3}$$

where it is assumed that $\widetilde{\gamma} = \gamma \in \mathrm{I}(\mathfrak{g}^*)$. Thus, one has two integrable flows (9.1) and (9.3) subject to the same Casimir function $\gamma \in \mathrm{I}(\mathfrak{g}^*)$, generating the same vector field d/dt on $\mathfrak{g}^*$. Now we pose the following problem : find the relationships between elements l and $\widetilde{l} \in \mathfrak{g}^*$ evolving with respect to flows (9.1) and (9.3) and describe their dual Hamiltonian properties. This problem is the subject of what follows in this short chapter.

9.2 Factorization properties

It follows from the Lax form of equations (9.1) and (9.3) that there exist one-parameter subgroups $a(t)$ and $\widetilde{a}(t) \in \exp \mathfrak{g}_+$, $t \in \mathbb{R}$, such that for any $l(0)$ and $\tilde{l}(0) \in \mathfrak{g}^*$

$$\begin{aligned} l(t) &= Ad^*_{a(t)} l(0) = a^{-1}(t) l(0) a(t), \\ \widetilde{l}(t) &= Ad^*_{\widetilde{a}(t)} l(0) = \widetilde{a}^{-1}(t)\widetilde{l}(0)\widetilde{a}(t), \end{aligned} \tag{9.4}$$

where it is obvious that

$$\begin{aligned} da(t)/dt &= -\ a(t)\nabla\gamma_+(l), \\ d\widetilde{a}(t)/dt &= -\ \widetilde{a}(t)\nabla\gamma_+(\widetilde{l}), \end{aligned} \tag{9.5}$$

for all $t \in \mathbb{R}$. From (9.4) it is seen that for all $t \in \mathbb{R}$

$$\begin{aligned} a(t)l(t)a^{-1}(t) &= l(0), \\ \widetilde{a}(t)\widetilde{l}(t)\widetilde{a}^{-1}(t) &= \widetilde{l}(0). \end{aligned} \tag{9.6}$$

Assume now that there exists an element $B(0) \in \exp \mathfrak{g}_+$ such that the expression

$$Ad^*_{B(0)} l(0) = \widetilde{l}(0)$$

holds, or equivalently

$$B^{-1}(0) l(0) B(0) = \widetilde{l}(0). \tag{9.7}$$

Whence, the equalities (9.6) give rise to the relationships

$$\widetilde{l} = B^{-1} l B, \tag{9.8}$$

where we assume by definition that $B \in \exp \mathfrak{g}_+$ is given as

$$B := B(t) = a(t)^{-1} B(0) \tilde{a}(t) \tag{9.9}$$

for all $t \in \mathbb{R}$.

Let us now assume that an element $A \in \exp \mathfrak{g}_+$ is defined as

$$A := lB. \tag{9.10}$$

It is evident that this is equivalent to the statement that the expression $A(0) = l(0)B(0) \in \exp \mathfrak{g}_+$ holds for the given element $l(0) \in \mathfrak{g}^*$. As a result of the representations (9.10) and (9.9), one can readily find the following evolution equations on $A, B \in \exp \mathfrak{g}_+$, introduced first in [102, 101]:

$$\begin{aligned} dA/dt &= \nabla\gamma_+(l)A - A\nabla\gamma_+(\widetilde{l}), \\ dB/dt &= \nabla\gamma_+(l)B - B\nabla\gamma_+(\widetilde{l}) \end{aligned} \tag{9.11}$$

for all $t \in \mathbb{R}$. This leads us to the following factorization theorem.

Theorem 9.1. *Let an element $l \in \mathfrak{g}^*$ admit the factorization $l = AB^{-1}$ with $A, B \in \exp \mathfrak{g}_+$. Then the Lax flows (9.1) and (9.3) are also factorized into two flows (9.11) with the element $\tilde{l} = B^{-1}A = A^{-1}lA \in \mathfrak{g}^*$.*

Proof. Only the last representation $\tilde{l} = B^{-1}A = A^{-1}lA \in \mathfrak{g}^*$ needs to be proved. In fact, owing to (9.8) $\tilde{l} = B^{-1}lB$, from (9.10) one immediately finds that $\widetilde{l} = B^{-1}A = I \cdot B^{-1}A \equiv A^{-1}lB \cdot B^{-1}A = A^{-1}lA \in \mathfrak{g}^*$, which completes the proof. $\square$

We have now constructed two factorized equations (9.11) subject to the representations $l = AB^{-1}$, $\widetilde{l} = B^{-1}A \in \mathfrak{g}^*$ with elements $A, B \in \exp \mathfrak{g}_+$ and the common invariant Casimir function $\gamma(l) = \gamma(\widetilde{l}) \in \mathrm{I}(\mathfrak{g}^*)$. Next we proceed to an analysis of the Hamiltonian properties of the flows (9.11).

9.3 Hamiltonian analysis

Consider the flows (9.1) and (9.2) to be Hamiltonian on $\mathfrak{g}^* \otimes \mathfrak{g}^*$ and subject to the following tensor doubled Poissonian structure, suggested in [43, 337]:

$$\vartheta : \begin{pmatrix} \nabla\gamma(l) \\ \nabla\gamma(l^*) \end{pmatrix} \longrightarrow \begin{pmatrix} [\nabla\gamma_+(l), l] - [\nabla\gamma(l), l]_+ \\ \left[\nabla\gamma_+(\widetilde{l}), \widetilde{l}\right] - \left[\nabla\gamma(\widetilde{l}), \widetilde{l}\right]_+ \end{pmatrix}, \tag{9.12}$$

where $\gamma \in \mathcal{D}(\mathfrak{g}^*)$ is any smooth functional on $\mathfrak{g}^* \otimes \mathfrak{g}^*$. Now employing the transformation

$$\Phi(A, B; l, \widetilde{l}) = 0 \Leftrightarrow l - AB^{-1} = 0, \quad \widetilde{l} - B^{-1}A = 0, \tag{9.13}$$

which obviously may be considered as a standard Bäcklund transformation, we can construct a new Poisson structure $\eta : T^*(G_+ \times G_+) \longrightarrow T(G_+ \times G_+)$ on the subgroup space $G_+ \times G_+$ with respect to the phase variables $(A, B) \in G_+ \times G_+$. With this, one finds [326] the transformed Poissonian structure

$\eta : T^*(G_+ \times G_+) \longrightarrow T(G_+ \times G_+)$ at $(A,B) \in G_+ \times G_+$, which corresponds to (9.12) and (9.13), where

$$\begin{aligned} \eta &= \tau\vartheta\tau^*, \\ \tau &= \Phi'_{(l,\tilde{l})}\Phi'^{-1}_{(A,B)}. \end{aligned} \tag{9.14}$$

Making use of the expressions

$$\Phi'_{(A,B)} = \begin{pmatrix} -(\cdot)B^{-1} & l(\cdot)B^{-1} \\ -B^{-1}(\cdot) & B^{-1}(\cdot)\tilde{l} \end{pmatrix}, \quad \Phi'_{(l,\tilde{l})} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\Phi'^{-1}_{(A,B)} = \begin{pmatrix} -(1 - l\otimes \tilde{l}^{-1})^{-1}(\cdot)B & (1 - l\otimes \tilde{l}^{-1})^{-1}lB(\cdot)\tilde{l}^{-1} \\ -(1 - l\otimes \tilde{l}^{-1})^{-1}(\cdot)B & (1 - l\otimes \tilde{l}^{-1})^{-1}B(.) \end{pmatrix},$$

$$(\Phi'^{*}_{(A,B)})^{-1} = \begin{pmatrix} -B(\cdot)(1-\tilde{l}^{-1}\otimes l)^{-1} & -B(\cdot)(1-\tilde{l}^{-1}\otimes l)^{-1} \\ \tilde{l}^{-1}(\cdot)Bl(1-\tilde{l}^{-1}\otimes l)^{-1} & (.)B(1-\tilde{l}^{-1}\otimes l)^{-1} \end{pmatrix} \tag{9.15}$$

together with the ϑ-structure (9.12), one finds from (9.14) that

$$\begin{aligned} \eta = &\begin{pmatrix} -(1 - l\otimes \tilde{l}^{-1})^{-1}(\cdot)B & (1 - l\otimes \tilde{l}^{-1})^{-1}lB(\cdot)\tilde{l}^{-1} \\ -(1 - l\otimes \tilde{l}^{-1})^{-1}(\cdot)B & (1 - l\otimes \tilde{l}^{-1})^{-1}B(\cdot) \end{pmatrix} \\ &\times \left(\begin{matrix} \begin{matrix} [l,((1 - l\otimes \tilde{l}^{-1})^{-1}(\cdot)B(1 - l\otimes \tilde{l}^{-1})^{-1}(\cdot))_+] \\ -[l,(1 - l\otimes \tilde{l}^{-1})^{-1}(\cdot)B(1 - l\otimes \tilde{l}^{-1})^{-1}(\cdot)]_+ \end{matrix} \\ \\ \begin{matrix} [(\tilde{l}^{-1}(\cdot)Bl(1-\tilde{l}^{-1}\otimes l)^{-1})_+,\tilde{l}] - [\tilde{l}^{-1}(\cdot)Bl(1-\tilde{l}^{-1}\otimes l)^{-1},\tilde{l}] \\ -[(\tilde{l}^{-1}(\cdot)Bl(1-\tilde{l}^{-1}\otimes l)^{-1}),\tilde{l}] + [((\cdot)B(1-\tilde{l}^{-1}\otimes l)^{-1})_+,\tilde{l}] \end{matrix} \end{matrix} \right. \end{aligned} \tag{9.16}$$

$$\left. \begin{matrix} \begin{matrix} -[(B(\cdot)(1-\tilde{l}^{-1}\otimes l)^{-1})_+, l] \\ +[(B(\cdot)(1-\tilde{l}^{-1}\otimes l)^{-1}), l]_+ \end{matrix} \\ \\ \begin{matrix} [((\cdot)B(1-\tilde{l}^{-1}\otimes l)^{-1})_+,\tilde{l}] \\ -[(\cdot)B(1-\tilde{l}^{-1}\otimes l)^{-1},\tilde{l}]_+ \end{matrix} \end{matrix} \right)$$

at $l = AB^{-1}$ and $\tilde{l} = B^{-1}A \in \mathfrak{g}^*$.

Now take any functional $\gamma \in \mathrm{I}(\mathfrak{g}^*)$ and define the functional $\tilde{\gamma} := \gamma_{l=AB^{-1}} \in \mathcal{D}(G_+ \times G_+)$. Then one can construct, owing to the Poissonian bracket (9.16), the following Hamiltonian flow on $G_+ \times G_+$:

$$\frac{d}{d\tau}(A,B)^\intercal = \eta\nabla\tilde{\gamma}(A,B), \tag{9.17}$$

where $(A,B) \in G_+ \times G_+$ and $\tau \in \mathbb{R}$ is an evolution parameter. The flow (9.17) is characterized by the following result.

Theorem 9.2. *The Hamiltonian vector field $d/d\tau$ on $G_+ \times G_+$ defined by (9.17) and the vector field d/dt defined by (9.11) coincide on $G_+ \times G_+$.*

Proof. This can be proved using a straightforward but tedious calculation starting with the definitions of the flows. □

This theorem completely solves a problem posed in [326] about the Hamiltonian formulation of factorized equations (9.11).

9.4 Tensor products of Poisson structures and source like factorized operator dynamical systems

Now let us assume that an operator pair $(A, B) \in G_+ \times G_+$ is transformed as $(A, B) :\longrightarrow (\tilde{A}, \tilde{B})$, where $\tilde{A} := \tilde{a}^{-1}A\tilde{a}$, $\tilde{B} := \tilde{b}^{-1}B\tilde{b}$ for some $\tilde{a}, \tilde{b} \in G_+$, depending on a scalar function $J \in C^\infty(\mathbb{R}; \mathbb{R})$. Take a suitable Poisson structure $\vartheta_{\tilde{J}}$ on a functional manifold $\tilde{M} \ni \tilde{J}$, and construct the following tensor Poisson structure on $\tilde{M} \times \tilde{G}_+ \times \tilde{G}_+$:

$$(\vartheta_{\tilde{J}} \otimes \vartheta_{(\tilde{A},\tilde{B})}) : T^*(\tilde{M} \times \tilde{G}_+ \times \tilde{G}_+) \to T(\tilde{M} \times \tilde{G}_+ \times \tilde{G}_+). \tag{9.18}$$

Since the extended mapping

$$(\tilde{J}; \tilde{A}, \tilde{B}) \xrightarrow{\mathcal{B}} (J = J; A = \tilde{a}\tilde{A}\tilde{a}^{-1}, B = \tilde{b}\tilde{B}\tilde{b}^{-1}) \tag{9.19}$$

is assumed to be a Bäcklund transformation of the corresponding Poisson structures, we can easily calculate the resulting Poisson structure on the manifold $(M; G_+ \times G_+) \ni (J; A, B)$. As for the construction of Lax integrable flows related to this mapping, it suffices to take a Casimir functional $\tilde{\gamma} := \gamma(\tilde{l})$, where $\tilde{l} := \tilde{A}\tilde{B}^{-1} \in \mathcal{G}^*$ and find the corresponding source-like flow on the manifold $(M; G_+ \times G_+)$, having taken into account that $\tilde{l} := \tilde{A}\tilde{B}^{-1} = \tilde{a}^{-1}A\tilde{a}\tilde{b}^{-1}B\tilde{b} \in \mathcal{G}^*$:

$$\frac{d}{dt}(J; A, B)^\intercal = -\vartheta_{(J;A,B)}\nabla\tilde{\gamma}(J; A, B), \tag{9.20}$$

where $\mathcal{B}_*\vartheta_{(J;A,B)} := \vartheta_{(\tilde{J})} \otimes \vartheta_{(\tilde{A},\tilde{B})}$. On the other hand, the flow (9.20) is globally equivalent to the Lax flow $d\tilde{l}/dt = [\nabla\gamma(\tilde{l}), \tilde{l}]$ on the space $\mathcal{G}^* \ni \tilde{l}$, which immediately shows that our flow possesses an infinite hierarchy of commuting conserved quantities, thereby guaranteeing that, with some additional conditions, its complete integrability by quadratures. The theoretical scheme presented above provides a complete explanation of a series of purely computational results announced in [73]. Other applications of this approach in hydrodynamics and plasma physics have recently been found.

9.5 Remarks

We have presented a purely Lie-algebraic solution to the problem concerning factorization of operator dynamical systems posed by Dickey [101, 102]. Our solution employs only the standard properties of tensor-multiplied Poisson structures and some specially constructed [43, 326] Bäcklund transformations. The approach presented in the work appears to be effective for many applications of factorized operator dynamical systems in diverse fields of mathematical physics, in particular in quantum computing mathematics [194, 262, 319, 320, 367], mathematical genetics and other applied fields.

Chapter 10

A Multi-dimensional Generalization of Delsarte–Lions Transmutation Operator Theory via Spectral and Differential Geometric Reduction

10.1 Spectral operators and generalized eigenfunctions expansions

Let $L \in \mathcal{L}(\mathcal{H})$ be a linear closeable operator with a dense domain $D(L) \subset \mathcal{H}$ in a Hilbert space $\mathcal{H}$. Consider the standard quasi-nucleous Gelfand rigging [28] of this Hilbert space $\mathcal{H}$ with the corresponding positive $\mathcal{H}_+$ and negative $\mathcal{H}_-$ Hilbert spaces as follows:

$$D(L) \subset \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- \subset D'(L), \tag{10.1}$$

which is useful for an investigation of the spectral properties of the operator L in $\mathcal{H}$. We shall use the following definition motivated by considerations from [28–30].

Definition 10.1. An operator $L \in \mathcal{L}(\mathcal{H})$ will be called *spectral* if for all Borel subsets $\Delta \subset \sigma(L)$ of the spectrum $\sigma(L) \subset \mathbb{C}$ and for all pairs $(u, v) \in \mathcal{H}_+ \times \mathcal{H}_+$ the following expressions are defined:

$$L = \int_{\sigma(L)} \lambda \mathrm{dE}(\lambda), \quad (u, \mathrm{E}(\Delta)v) = \int_{\Delta} (u, \mathrm{P}(\lambda)v) \mathrm{d}\rho_\sigma(\lambda), \tag{10.2}$$

where ρ_σ is a finite Borel measure on the spectrum $\sigma(L)$, E is a self-adjoint projection operator measure on the spectrum $\sigma(L)$ such that $\mathrm{E}(\Delta)\mathrm{E}(\Delta') = \mathrm{E}(\Delta \cap \Delta')$ for any Borel subsets $\Delta, \Delta' \subset \sigma(L)$, and $\mathrm{P}(\lambda) : \mathcal{H}_+ \to \mathcal{H}_-$, $\lambda \in \sigma(L)$, is the corresponding family of nucleous integral operators from $\mathcal{H}_+$ into $\mathcal{H}_-$.

It follows from the expression (10.2) that one can write down formally in the weak topology of $\mathcal{H}$ that

$$\mathrm{E}(\Delta) = \int_{\Delta} \mathrm{P}(\lambda) \mathrm{d}\rho_{\sigma}(\lambda) \tag{10.3}$$

for any Borel subset $\Delta \subset \sigma(L)$.

Similarly, one has the corresponding expressions for the adjoint spectral operator $L^* \in \mathcal{L}(\mathcal{H})$ whose domain $\mathrm{D}(L^*) \subset \mathcal{H}$ is also assumed to be dense in $\mathcal{H}$:

$$(\mathrm{E}^*(\Delta)u, v) = \int_{\Delta} (P^*(\lambda)u, v) \mathrm{d}\rho^*_{\sigma}(\lambda), \tag{10.4}$$

$$\mathrm{E}^*(\Delta) = \int_{\Delta} P^*(\lambda) \mathrm{d}\rho^*_{\sigma}(\lambda),$$

where E^* is the corresponding projection spectral measure on Borel subsets $\Delta \in \sigma(L^*)$, $\mathrm{P}^*(\lambda) : \mathcal{H} \to \mathcal{H}$, $\lambda \in \sigma(L^*)$, is the corresponding family of nucleous integral operators in $\mathcal{H}$ and ρ^*_{σ} is a finite Borel measure on the spectrum $\sigma(L^*)$. We also assume that the following conditions

$$\mathrm{P}(\mu)(L - \mu \mathrm{I})v = 0, \quad \mathrm{P}^*(\lambda)(L^* - \bar{\lambda}\mathrm{I})u = 0 \tag{10.5}$$

hold for all $u \in \mathrm{D}(L^*)$, $v \in \mathrm{D}(L)$, where $\bar{\lambda} \in \sigma(L^*)$, $\mu \in \sigma(L)$. In particular, it is assumed that $\sigma(L^*) = \bar{\sigma}(L)$.

Next, we describe the generalized eigenfunctions [190, 359, 350] corresponding to operators L and L^* via the approach devised in [28]. We shall say that an operator $L \in \mathcal{L}(\mathcal{H})$ with a dense domain $\mathrm{D}(L)$ allows a *rigging continuation*, if there is another topological subspace dense in $\mathcal{H}_+$, $\mathrm{D}_+(L^*) \subset \mathrm{D}(L^*)$, such that the adjoint operator $L^* \in \mathcal{L}(\mathcal{H})$ maps it continuously into $\mathcal{H}_+$.

Definition 10.2. A vector $\psi_{\lambda} \in \mathcal{H}_-$ is called a *generalized eigenfunction* of the operator $L \in \mathcal{L}(\mathcal{H})$ corresponding to an eigenvalue $\lambda \in \sigma(L)$ if

$$((L^* - \bar{\lambda}\mathrm{I})u, \psi_{\lambda}) = 0 \tag{10.6}$$

for all $u \in \mathrm{D}_+(L^*)$.

It is evident that when $\psi_{\lambda} \in \mathrm{D}(L)$, $\lambda \in \sigma(L)$, then $L\psi_{\lambda} = \lambda\psi_{\lambda}$ as usual. The definition (10.6) is related [28] to an extension of the operator $L : \mathcal{H} \mapsto \mathcal{H}$. As the operator $L^* : \mathrm{D}_+(L^*) \to \mathcal{H}_+$ is continuous, one can define the adjoint operator $L_{ext} := L^{*,+} : \mathcal{H}_- \to \mathrm{D}_-(L^*)$ with respect to the standard scalar product in $\mathcal{H}$, that is

$$(L^* v, u) = (v, L^{*,+} u) \tag{10.7}$$

for any $v \in \mathrm{D}_+(L^*)$ and $u \in \mathcal{H}_-$ and it coincides with the operator $L : \mathcal{H} \to \mathcal{H}$ on $\mathrm{D}(L)$. The definition (10.6) of a generalized eigenfunction $\psi_{\lambda} \in \mathcal{H}_-$ for $\lambda \in \sigma(L)$ is equivalent to the standard expression

$$L_{ext}\psi_{\lambda} = \lambda\psi_{\lambda}. \tag{10.8}$$

If we define the scalar product

$$(u,v) := (u,v)_+ + (L^*u, L^*v)_+ \tag{10.9}$$

on the dense subspace $\mathrm{D}_+(L^*) \subset \mathcal{H}_+$, then this subspace can be transformed naturally into the Hilbert space $\mathrm{D}_+(L^*)$, whose adjoint "negative" space $\mathrm{D}'_+(L^*) := \mathrm{D}_-(L^*) \supset \mathcal{H}_-$. Take now any generalized eigenfunction $\psi_\lambda \in Im\ \mathrm{P}(\lambda) \subset \mathcal{H}_-,\ \lambda \in \sigma(L)$, of the operator $L : \mathcal{H} \to \mathcal{H}$. Then, as one can see from (10.5), $L^*_{ext}\varphi_\lambda = \bar{\lambda}\varphi_\lambda$ for some function $\varphi_\lambda \in Im\ \mathrm{P}^*(\lambda) \subset \mathcal{H}_-$, $\bar{\lambda} \in \sigma(L^*)$, and $L^*_{ext} : \mathcal{H}_- \to \mathrm{D}_-(L)$ is the corresponding extension of the adjoint operator $L^* : \mathcal{H} \to \mathcal{H}$ by means of reducing, as above, the domain $\mathrm{D}(L)$ to a new domain $\mathrm{D}_+(L) \subset \mathrm{D}(L)$ dense in $\mathcal{H}_+$ on which the operator $L : \mathrm{D}_+(L) \to \mathcal{H}_+$ is continuous.

10.2 Semilinear forms, generalized kernels and congruence of operators

Let $\mathrm{K} : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ be a semilinear form in a Hilbert space $\mathcal{H}$, then we have the following well-known classical result.

Theorem 10.1 (Schwartz [28]**).** *Consider a standard Gelfand rigged chain of Hilbert spaces (10.1) that is invariant under the complex involution $\mathbb{C} :\to \mathbb{C}^*$. Then any continuous semilinear form $\mathrm{K} : \mathcal{H}\times\mathcal{H} \to \mathbb{C}$ can be written, by means of a generalized kernel $\hat{\mathrm{K}} \in \mathcal{H}_- \otimes \mathcal{H}_-$, as*

$$\mathrm{K}[u,v] = (\hat{\mathrm{K}}, v \otimes u)_{\mathcal{H}\otimes\mathcal{H}} \tag{10.10}$$

for any $u, v \in \mathcal{H}_+ \subset \mathcal{H}$. The kernel $\hat{\mathrm{K}} \in \mathcal{H}_- \otimes \mathcal{H}_-$ has the representation

$$\hat{\mathrm{K}} = (\mathrm{D} \otimes \mathrm{D})\bar{\mathrm{K}},$$

where $\bar{\mathrm{K}} \in \mathcal{H} \otimes \mathcal{H}$ is the usual kernel and $\mathrm{D} : \mathcal{H} \to \mathcal{H}_-$ is the square root $\sqrt{\mathrm{J}^*}$ of the positive operator $\mathrm{J}^* : \mathcal{H} \ \to \ \mathcal{H}_-$, which is a Hilbert–Schmidt embedding of $\mathcal{H}_+$ into $\mathcal{H}$ with respect to the chain (10.1). Moreover, the related kernels $(\mathrm{D} \otimes \mathrm{I})\bar{\mathrm{K}}$, $(\mathrm{I} \otimes \mathrm{D})\bar{\mathrm{K}} \in \mathcal{H} \times \mathcal{H}$ are the usual ones.

Now, consider again the operator $L : \mathcal{H} \to \mathcal{H}$ with a dense domain $\mathrm{D}(L) \subset \mathcal{H}$ allowing the Gelfand rigging continuation (10.1) introduced above, and let $\mathrm{D}_+(L^*) \subset \mathrm{D}(L^*)$ the related dense subspace of $\mathcal{H}_+$.

Definition 10.3. A set of generalized kernels $\hat{\mathrm{Z}}_\lambda \subset \mathcal{H}_- \otimes \mathcal{H}_-$ for $\lambda \in \sigma(L) \cap \bar{\sigma}(L^*)$ will be called *elementary* for the operator $L : \mathcal{H} \to \mathcal{H}$ if for any $\lambda \in \sigma(L) \cap \bar{\sigma}(L^*)$, the norm $||\hat{\mathrm{Z}}_\lambda||_{\mathcal{H}_-\otimes\mathcal{H}_-} < \infty$ and

$$(\hat{\mathrm{Z}}_\lambda, ((\Delta - \lambda \mathrm{I})v) \otimes u) = 0, \quad (\hat{\mathrm{Z}}_\lambda, v \otimes (\mathbb{L}^* - \lambda\ \mathrm{I})u) = 0 \tag{10.11}$$

for all $(u,v) \in \mathcal{H}_- \otimes \mathcal{H}_-$.

Assume further, that all our functional spaces are invariant with respect to the involution $\mathbb{C} :\to \mathbb{C}^*$ and set $\mathrm{D}_+ := \mathrm{D}_+(L^*) = \mathrm{D}_+(L) \subset \mathcal{H}_+$. Then one can construct the corresponding extensions $L_{ext} \supset L$ and $L^*_{ext} \supset L^*$, which are linear operators continuously acting from $\mathcal{H}_-$ into $\mathcal{D}_- := \mathrm{D}'_+$. The chain (10.1) is now extended to the chain

$$\mathrm{D}_+ \subset \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- \subset \mathrm{D}_- \tag{10.12}$$

and it is also assumed that the unit operator $\mathrm{I} : \mathcal{H}_- \to \mathcal{H}_- \subset \mathrm{D}_-$ is extended naturally as the embedding operator from $\mathcal{H}_-$ into D_-. Then equalities (10.11) can be equivalently written [28] as

$$(L_{ext} \otimes \mathrm{I})\ \hat{\mathrm{Z}}_\lambda = \lambda \hat{\mathrm{Z}}_\lambda, \quad (\mathrm{I} \otimes L^*_{ext})\ \hat{\mathrm{Z}}_\lambda = \lambda \hat{\mathrm{Z}}_\lambda \tag{10.13}$$

for any $\lambda \in \sigma(L) \cap \bar{\sigma}(L^*)$. Take now a kernel $\hat{\mathrm{K}} \in \mathcal{H}_- \otimes \mathcal{H}_-$ and suppose that the following operator equality

$$(L_{ext} \otimes \mathrm{I})\ \hat{\mathrm{K}} = (\mathrm{I} \otimes L^*_{ext})\ \hat{\mathrm{K}} \tag{10.14}$$

holds. Since equation (10.13) can be written in the form

$$(L_{ext} \otimes \mathrm{I})\ \hat{\mathrm{Z}}_\lambda = (\mathrm{I} \otimes L^*_{ext})\ \hat{\mathrm{Z}}_\lambda \tag{10.15}$$

for any $\lambda \in \sigma(L) \cap \bar{\sigma}(L^*)$, the following characteristic theorem [28] holds.

Theorem 10.2 (Chapter 8, p.621 of [28]**).** *Let a kernel* $\hat{\mathrm{K}} \in \mathcal{H}_- \otimes \mathcal{H}_-$ *satisfy the condition (10.14). Then owing to (10.15), there exists a finite Borel measure* ρ_σ *defined on Borel subsets* $\Delta \subset \sigma(L) \cap \bar{\sigma}(L^*)$ *such that the following weak spectral representation*

$$\hat{\mathrm{K}} = \int_{\sigma(L)\cap\bar{\sigma}(L^*)} \hat{\mathrm{Z}}_\lambda \mathrm{d}\rho_\sigma(\lambda) \tag{10.16}$$

holds. Moreover, it follows from (10.13) that we have the following representation

$$\hat{\mathrm{Z}}_\lambda = \psi_\lambda \otimes \varphi_\lambda,$$

where $L_{ext}\psi_\lambda = \lambda\psi_\lambda$, $L^*_{ext}\varphi_\lambda = \bar{\lambda}\varphi_\lambda$, $(\psi_\lambda, \varphi_\lambda) \in \mathcal{H}_{--} \times \mathcal{H}_{--}$ *and* $\lambda \in \sigma(L) \cap \bar{\sigma}(L^*)$, *where*

$$\mathcal{H}_{++} \subset \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- \subset \mathcal{H}_{--} \tag{10.17}$$

is an appropriate nucleus rigging extension of the Hilbert space $\mathcal{H}$.

Proof. It is easy to see from (10.15) that the kernel (10.16) satisfies equation (10.14). Consider now the Hilbert–Schmidt rigged chain (10.17) and construct a new Hilbert space $\mathcal{H}_{\mathrm{K}}$ with scalar product

$$(u,v)_{\mathrm{K}} := (|\hat{\mathrm{K}}|, v\otimes u)_{\mathcal{H}\otimes\mathcal{H}} \tag{10.18}$$

for all $u,v \in \mathcal{H}_+$, where $|\hat{\mathrm{K}}| := \sqrt{\hat{\mathrm{K}}^*\hat{\mathrm{K}}} \in \mathcal{H}_- \otimes \mathcal{H}_-$ is a positive-definite kernel. The norm $||\cdot||_{\mathrm{K}}$ in $\mathcal{H}_+$ is weaker than $||\cdot||_+$ in $\mathcal{H}_+$ since for any $u \in \mathcal{H}_+,\quad ||u||_+^2 = (|\hat{\mathrm{K}}|, u\otimes u)_{\mathcal{H}\otimes\mathcal{H}} \leq ||\hat{\mathrm{K}}||_- ||u\otimes u||_+ = ||\hat{\mathrm{K}}||_- ||u||_+^2$, so we have the embedding $\mathcal{H}_+ \subset \mathcal{H}_{\mathrm{K}}$. Now we define the following rigged chain related to $\mathcal{H}_{\mathrm{K}}$:

$$\mathcal{H}_{++} \subset (\mathcal{H}_+) \subset \mathcal{H}_{\mathrm{K}} \subset \mathcal{H}_{-\,-,\mathrm{K}} \tag{10.19}$$

and consider (see [28, 32]) the following expression

$$(u, \mathrm{E}(\Delta)v)_{\mathrm{K}} = \int_{\Delta} \mathrm{d}\rho_\sigma(\lambda)(\mathrm{u}, \mathrm{P}(\lambda)v)_{\mathrm{K}}, \tag{10.20}$$

where $\Delta \subset \mathbb{C}$, $u,v \in \mathcal{H}_{++}$, $\mathrm{E}:\mathbb{C}\to\mathcal{B}(\mathcal{H}_{++})$ is a generalized unity spectral expansion for the spectral operator $L:\mathcal{H}\to\mathcal{H}$ whose domain $D(L)$ is reduced to $\mathcal{H}_{++} \subset D(L)$, $\mathrm{P}:\mathbb{C}\to\mathcal{B}_2(\mathcal{H}_{++};\mathcal{H}_{-\,-,\mathrm{K}})$ is a Hilbert–Schmidt operator of generalized projecting and ρ_σ is a finite Borel measure on $\mathbb{C}$. Assume that the kernel $\hat{\mathrm{K}}\in \mathcal{H}_- \otimes \mathcal{H}_-$ has the representation

$$(\hat{\mathrm{K}}, v\otimes\mathrm{u})_{\mathcal{H}\otimes\mathcal{H}} = (|\mathrm{K}|, \mathrm{S}_{\mathrm{K}}^{(r)}v\otimes\mathrm{S}_{\mathrm{K}}^{(\mathrm{l})}\mathrm{u})_{\mathcal{H}\otimes\mathcal{H}} = (\mathrm{S}_{\mathrm{K}}^{(\mathrm{l})}\mathrm{u}, \mathrm{S}_{\mathrm{K}}^{(\mathrm{r})}v)_{\mathrm{K}} \tag{10.21}$$

for all $u,v \in \mathcal{H}_{++}$, where $\mathrm{S}_{\mathrm{K}}^{(l)}$ and $\mathrm{S}_{\mathrm{K}}^{(\mathrm{r})} \in \mathcal{B}(\mathcal{H}_{++})$ are appropriate bounded operators in $\mathcal{H}_{++}$. Then making use of (10.19) and (10.20), one finds that

$$(\hat{\mathrm{K}}, v\otimes\mathrm{u})_{\mathcal{H}\otimes\mathcal{H}} = \int_{\sigma(L)\cap\bar{\sigma}(L^*)} \mathrm{d}\rho_\sigma(\lambda)(\mathrm{S}_{\mathrm{K}}^{(\mathrm{l}),-1}u, \mathrm{P}(\lambda)\mathrm{S}_{\mathrm{K}}^{(\mathrm{r}),-1}v)_{\mathrm{K}}. \tag{10.22}$$

Since there exist [28, 32] two isometries $\mathrm{J}_{\mathrm{K}}:\mathcal{H}_{-\,-,\mathrm{K}} \to \mathcal{H}_{++}$ and $\mathrm{J}:\mathcal{H}_{--} \to \mathcal{H}_{++}$ related, respectively, with rigged chains (10.19) and (10.17), the scalar product on the right-hand side of (10.22) can be transformed as

$$(\mathrm{S}_{\mathrm{K}}^{(\mathrm{l}),-1}u, \mathrm{P}(\lambda)\mathrm{S}_{\mathrm{K}}^{(\mathrm{r}),-1}v)_{\mathrm{K}} = (\mathrm{S}_{\mathrm{K}}^{(\mathrm{l}),-1}u, \mathrm{J}_{\mathrm{K}}\mathrm{P}(\lambda)\mathrm{S}_{\mathrm{K}}^{(\mathrm{r}),-1}v)_{++} \tag{10.23}$$

for all $u,v \in \mathcal{H}_{++}$, where $\mathrm{J}_{\mathrm{K}}\mathrm{S}_{\mathrm{K}}\mathrm{P}(\lambda) \in\mathcal{B}_2(\mathcal{H}_{++})$ as a product of the Hilbert–Schmidt operator $\mathrm{P}(\lambda) \in\mathcal{B}_2(\mathcal{H}_{++};\mathcal{H}_{-\,-,\mathrm{K}})$, $\lambda \in \sigma(L)\cap\bar{\sigma}(L^*)$, and bounded operators J_{K} and S_{K}. Then from a simple corollary of the Schwartz Theorem 10.1, it can be readily seen that there exists a kernel $\hat{\mathrm{Q}}_\lambda \in \mathcal{H}_{++}\otimes\mathcal{H}_{++}$ for any $\lambda \in \sigma(L)\cap\bar{\sigma}(L^*)$, such that

$$(\mathrm{S}_{\mathrm{K}}^{(\mathrm{l}),-1}u, \mathrm{J}_{\mathrm{K}}\mathrm{P}(\lambda)\mathrm{S}_{\mathrm{K}}^{(\mathrm{r}),-1}v)_{++} = (\hat{\mathrm{Q}}_\lambda, v\otimes u)_{\mathcal{H}\otimes\mathcal{H}} \tag{10.24}$$

for all $u,v \in \mathcal{H}_{++}$. Defining now the kernel $\hat{\mathrm{Z}}_\lambda := (\mathrm{J}^{-1}\otimes\mathrm{J}^{-1})\hat{\mathrm{Q}}_\lambda \in \mathcal{H}_{--}\otimes\mathcal{H}_{--}$, $\lambda \in \sigma(L)\cap\bar{\sigma}(L^*)$, one finds finally from (10.22)-(10.24) that for all $u,v \in \mathcal{H}_{++}$,

$$(\hat{\mathrm{K}},v\otimes u)_{\mathcal{H}\otimes\mathcal{H}} = \int_{\sigma(L)\cap\bar{\sigma}(L^*)} \mathrm{d}\rho_\sigma(\lambda)(\hat{\mathrm{Z}}_\lambda, v\otimes u)_{\mathcal{H}\otimes\mathcal{H}}. \tag{10.25}$$

As the constructed kernel $\hat{\mathrm{Z}}_\lambda \in \mathcal{H}_{--}\otimes\mathcal{H}_{--}$ obviously satisfies the relationship (10.15) for every $\lambda \in \sigma(L)\cap\bar{\sigma}(L^*)$, the theorem is proved. $\square$

The above construction for a self-similar congruent kernel $\hat{\mathrm{K}} \in \mathcal{H}_-\otimes\mathcal{H}_-$ in the form (10.16) related to an operator $L \in \mathcal{L}(\mathcal{H})$ can be modified for the case when self-similarity is replaced by simple similarity as we shall show in the succeeding section.

10.3 Congruent kernel operators and related Delsarte transmutation maps

Consider in a Hilbert space $\mathcal{H}$ a pair of densely defined linear differential operators L and $\tilde{L} \in \mathcal{L}(\mathcal{H})$. In this context, we shall find the definition below to be quite useful.

Let a pair of kernels $\hat{\mathrm{K}}_{\mathrm{s}} \in \mathcal{H}_-\otimes\mathcal{H}_-$, $s=\pm$, satisfy the congruence relationships

$$(\tilde{L}_{ext}\otimes 1)\hat{\mathrm{K}}_{\mathrm{s}} = (1\otimes L^*_{ext})\hat{\mathrm{K}}_{\mathrm{s}} \tag{10.26}$$

for the extended linear operators $L,\ \tilde{L} \in \mathcal{L}(\mathcal{H})$. Then the kernels $\hat{\mathrm{K}}_{\mathrm{s}} \in \mathcal{H}_-\otimes\mathcal{H}_-$, $s=\pm$, will be called *congruent* to this pair $(L,\tilde{L})$ of operators in $\mathcal{H}$.

Since not every pair $L,\ \tilde{L} \in \mathcal{L}(\mathcal{H})$ can be congruent, the natural problem is how to describe the set of corresponding kernels $\hat{\mathrm{K}}_{\mathrm{s}} \in \mathcal{H}_-\otimes\mathcal{H}_-$, $s=\pm$, congruent to a given pair $(L,\tilde{L})$ of operators in $\mathcal{H}$. The next question is that of existence of kernels $\hat{\mathrm{K}}_{\mathrm{s}} \in \mathcal{H}_-\otimes\mathcal{H}_-$, $s=\pm$, congruent to this pair. The question has an obvious answer when $\tilde{L}=L$ and the congruence is then self-similar. The interesting case when $\tilde{L}\neq L$ is nontrivial and can be treated more or less successfully if there exist suitably defined bounded and invertible operators $\Omega_{\mathrm{s}} \in \mathcal{L}(\mathcal{H})$, $s=\pm$, satisfying the transmutation conditions

$$\tilde{L}\boldsymbol{\Omega}_{\mathrm{s}} = \boldsymbol{\Omega}_{\mathrm{s}}L. \tag{10.27}$$

Definition 10.4. (Delsarte [93], Delsarte and Lions [95]) Let a pair of densely defined differential closeable operators $L,\tilde{L} \in \mathcal{L}(\mathcal{H})$ in a Hilbert space $\mathcal{H}$ be endowed with a pair of closed subspaces $\mathcal{H}_0, \tilde{\mathcal{H}}_0 \subset \mathcal{H}_-$ subject to a rigged Hilbert spaces chain (10.1). Then suitably defined and invertible operators $\Omega_s \in \mathcal{L}(\mathcal{H})$, $s = \pm$, are called *Delsarte transmutations* if the following conditions hold: i) the operators $\mathbf{\Omega}_s$ and their inverses $\mathbf{\Omega}_s^{-1}$, $s = \pm$, are continuous; ii) the images $\operatorname{Im} \Omega_s|_{\mathcal{H}_0} = \tilde{\mathcal{H}}_0, s = \pm$; and iii) the relationships (10.27) obtain.

Suppose now that an operator pair $(L, \tilde{L}) \subset \mathcal{L}(\mathcal{H})$ is differentiable of the same order $n(L) \in \mathbb{Z}_+$, that is

$$L := \sum_{|\alpha|=0}^{n(L)} a_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}, \qquad \tilde{L} := \sum_{|\alpha|=0}^{n(L)} \tilde{a}_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}, \tag{10.28}$$

where $x \in \mathrm{Q}$, $\mathrm{Q} \subset \mathbb{R}^m$ is an open connected region in $\mathbb{R}^m$, the smooth coefficients a_α, $\tilde{a}_\alpha \in C^\infty(\mathrm{Q}; End\ \mathbb{C}^{\mathrm{N}})$ for all $\alpha \in \mathbb{Z}_+^m$, $0 \le |\alpha| \le n(L)$ and $\mathrm{N} \in \mathbb{Z}_+$. The differential expressions (10.28) are defined and closeable on the dense domains in the Hilbert space $\mathcal{H} := L^2(\mathrm{Q}; \mathbb{C}^{\mathrm{N}})$ and $\mathrm{D}(L)$, $\mathrm{D}(\tilde{L}) \subset \mathrm{W}_2^{n(L)}(\mathrm{Q}; \mathbb{C}^{\mathrm{N}}) \subset \mathcal{H}$. Accordingly there exists the corresponding to (10.28) pair of adjoint operators $L^*, \tilde{L}^* \in \mathcal{L}(\mathcal{H})$, which are also defined on dense domains $\mathrm{D}(L^*)$, $\mathrm{D}(\tilde{L}^*) \subset \mathrm{W}_2^{n(L)}(\mathrm{Q}; \mathbb{C}^{\mathrm{N}}) \subset \mathcal{H}$.

Take a pair of invertible bounded linear operators $\mathbf{\Omega}_s \in \mathcal{L}(\mathcal{H})$, $s = \pm$, and consider the following Delsarte transformed operators

$$\tilde{L}_s := \mathbf{\Omega}_s L \mathbf{\Omega}_s^{-1}, \tag{10.29}$$

$s = \pm$, which, by definition continues to be differentiable. An additional natural constraint on the operators $\Omega_s \in \mathcal{L}(\mathcal{H})$, $s = \pm$, is the independence [117, 121] of the differential expressions for operators (10.29) on the indices $s = \pm$. The problem of constructing such Delsarte transmutation operators $\Omega_s \in \mathcal{L}(\mathcal{H})$, $s = \pm$, is very complicated but can be extraordinarily useful in light of the special results obtained in [117, 284] for two-dimensional Dirac and three-dimensional Laplacian operators.

Before attacking this problem, we consider some formal generalizations of the results described before. Take an elementary kernel $\hat{\tilde{Z}}_\lambda \in \mathcal{H}_- \otimes \mathcal{H}_-$ satisfying the conditions generalizing (10.13):

$$(\tilde{L}_{ext} \otimes \mathrm{I})\ \hat{\tilde{\mathrm{Z}}}_\lambda = \lambda \hat{\tilde{\mathrm{Z}}}_\lambda, \qquad (\mathrm{I} \otimes L^*_{ext})\ \hat{\tilde{\mathrm{Z}}}_\lambda = \lambda \hat{\tilde{\mathrm{Z}}}_\lambda \tag{10.30}$$

for $\lambda \in \sigma(\tilde{L}) \cap \bar{\sigma}(L^*)$, which are well suited to treating the equation (10.26). Then one has an elementary kernel $\widehat{\mathrm{Z}}_\lambda \in \mathcal{H}_- \otimes \mathcal{H}_-$ for any $\lambda \in \sigma(\widetilde{L}) \cap \sigma(\widetilde{L}^*)$ that solves equation (10.26), that is

$$(\widetilde{L}_{ext} \otimes 1)\ \hat{\tilde{\mathrm{Z}}}_\lambda = (1 \otimes L^*_{ext})\ \hat{\tilde{\mathrm{Z}}}_\lambda. \tag{10.31}$$

Thus, one can expect that for kernels $\hat{\mathrm{K}}_{\mathrm{s}} \in \mathcal{H}_- \otimes \mathcal{H}_-$, $s = \pm$, there exist the (similar to (10.18)) spectral representations

$$\hat{\mathrm{K}}_{\mathrm{s}} = \int_{\sigma(\widetilde{L})\cap\bar{\sigma}(L^*)} \widehat{\tilde{\mathrm{Z}}}_\lambda \mathrm{d}\rho_{\sigma,\mathrm{s}}(\lambda), \tag{10.32}$$

$s = \pm$, with finite spectral measures $\rho_{\sigma,\mathrm{s}}$, $s = \pm$, localized on the Borel subsets of the common spectrum $\sigma(\tilde{L}) \cap \bar{\sigma}(L^*)$. Using the spectral representation like (10.17) applied separately to operators $\tilde{L} \in \mathcal{L}(\mathcal{H})$ and $L^* \in \mathcal{L}(\mathcal{H})$, we easily obtain the following result.

Theorem 10.3. *The equations (10.30) are compatible for any $\lambda \in \sigma(\widetilde{L}) \cap \bar{\sigma}(L^*)$ and, moreover, for kernels $\hat{\mathrm{K}}_{\mathrm{s}} \in \mathcal{H}_- \otimes \mathcal{H}_-$, $s = \pm$, satisfying the congruence condition (10.26), there exists a kernel $\widehat{\tilde{\mathrm{Z}}}_\lambda \in \mathcal{H}_{-\,-} \otimes \mathcal{H}_{-\,-}$ for a suitable Gelfand rigged Hilbert spaces chain (10.20) with the spectral representations (10.32).*

Now we turn to the inverse problem of constructing kernels $\hat{\mathrm{K}}_s \in \mathcal{H}_- \otimes \mathcal{H}_-$, $s = \pm$, like (10.32) that satisfy the congruence conditions (10.26) subject to the same pair $(L, \tilde{L})$ of differential operators in $\mathcal{H}$ that are related via the Delsarte transmutation condition (10.27). In some sense, we shall state that only for such Delsarte related operator pairs $(L, \tilde{L})$ in $\mathcal{H}$ one can construct a dual pair $\{\hat{\mathrm{K}}_s \in \mathcal{H}_- \otimes \mathcal{H}_- : s = \pm\}$ of the corresponding congruent kernels satisfying the conditions like (10.26), that is

$$(\tilde{L}_{ext} \otimes 1)\ \hat{\mathrm{K}}_\pm = \hat{\mathrm{K}}_\pm(1 \otimes L^*_{ext}). \tag{10.33}$$

Suppose now that there exists another pair of Delsarte transmutation operators $\mathbf{\Omega}_s$ and $\mathbf{\Omega}_s^{\circledast} \in \mathcal{L}(\mathcal{H})$, $s = \pm$, satisfying condition ii) of Definition 10.4 in terms of the corresponding two pairs of differential operators $(L, \tilde{L})$ and $(L^*, \tilde{L}^*) \subset \mathcal{L}(\mathcal{H})$. Consequently, there exists an additional pair of closed subspace $\mathcal{H}_0^{\circledast}$ and $\tilde{\mathcal{H}}_0^{\circledast} \subset \mathcal{H}_-$ such that

$$\mathrm{Im}\ \mathbf{\Omega}_s^{\circledast}|_{\mathcal{H}_0^{\circledast}} = \tilde{\mathcal{H}}_0^{\circledast} \tag{10.34}$$

$s = \pm$, for the Delsarte transmutation operator $\mathbf{\Omega}_s^{\circledast} \in \mathcal{L}(\mathcal{H})$, $s = \pm$, satisfying the obvious conditions

$$\tilde{L}^* \cdot \mathbf{\Omega}_s^{\circledast} = \mathbf{\Omega}_s^{\circledast} \cdot L^* \tag{10.35}$$

$s = \pm$, involving the adjoint operators $\tilde{L}^*, L^* \in \mathcal{L}(\mathcal{H})$ defined above and given, as follows from (10.28), by the usual differential expressions

$$L^* = \sum_{|\alpha|=0}^{n(L)} (-1)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x^\alpha} \cdot \bar{a}_\alpha^{\intercal}(x),\ \ \tilde{L}^* = \sum_{|\alpha|=0}^{n(L)} (-1)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x^\alpha} \cdot \bar{\tilde{a}}_\alpha^{\intercal}(x) \tag{10.36}$$

for all $x \in \mathrm{Q} \subset \mathbb{R}^m$.

Now, following [145, 272], we construct the Delsarte transmutation operators of Volterra type

$$\mathbf{\Omega}_{\pm} := 1 \ + \mathrm{K}_{\pm}(\mathbf{\Omega}), \tag{10.37}$$

corresponding to two different kernels $\hat{\mathrm{K}}_{+}$ and $\hat{\mathrm{K}}_{-} \in \mathcal{H}_{-} \otimes \mathcal{H}_{-}$, of integral Volterra operators $\mathrm{K}_{+}(\mathbf{\Omega})$ and $\mathrm{K}_{-}(\mathbf{\Omega})$, related as follows

$$(u, \mathrm{K}_{\pm}(\mathbf{\Omega})v) := (u\ \chi(S_{x,\pm}^{(m)}), \hat{\mathrm{K}}_{\pm} v) \tag{10.38}$$

for all $(u, v) \in \mathcal{H}_{+} \times \mathcal{H}_{+}$, where $\chi(S_{x,\pm}^{(m)})$ are characteristic functions of two m-dimensional smooth hypersurfaces $S_{x,+}^{(m)}$ and $S_{x,-}^{(m)} \in \mathcal{K}(\mathrm{Q})$ from a singular simplicial complex $\mathcal{K}(\mathrm{Q})$ of the open set $\mathrm{Q} \subset \mathbb{R}^m$, chosen so that the boundary $\partial(S_{x,+}^{(m)} \cup S_{x,-}^{(m)}) = \partial \mathrm{Q}$. When $\mathrm{Q} := \mathbb{R}^m$, it is assumed naturally that $\partial \mathbb{R}^m = \varnothing$. Making use of the Delsarte operators (10.37) and relationship such as (10.27), one obtains the following differential operator expressions:

$$\tilde{L}_{\pm} - L = \mathrm{K}_{\pm}(\mathbf{\Omega})L - \tilde{L}_{\pm}\mathrm{K}_{\pm}(\mathbf{\Omega}). \tag{10.39}$$

Since the left-hand sides of (10.39) are purely differential expressions, it follows immediately that the local kernel relationships like (10.32) hold:

$$(\tilde{L}_{ext,\pm} \otimes 1)\ \hat{\mathrm{K}}_{\pm} = (1 \otimes L_{ext}^{*})\ \hat{\mathrm{K}}_{\pm}. \tag{10.40}$$

The expressions (10.39) define, in general, two different differential expressions $\tilde{L}_{\pm} \in \mathcal{L}(\mathcal{H})$ depending both on the kernels $\hat{\mathrm{K}}_{\pm} \in \mathcal{H}_{-} \otimes \mathcal{H}_{-}$ and on the chosen hypersurfaces $S_{x,\pm}^{(m)} \in \mathcal{K}(\mathrm{Q})$. As we prove later, the following important theorem holds.

Theorem 10.4. *Let smooth hypersurfaces $S_{x,\pm}^{(m)} \in \mathcal{K}(\mathrm{Q})$ be chosen such that $\partial(S_{x,+}^{(m)} \cup S_{x,-}^{(m)}) = \partial \mathrm{Q}$ and $\partial S_{x,\pm}^{(m)} = \mp\sigma_x^{(m-1)} + \sigma_{x\pm}^{(m-1)}$, where $\sigma_x^{(m-1)}$ and $\sigma_{x\pm}^{(m-1)}$ are homologous simplicial chains in the homology group $\mathrm{H}_{m-1}(\mathrm{Q};\mathbb{C})$ and parametrized, respectively, by a running point $x \in \mathrm{Q}$ and fixed points $x_{\pm} \in \partial \mathrm{Q}$, and satisfying the following homotopy condition: $\lim_{x \to x_{\pm}} \sigma_x^{(m-1)} = \mp\sigma_{x_{\pm}}$. Then the operator equalities*

$$\tilde{L}_{+} := \mathbf{\Omega}_{+} L \mathbf{\Omega}_{+}^{-1} = \tilde{L} = \mathbf{\Omega}_{-} L \mathbf{\Omega}_{-}^{-1} := \tilde{L}_{-} \tag{10.41}$$

are satisfied if either of the following obtains: the commutation property

$$[\mathbf{\Omega}_{+}^{-1}\ \mathbf{\Omega}_{-}, L] = 0 \tag{10.42}$$

or, equivalently, the kernel relationship

$$(L_{ext} \otimes 1)\hat{\mathbf{\Omega}}_{+}^{-1} * \hat{\mathbf{\Omega}}_{-} = (1 \otimes L_{ext}^{*})\hat{\mathbf{\Omega}}_{+}^{-1} * \hat{\mathbf{\Omega}}_{-}. \tag{10.43}$$

Remark 10.1. It is noted here that special degenerate cases of Theorem 10.4 have been proved in [117, 283] for two-dimensional Dirac and three-dimensional Laplace type differential operators. The constructions and tools devised there were modified for the approach developed here.

Consider now a pair $(\mathbf{\Omega}_+, \mathbf{\Omega}_-)$ of Delsarte transmutation operators in the form (10.37) and satisfying all of the conditions from Theorem 10.4. Then the following is true.

Lemma 10.1. *Let an invertible Fredholm operator* $\mathbf{\Omega} := 1 + \Phi \in \mathcal{B}(\mathcal{H}) \cap Aut(\mathcal{H})$ *with* $\Phi \in \mathcal{B}_\infty(\mathcal{H})$ *have the factorization*

$$\mathbf{\Omega} = \mathbf{\Omega}_+^{-1}\mathbf{\Omega}_- \tag{10.44}$$

by means of two Delsarte operators $\mathbf{\Omega}_+$ *and* $\mathbf{\Omega}_- \in \mathcal{L}(\mathcal{H})$ *in the form (10.37). Then there exists a unique operator kernel* $\hat{\Phi} \in \mathcal{H}_- \otimes \mathcal{H}_-$ *corresponding naturally to the compact operator* $\Phi(\mathbf{\Omega}) \in \mathcal{B}_\infty(\mathcal{H})$ *and satisfying the self-similar congruence condition*

$$(L_{ext} \otimes 1)\ \hat{\Phi} = (1 \otimes L_{ext}^*)\ \hat{\Phi}, \tag{10.45}$$

related to the properties (10.42) and (10.43).

From the equality (10.45) and Theorem 10.2 one obtains the following corollary.

Corollary 10.1. *There exists a finite Borel measure* ρ_σ *defined on the Borel subsets of* $\sigma(\mathrm{L}) \cap \overline{\sigma}(\mathrm{L}^*)$ *such that the following weak equality holds:*

$$\widehat{\Phi} = \int_{\sigma(\mathrm{L}) \cap \overline{\sigma}(\mathrm{L}^*)} \widehat{\mathrm{Z}}_\lambda \mathrm{d}\rho_\sigma(\lambda). \tag{10.46}$$

As for the differential expression $L \in \mathcal{L}(\mathcal{H})$ and the corresponding Volterra type Delsarte transmutation operators $\mathbf{\Omega}_\pm \in \mathcal{L}(\mathcal{H})$, the conditions (10.42) and (10.45) are equivalent to the operator equation

$$[\Phi(\mathbf{\Omega}), L] = 0. \tag{10.47}$$

In fact, since equalities (10.41) hold, one obtains

$$\begin{aligned} L(1 + \Phi(\mathbf{\Omega})) &= L(\mathbf{\Omega}_+^{-1}\mathbf{\Omega}_-) = \mathbf{\Omega}_+^{-1}(\mathbf{\Omega}_+ L \mathbf{\Omega}_+^{-1})\mathbf{\Omega}_- \\ &= \mathbf{\Omega}_+^{-1}(\mathbf{\Omega}_- L \mathbf{\Omega}_-)\mathbf{\Omega}_- = \mathbf{\Omega}_+^{-1}\mathbf{\Omega}_- L = (1 + \Phi(\mathbf{\Omega}))L, \end{aligned} \tag{10.48}$$

meaning exactly (10.47).

Suppose also that for another Fredholm operator $\mathbf{\Omega}^{\circledast} = 1 + \Phi^{\circledast}(\mathbf{\Omega}) \in \mathcal{L}(\mathcal{H})$ with $\Phi^{\circledast}(\mathbf{\Omega}) \in \mathcal{B}_{\infty}(\mathcal{H})$, there exist two Delsarte transmutation Volterra type operators $\mathbf{\Omega}_{\pm}^{\circledast} \in \mathcal{L}(\mathcal{H})$ in the form

$$\mathbf{\Omega}_{\pm}^{\circledast} = 1 + \mathrm{K}_{\pm}^{\circledast}(\mathbf{\Omega}) \tag{10.49}$$

with Volterra [145, 78] integral operators $\mathrm{K}_{\pm}^{\circledast}(\mathbf{\Omega})$ related naturally to kernels $\hat{\mathrm{K}}_{\pm} \in \mathcal{H}_{-} \otimes \mathcal{H}_{-}$, and the factorization condition

$$1 + \Phi^{\circledast}(\mathbf{\Omega}) = \mathbf{\Omega}_{+}^{\circledast,-1}\mathbf{\Omega}_{-}^{\circledast} \tag{10.50}$$

is satisfied. Then one has the following result.

Theorem 10.5. *Let a pair of hypersurfaces $S_{x,\pm}^{(m)} \subset \mathcal{K}(\mathrm{Q})$ satisfy all of the conditions from Theorem 3.2. Then the Delsarte transformed operators $\tilde{L}_{\pm}^{*} \in \mathcal{L}(\mathcal{H})$ are differential and equal; that is,*

$$\tilde{L}_{+}^{*} = \mathbf{\Omega}_{+}^{\circledast} L^{*} \mathbf{\Omega}_{+}^{\circledast,-1} = \tilde{L}^{*} = \mathbf{\Omega}_{-}^{\circledast} L^{*} \mathbf{\Omega}_{-}^{\circledast,-1} = \tilde{L}_{-}^{\circledast}, \tag{10.51}$$

iff the following commutation condition

$$[\Phi^{\circledast}(\mathbf{\Omega}), L^{*}] = 0 \tag{10.52}$$

holds.

Proof. We argue as before when analyzing the congruence condition for a given pair $(L, \tilde{L}) \subset \mathcal{L}(\mathcal{H})$ of differential operators and their adjoints in $\mathcal{H}$. Using the Delsarte transmutation from the differential operators L and $L^{*} \in \mathcal{L}(\mathcal{H})$, we have obtained above two differential operators

$$\tilde{L} = \mathbf{\Omega}_{\pm} L \mathbf{\Omega}_{\pm}^{-1}, \quad \tilde{L}^{*} = \mathbf{\Omega}_{\pm}^{\circledast} L^{*} \mathbf{\Omega}_{\pm}^{\circledast,-1}, \tag{10.53}$$

which must be compatible and so related as

$$(\tilde{L})^{*} = \widetilde{(L^{*})}. \tag{10.54}$$

The condition (10.54) due to (10.53) gives rise to the following additional commutation expressions for the kernels $\mathbf{\Omega}_{\pm}^{\circledast}$ and $\mathbf{\Omega}_{\pm}^{*} \in \mathcal{L}(\mathcal{H})$:

$$[L^{*}, \mathbf{\Omega}_{\pm}^{*}\mathbf{\Omega}_{\pm}^{\circledast}] = 0, \tag{10.55}$$

which are obviously equivalent to the commutation relationship

$$[L, \mathbf{\Omega}_{\pm}^{\circledast,*}\mathbf{\Omega}_{\pm}] = 0, \tag{10.56}$$

so the proof is complete. □

As a result of representations (10.56), one can readily prove the following corollary.

Corollary 10.2. *There exist finite Borel measures $\rho_{\sigma,\pm}$ localized on $\sigma(L)\cap\bar{\sigma}(L^*)$ such that the following weak kernel representations*

$$\hat{\mathbf{\Omega}}_{\pm}^{\circledast,*} * \hat{\mathbf{\Omega}}_{\pm} = \int_{\sigma(L)\cap\bar{\sigma}(L^*)} \hat{\mathrm{Z}}_{\lambda} d\rho_{\sigma,\pm}(\lambda) \tag{10.57}$$

hold, where $\hat{\mathbf{\Omega}}_{\pm}^{\circledast,*}$ and $\hat{\mathbf{\Omega}}_{\pm} \in \mathcal{H}_{-}\otimes\mathcal{H}_{-}$ are the corresponding kernels of integral Volterra operators $\mathbf{\Omega}_{\pm}^{\circledast,*}$ and $\mathbf{\Omega}_{\pm} \in \mathcal{L}(\mathcal{H})$.

The integral operators of Volterra type (10.37) constructed above by means of kernels in the form (10.32) are, as is well known [78, 117, 232, 233, 247], very important for studying many problems of spectral analysis and related integrable nonlinear dynamical systems [121, 247, 283, 282, 326, 406] on functional manifolds. In particular, they serve as factors for a class of Fredholm operators entering the fundamental Gelfand–Levitan–Marchenko operator equations [121, 233, 247], whose solutions are kernels of Delsarte transmutation operators of Volterra type related to the corresponding congruent kernels subject to given pairs of closeable differential operators in a Hilbert space $\mathcal{H}$. Thus, it is natural to learn more of their structural properties subject to their representations both in the form (10.32), (10.37) and in the dual form in the general Gokhberg–Krein theory [121, 145, 272] of Volterra operators.

To continue, we need to introduce some additional notions and definitions from [78, 145] that are important for what will follow below. Define a set $\mathcal{P}$ of projectors $\mathrm{P}^2 = \mathrm{P} : \mathcal{H} \to \mathcal{H}$ called a *projector chain* if for any pair $\mathrm{P}_1, \mathrm{P}_2 \in \mathcal{P}$, $\mathrm{P}_1 \neq \mathrm{P}_2$, one has either $\mathrm{P}_1 < \mathrm{P}_2$ or $\mathrm{P}_2 < \mathrm{P}_1$, and $\mathrm{P}_1\mathrm{P}_2 = \min(\mathrm{P}_1, \mathrm{P}_2)$. The ordering $\mathrm{P}_1 < \mathrm{P}_2$ means, as usual, that $\mathrm{P}_1\mathcal{H} \subset \mathrm{P}_2\mathcal{H}$, $\mathrm{P}_1\mathcal{H} \neq \mathrm{P}_2\mathcal{H}$. If $\mathrm{P}_1\mathcal{H} \subset \mathrm{P}_2\mathcal{H}$, then one writes $\mathrm{P}_1 \leq \mathrm{P}_2$. The closure $\overline{\mathcal{P}}$ of a chain $\mathcal{P}$ is set of all operators that are weak limits of sequences from $\mathcal{P}$. The inclusion relation $\mathcal{P}_1 \subset \mathcal{P}_2$ of any two sets of projector chains is obviously transitive allowing to consider the set of all projector chains as a partially ordered set. A chain $\mathcal{P}$ is called *maximal* if it cannot be extended. It is evident that a maximal chain is closed and contains zero $0 \in \mathcal{P}$ and unity $1 \in \mathcal{P}$ operators. A pair of projectors $(\mathrm{P}^-, \mathrm{P}^+) \subset \mathcal{P}$ is called a *break* of the chain $\mathcal{P}$ if $\mathrm{P}_- < \mathrm{P}_+$ and for all $\mathrm{P} \in \mathcal{P}$ either $\mathrm{P} < \mathrm{P}^-$ or $\mathrm{P}^+ < \mathrm{P}$. A closed chain is called *continuous* if for any pair of projectors $\mathrm{P}_1, \mathrm{P}_2 \subset \mathcal{P}$ there exists a projector $\mathrm{P} \in \mathcal{P}$ such that $\mathrm{P}_1 < \mathrm{P} < \mathrm{P}_2$. A maximal chain $\mathcal{P}$ will be called *complete* if it is continuous. An increasing (with respect to inclusion) projector valued function $\mathrm{P} : \mathrm{Q} \ni \Delta \to \mathcal{P}$ is called a *parametrization* of a chain $\mathcal{P}$, if the chain $\mathcal{P} = \mathrm{Im}(\mathrm{P} : \mathrm{Q} \ni \Delta \to \mathcal{P})$. A parametrization of the self-adjoint chain $\mathcal{P}$ is *smooth* if for any $u \in \mathcal{H}$, the positive valued measure $\Delta \to (u, \mathrm{P}(\Delta)u)$ is absolutely continuous.

It is well known [121, 145, 272] that every complete projector chain has a smooth parametrization. In what follows, a projector chain $\mathcal{P}$ will be self-adjoint, complete and endowed with a fixed smooth parametrization with respect to an operator valued function $\mathrm{F} : \mathcal{P} \to \mathcal{B}(\mathcal{H})$, and expressions like $\int_{\mathcal{P}} \mathrm{F}(\mathrm{P})\mathrm{dP}$ and $\int_{\mathcal{P}} \mathrm{dPF}(\mathrm{P})$ will be used for the corresponding [145] Riemann–Stieltjes integrals with respect to the corresponding projector chain. Now, consider a linear compact operator $\mathrm{K} \in \mathcal{B}_\infty(\mathcal{H})$ acting in a separable Hilbert space $\mathcal{H}$ endowed with a projector chain $\mathcal{P}$. A chain $\mathcal{P}$ is called *proper* with respect to an operator $\mathrm{K} \in \mathcal{B}_\infty(\mathcal{H})$ if $\mathrm{PKP} = \mathrm{KP}$ for any projector $\mathrm{P} \in \mathcal{P}$, meaning obviously that subspace $\mathrm{P}\mathcal{H}$ is invariant with respect to the operator $\mathrm{K} = \mathcal{B}_\infty(\mathcal{H})$ for any $\mathrm{P} \in \mathcal{P}$. As before, we denote by $\sigma(\mathrm{K})$ the spectrum of an operator $\mathrm{K} \in \mathcal{L}(\mathcal{H})$.

Definition 10.5. An operator $\mathrm{K} \in \mathcal{B}_\infty(\mathcal{H})$ is called *Volterra* if $\sigma(\mathrm{K}) = \{0\}$.

As shown in [145], a Volterra operator $\mathrm{K} \in \mathcal{B}_\infty(\mathcal{H})$ possesses a maximal proper projector chain $\mathcal{P}$ such that for any break $(\mathrm{P}^-, \mathrm{P}^-)$ the following relationship

$$(\mathrm{P}^+ - \mathrm{P}^-)\ \mathrm{K}\ (\mathrm{P}^+ - \mathrm{P}^-) = 0 \tag{10.58}$$

holds. Since the integral operators (10.37) are of Volterra type and congruent to a pair $(L, \widetilde{L})$ of closeable differential operators in $\mathcal{H}$, we will be interested in their properties with respect to the definition given above and to the corresponding proper maximal projector chains $\mathcal{P}(\mathbf{\Omega})$.

Suppose we have a Fredholm operator $\mathbf{\Omega} \in \mathcal{B}(\mathcal{H}) \cap Aut(\mathcal{H})$ self-congruent to a closeable differential operator $L \in \mathcal{L}(\mathcal{H})$. As we also have an elementary kernel (10.15) in the spectral form (10.16), our present task will be a description of elementary kernels $\hat{\mathrm{Z}}_\lambda$, $\lambda \in \sigma(L) \cap \bar{\sigma}(L^*)$, by means of a suitable smooth and complete parametrization. To treat this problem, we make use of very interesting recent results obtained in [272] that are devoted to the factorization of Fredholm operators. This work contains some aspects of our factorization problem for Delsarte transmutation operators $\mathbf{\Omega} \in (\mathcal{L}(\mathcal{H}))$ in the form (10.37).

We now formulate some preliminary results from [145, 272] for the problem at hand. As before, we denote by $\mathcal{B}(\mathcal{H})$ the Banach algebra of all linear continuous operators in $\mathcal{H}$, and also by $\mathcal{B}_\infty(\mathcal{H})$ the Banach algebra of all compact operators from $\mathcal{B}(\mathcal{H})$ and by $\mathcal{B}_0(\mathcal{H})$ the subspace of all finite-dimensional operators in $\mathcal{B}_\infty(\mathcal{H})$. Define

$$\begin{aligned} \mathcal{B}^-(\mathcal{H}) &= \{\mathrm{K} \in \mathcal{B}(\mathcal{H}) : (1-\mathrm{P})\mathrm{KP} = 0,\ \mathrm{P} \in \mathcal{P}\}, \\ \mathcal{B}^+(\mathcal{H}) &= \{\mathrm{K} \in \mathcal{B}(\mathcal{H}) : \mathrm{PK}(1-\mathrm{P}) = 0,\ \mathrm{P} \in \mathcal{P}\}, \end{aligned} \tag{10.59}$$

and call an operator $\mathrm{K} \in \mathcal{B}^+$, $(\mathrm{K} \in \mathcal{B}^-)$ *up-triangle* (respectively *down-triangle*) with respect to the projector chain $\mathcal{P}$. Denote also by $\mathcal{B}_p(\mathcal{H})$, $p \in [1,\infty]$, the so-called Neumann–Shattin ideals and set

$$\mathcal{B}_\infty^+(\mathcal{H}) := \mathcal{B}_\infty(\mathcal{H}) \cap \mathcal{B}^+(\mathcal{H}), \quad \mathcal{B}_\infty^-(\mathcal{H}) := \mathcal{B}_\infty(\mathcal{H}) \cap \mathcal{B}^-(\mathcal{H}). \tag{10.60}$$

According to Definition 10.5, the Banach subspaces (10.60) are Volterra, since they are closed in $\mathcal{B}_\infty(\mathcal{H})$ and satisfy

$$\mathcal{B}_\infty^+(\mathcal{H}) \cap \mathcal{B}_\infty^-(\mathcal{H}) = \varnothing. \tag{10.61}$$

Denote by $\mathcal{P}^+$ $(\mathcal{P}^-)$ the corresponding projectors of the linear space

$$\widetilde{\mathcal{B}}_\infty(\mathcal{H}) := \mathcal{B}_\infty^+(\mathcal{H}) \oplus \mathcal{B}_\infty^-(\mathcal{H}) \subset \mathcal{B}_\infty(\mathcal{H})$$

in $\mathcal{B}_\infty^+(\mathcal{H})$ $(\mathcal{B}_\infty^-(\mathcal{H}))$, and call them after [145] *transformators* of a *triangle shear*. The transformators $\mathcal{P}^+$ and $\mathcal{P}^-$ are known [145] to be continuous operators in the ideals $\mathcal{B}_p(\mathcal{H})$, $p \in [1,\infty]$. It follows from these definitions that

$$\mathcal{P}^+(\Phi) + \mathcal{P}^-(\Phi) = \Phi, \quad \mathcal{P}^\pm(\Phi) = \tau\mathcal{P}^\mp\tau(\Phi) \tag{10.62}$$

for any $\Phi \in \mathcal{B}(\mathcal{H})$, where $\tau : \mathcal{B}_p(\mathcal{H}) \rightarrow \mathcal{B}_p(\mathcal{H})$ is the standard involution in $\mathcal{B}_p(\mathcal{H})$ acting as $\tau(\Phi) := \Phi^*$.

Remark 10.2. It is clear and important that the transformators $\mathcal{P}^+$ and $\mathcal{P}^-$ strongly depend on a fixed projector chain $\mathcal{P}$.

Now define

$$\mathcal{V}_f^\pm := \{1 + \mathrm{K}_\pm : \mathrm{K}_\pm \in \mathcal{B}_\infty^\pm(\mathcal{H})\} \tag{10.63}$$

and

$$\mathcal{V}_f := \{\boldsymbol{\Omega}_+^{-1} \cdot \boldsymbol{\Omega}_- : \boldsymbol{\Omega}_\pm \in \mathcal{V}_f^\pm\}. \tag{10.64}$$

It is easy to check that $\mathcal{V}_f^+$ and $\mathcal{V}_f^-$ are subgroups of invertible operators from $\mathcal{L}(\mathcal{H})$ and, moreover, $\mathcal{V}_f^+ \cap \mathcal{V}_f^- = \{1\}$. Consider also the two operator sets

$$\begin{aligned} \mathcal{W} &:= \{\Phi \in \mathcal{B}_\infty(\mathcal{H}) :\ \mathrm{Ker}(1 + \mathrm{P}\Phi\mathrm{P}) = \{0\},\quad \mathrm{P} \in \mathcal{P}\}, \\ \mathcal{W}_f &:= \{\Phi \in \mathcal{B}_\infty(\mathcal{H}) :\quad \boldsymbol{\Omega} := 1 + \Phi \in \mathcal{V}_f\}, \end{aligned} \tag{10.65}$$

which are characterized by the following (see [145, 272]) theorem.

Theorem 10.6 (Gokhberg and Krein [145]**).** *The following conditions hold: i) $\mathcal{W}_f \subset \mathcal{W}$; ii) $\mathcal{B}_\omega(\mathcal{H}) \cap \mathcal{W} \subset \mathcal{W}_f$ where $\mathcal{B}_\omega(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ is the so-called Macaev ideal; iii) for any $\Phi \in \mathcal{W}_f$ it is necessary and sufficient that at least one of the integrals*

$$\mathcal{K}_+(\mathbf{\Omega}) = -\int_{\mathcal{P}} \mathrm{dP\Phi P}(1 + \mathrm{P\Phi P})^{-1}, \ \textit{or}$$

$$(1 + \mathrm{K}_-(\mathbf{\Omega}))^{-1} = 1 - \int_{\mathcal{P}} (1 + \mathrm{P\Phi P})^{-1}\mathrm{P\Phi dP} \tag{10.66}$$

is convergent in the uniform operator topology, and, moreover, if one integral of (10.49) is convergent then the other one is convergent too; and iv) the factorization representation

$$\mathbf{\Omega} = 1 + \Phi = (1 + \mathrm{K}_+(\mathbf{\Omega}))^{-1}(1 + \mathrm{K}_-(\mathbf{\Omega})) \tag{10.67}$$

for $\Phi \in \mathcal{W}_f$ is satisfied.

This theorem is too general since it does not take into account the crucial relationship (10.47) connecting the operators representation (10.67) with a given differential operator $L \in \mathcal{L}(\mathcal{H})$. Thus, it is necessary to satisfy the condition (10.47). It follows from (10.26) and (10.41) that if this condition is satisfied, the following crucial equalities

$$(1+\mathrm{K}_+(\mathbf{\Omega}))L(1+\mathrm{K}_+(\mathbf{\Omega}))^{-1} = \tilde{L} = (1+\mathrm{K}_-(\mathbf{\Omega}))L(1+\mathrm{K}_-(\mathbf{\Omega}))^{-1} \tag{10.68}$$

in $\mathcal{H}$ and the corresponding congruence relationships

$$(\tilde{L}_{ext} \otimes 1)\widehat{\mathrm{K}}_\pm = (1 \otimes L^*_{ext})\widehat{\mathrm{K}}_\pm \tag{10.69}$$

in $\mathcal{H}_+ \otimes \mathcal{H}_+$ hold. Here by $\hat{\mathrm{K}}_\pm \in \mathcal{H}_- \otimes \mathcal{H}_-$ we mean the corresponding kernels of Volterra operators $\mathrm{K}_\pm(\mathbf{\Omega}) \in \mathcal{B}^\pm_\infty(\mathcal{H})$. Since the factorization (10.67) is unique, the corresponding kernels satisfy the conditions (10.68) and (10.69). Accordingly the self-similar congruence condition must be solved with respect to a kernel $\hat{\Phi} \in \mathcal{H}_- \otimes \mathcal{H}_-$ corresponding to the integral operator $\Phi \in \mathcal{B}_\infty(\mathcal{H})$, and then the corresponding unique factorization (10.67) must be found so that it satisfies conditions (10.68) and (10.69).

Toward this end, we first define a unique positive Borel finite measure on the Borel subsets $\Delta \subset \mathrm{Q}$ of the open set $\mathrm{Q} \subset \mathbb{R}^m$, which satisfies the following condition for any projector $\mathrm{P}_x \in \mathcal{P}_x$ of a chain $\mathcal{P}_x$ marked by a running point $x \in \mathrm{Q}$:

$$(u, \mathrm{P}_x(\Delta)v)_{\mathcal{H}} = \int_{\Delta \subset Q} (u, \mathcal{X}_x(y)v)\mathrm{d}\mu_{\mathcal{P}_x}(y) \tag{10.70}$$

for all $u, v \in \mathcal{H}_+$, where $\mathcal{X}_x : \mathrm{Q} \rightarrow \mathcal{B}_2(\mathcal{H}_+, \mathcal{H}_-)$ is for any $x \in \mathrm{Q}$ a measurable (with respect to some Borel measure $\mu_{\mathcal{P}_x}$ on Borel subsets of Q) operator-valued mapping of Hilbert–Schmidt type. The representation (10.70) follows due to the reasoning similar to that in [28], which is based on the standard Radon–Nikodym theorem [28, 115]. Thus, in the weak sense

$$\mathrm{P}_x(\Delta) = \int_\Delta \mathcal{X}_x(y) \mathrm{d}\mu_{\mathcal{P}_x}(y) \tag{10.71}$$

for any Borel set $\Delta \in \mathrm{Q}$ and a running point $x \in \mathrm{Q}$. Thus, using the weak representation (10.71), we readily see that an integral expression like $\mathrm{I}_{f,g}(x) = \int_{\mathcal{P}_x} f(\mathrm{P}_x) \mathrm{dP}_x g(\mathrm{P}_x)$, $x \in \mathrm{Q}$, for any continuous mappings $f, g : \mathcal{P}_x \rightarrow \mathcal{B}(\mathcal{H})$ can be represented as

$$\mathrm{I}_{f,g}(x) = \int_\mathrm{Q} f(\mathrm{P}(y)) \chi_x(y) g(\mathrm{P}(y)) \mathrm{d}\mu_{\mathcal{P}_x}(y). \tag{10.72}$$

Consequently, for the Volterra operators (10.66) one has

$$\begin{gathered} \mathrm{K}_{+,x}(\boldsymbol{\Omega}) = -\int_\mathrm{Q} (1 + \mathrm{P}_x(y) \Phi \mathrm{P}_x(y))^{-1} \mathrm{P}_x(y) \Phi \mathrm{d}\mu_{\mathcal{P}_{x,+}}(y), \\ (1 + \mathrm{K}_{+,x}(\boldsymbol{\Omega}))^{-1} = 1 - \int_\mathrm{Q} \mathrm{d}\mu_{\mathcal{P}_{x,+}}(y) \Phi \mathrm{P}_x(y) (1 + \mathrm{P}_x(y) \Phi \mathrm{P}_x(y))^{-1} \end{gathered} \tag{10.73}$$

for some Borel measure $\mu_{\mathcal{P}_{x,+}}$ on Q and a given operator $\Phi \in \mathcal{B}_\infty(\mathcal{H})$. The first expression of (10.73) can be written for the corresponding kernels $\hat{\mathrm{K}}_{+,x}(y) \in \mathcal{H}_- \otimes \mathcal{H}_-$ as

$$\hat{\mathrm{K}}_{+,x}(y) = -\int_{\sigma(L) \cap \bar{\sigma}(L^*)} \mathrm{d}\rho_{\sigma,+}(\lambda) \tilde{\psi}_\lambda(x) \otimes \varphi_\lambda(y), \tag{10.74}$$

where, in view of the representation (10.46) and Theorem 10.2, we define the following convolution of two kernels for any running points x, y and $x' \in \mathrm{Q}$:

$$((1 + \mathrm{P}_x(x') \Phi \mathrm{P}_x(x'))^{-1}) * (\psi_\lambda(x') \otimes \varphi_\lambda(y)) := \tilde{\psi}_\lambda(x) \otimes \varphi_\lambda(y) \tag{10.75}$$

for $\lambda \in \sigma(\tilde{L}) \cap \bar{\sigma}(L^*)$ and some $\tilde{\psi}_\lambda \in \mathcal{H}_-$. By virtue of the representation (10.32) at $s =$ "+", from (10.74) one easily concludes that the elementary congruent kernel

$$\hat{\hat{\mathrm{Z}}}_\lambda = \tilde{\psi}_\lambda \otimes \varphi_\lambda \tag{10.76}$$

satisfies the conditions $(\tilde{L}_{ext} \otimes 1) \hat{\hat{\mathrm{Z}}}_\lambda = \lambda \hat{\hat{\mathrm{Z}}}_\lambda$ and $(1 \otimes L^*) \hat{\hat{\mathrm{Z}}}_\lambda = \lambda \hat{\hat{\mathrm{Z}}}_\lambda$ for any $\lambda \in \sigma(\tilde{L}) \cap \bar{\sigma}(L^*)$. Now for the operator $\mathrm{K}_+(\boldsymbol{\Omega}) \in \mathcal{B}_\infty^+(\mathcal{H})$, one finds the following integral representation

$$\mathrm{K}_+(\boldsymbol{\Omega}) = -\int_{S_{+,x}^{(m)}} \mathrm{d}y \int_{\sigma(\tilde{L}) \cap \bar{\sigma}(L^*)} \mathrm{d}\rho_{\sigma,+}(\lambda) \tilde{\psi}_\lambda(x) \bar{\varphi}_\lambda^\intercal(y)(\cdot), \tag{10.77}$$

which evidently satisfies the congruence condition (10.26), where we set

$$d\mu_{\mathcal{P}_{x,+}}(y) = \chi_{S^{(m)}_{+,x}} dy, \tag{10.78}$$

with $\chi_{S^{(m)}_{+,x}}$ being the characteristic function of the support of the measure $d\mu_{\mathcal{P}_x}$, that is supp $\mu_{\mathcal{P}_{x,+}} := S^{(m)}_{+,x} \in \mathcal{K}(\mathrm{Q})$. Analogous reasoning can be applied to describe the structure of the second factor operator $\mathrm{K}_-(\mathbf{\Omega}) \in \mathcal{B}^-_\infty(\mathcal{H})$:

$$\mathrm{K}_-(\mathbf{\Omega}) = -\int_{S^{(m)}_{-,x}} dy \int_{\sigma(\widetilde{L})\cap\bar{\sigma}(L^*)} d\rho_{\sigma,-}(\lambda)\tilde{\psi}_\lambda(x)\bar{\varphi}^\intercal_\lambda(y)(\cdot), \tag{10.79}$$

where, $S^{(m)}_{-,x} \subset \mathrm{Q}$, $x \in \mathrm{Q}$, is, as before, the support supp$\mu_{\mathcal{P}_{x,-}} := S^{(m)}_{-,x} \in \mathcal{K}(\mathrm{Q})$ of the finite Borel measure $\mu_{\mathcal{P}_{x,-}}$ corresponding to the operator (10.79) and defined on the Borel subsets of $\mathrm{Q} \subset \mathbb{R}^m$.

It is natural now to set $x \in \partial S^{(m)}_{+,x} \cap \partial S^{(m)}_{-,x}$ as an intrinsic point of the boundary $\partial S^{(m)}_{+,x} \backslash \partial Q = -\partial S^{(m)}_{-,x} \backslash \partial Q := \sigma^{(m-1)}_x \in \mathcal{K}(\mathrm{Q})$, where $\mathcal{K}(\mathrm{Q})$ is, as before, a singular simplicial complex generated by the open set $\mathrm{Q} \subset \mathbb{R}^m$. Thus, for our Fredholm operator $\mathbf{\Omega} := 1 + \Phi \in \mathrm{V}_f$ the corresponding factorization is

$$\mathbf{\Omega} = (1 + \mathrm{K}_+(\mathbf{\Omega}))^{-1}(1 + \mathrm{K}_-(\mathbf{\Omega})) := \mathbf{\Omega}^{-1}_+\mathbf{\Omega}_-, \tag{10.80}$$

where the integral operators $\mathrm{K}_\pm(\mathbf{\Omega}) \in \mathcal{B}^\pm_\infty(\mathcal{H})$ are given by expression (10.77) and (10.79) parametrized by a running intrinsic point $x \in \mathrm{Q}$.

10.4 Differential-geometric structure of the Lagrangian identity and related Delsarte transmutation operators

In the preceding section, we studied the spectral structure of Delsarte transmutation Volterra operators $\mathbf{\Omega}_\pm \in \mathcal{L}(\mathcal{H})$ factorizing a Fredholm operator $\mathbf{\Omega} = \mathbf{\Omega}^{-1}_+\mathbf{\Omega}_-$ and determined their relationships with the approach suggested in [145, 272]. In particular, we demonstrated the existence of a Borel measures $\mu_{\mathcal{P}_{x,\pm}}$ localized upon hypersurfaces $S^{(m)}_{\pm,x} \in \mathcal{K}(\mathrm{Q})$ and naturally related to the corresponding integral operators $\mathrm{K}_\pm(\mathbf{\Omega})$, whose kernels $\hat{\mathrm{K}}_\pm(\mathbf{\Omega}) \in \mathcal{H}_- \otimes \mathcal{H}_-$ are congruent to a pair of given differential operators $(L, \tilde{L}) \subset \mathcal{L}(\mathcal{H})$ satisfying the relationships (10.37). In what follows, we shall study some differential-geometric properties of the Lagrange identity associated with two Delsarte related differential operators L and $\tilde{L}$ in $\mathcal{H}$ and describe by means of specially constructed integral operator kernels the corresponding Delsarte transmutation operators exactly in the same spectral form as above.

Let a multi-dimensional linear differential operator $\mathrm{L} : \mathcal{H} \to \mathcal{H}$ of order $n(\mathrm{L}) \in \mathbb{Z}_+$ have the form

$$\mathrm{L}(x|\partial) := \sum_{|\alpha|=0}^{n(\mathrm{L})} a_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}, \tag{10.81}$$

and be defined on a dense domain $D(\mathrm{L}) \subset \mathcal{H}$, where, as usual, $\alpha \in \mathbb{Z}_+^m$ is a multi-index, $x \in \mathbb{R}^m$, and for brevity we assume that the coefficients $a_\alpha \in \mathcal{S}(\mathbb{R}^m; End\mathbb{C}^N)$, $\alpha \in \mathbb{Z}_+^m$. Consider the following easily derivable generalized Lagrangian identity for the differential expression (10.81):

$$< \mathrm{L}^*\varphi, \psi > - < \varphi, \mathrm{L}\psi >= \sum_{i=1}^{m} (-1)^{i+1} \frac{\partial}{\partial x_i} Z_i[\varphi, \psi], \tag{10.82}$$

where $(\varphi, \psi) \in \mathcal{H}^* \times \mathcal{H}$, the maps $Z_i : \mathcal{H}^* \times \mathcal{H} \to \mathbb{C}$, $1 \le i \le m$, are semilinear by construction and $\mathrm{L}^* : \mathcal{H}^* \to \mathcal{H}^*$ is the differential expression that is correspondingly formally conjugated to (10.81):

$$\mathrm{L}^*(x|\partial) := \sum_{|\alpha|=0}^{n(\mathcal{L})} (-1)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x^\alpha} \cdot \bar{a}_\alpha^\intercal(x).$$

By multiplying the identity (10.82) by the usual oriented Lebesgue measure $dx = \wedge_{j=\overrightarrow{1,m}} dx_j$, we obtain

$$< \mathrm{L}^*\varphi, \psi > dx - < \varphi, \mathrm{L}\psi > dx = dZ^{(m-1)}[\varphi, \psi] \tag{10.83}$$

for all $(\varphi, \psi) \in \mathcal{H}^* \times \mathcal{H}$, where

$$Z^{(m-1)}[\varphi, \psi] := \sum_{i=1}^{m} dx_1 \wedge dx_2 \wedge ... \wedge dx_{i-1} \wedge Z_i[\varphi, \psi] dx_{i+1} \wedge ... \wedge dx_m \tag{10.84}$$

is an $(m-1)$-differential form on $\mathbb{R}^m$.

Consider now all such pairs $(\varphi(\lambda), \psi(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}_0 \subset \mathcal{H}_- \times \mathcal{H}_-$, $\lambda, \mu \in \Sigma$, where as before

$$\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- \tag{10.85}$$

is the usual Gelfand triple of Hilbert spaces [28, 32] related to the Hilbert–Schmidt rigged Hilbert space $\mathcal{H}$. Here $\Sigma \in \mathbb{C}^p$, $p \in \mathbb{Z}_+$, is a fixed measurable space of parameters endowed with a finite Borel measure ρ, and the differential form (10.84) is exact; that is, there is a set of $(m-2)$-differential forms $\Omega^{(m-2)}[\varphi(\lambda), \psi(\mu)] \in \Lambda^{m-2}(\mathbb{R}^m; \mathbb{C})$, $\lambda, \mu \in \Sigma$, on $\mathbb{R}^m$ satisfying

$$Z^{(m-1)}[\varphi(\lambda), \psi(\mu)] = d\Omega^{(m-2)}[\varphi(\lambda), \psi(\mu)]. \tag{10.86}$$

A way to realize this condition is to take closed subspaces $\mathcal{H}_0^*$ and $\mathcal{H}_0 \subset \mathcal{H}_-$ as solutions to the corresponding linear differential equations under some boundary conditions:

$$\mathcal{H}_0 := \{\psi(\lambda) \in \mathcal{H}_- : \mathrm{L}\psi(\lambda) = 0, \quad \psi(\lambda)|_{x\in\Gamma} = 0, \ \lambda \in \Sigma\},$$
$$\mathcal{H}_0^* := \{\varphi(\lambda) \in \mathcal{H}_-^* : \mathrm{L}^*\varphi(\lambda) = 0, \quad \varphi(\lambda)|_{x\in\Gamma} = 0, \ \lambda \in \Sigma\}.$$

The triple (10.85) allows us to determine a set of generalized eigenfunctions for the extended operators L, $\mathrm{L}^* : \mathcal{H}_- \to \mathcal{H}_-$, if $\Gamma \subset \mathbb{R}^m$ is taken as an $(n-1)$-dimensional piecewise smooth hypersurface embedded in the configuration space $\mathbb{R}^m$. Obviously, there can exist situations [117, 233, 232] when boundary conditions are not necessary.

Now let $S_\pm(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)}) \in C_{m-1}(M;\mathbb{C})$ denote two non-intersecting $(m-1)$-dimensional piecewise smooth hypersurfaces from the singular simplicial chain group $C_{m-1}(M;\mathbb{C}) \subset \mathcal{K}(M)$ of a topological compactification $M := \bar{\mathbb{R}}^m$, such that their boundaries are the same, that is $\partial S_\pm(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)}) = \sigma_x^{(m-2)} - \sigma_{x_0}^{(m-2)}$. In addition, $\partial(S_+(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)}) \cup S_-(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)})) = \varnothing$, where $\sigma_x^{(m-2)}$ and $\sigma_{x_0}^{(m-2)} \in C_{m-2}(\mathbb{R}^m;\mathbb{C})$ are $(m-2)$-dimensional homologous cycles from a suitable chain complex $\mathcal{K}(M)$ parametrized (still formally) by means of two points $x, x_0 \in M$ and related to the chosen hypersurface $\Gamma \subset M$. Then from (10.86) and Stokes' theorem [96, 97, 144, 386, 393], one finds that

$$\begin{aligned}
&\int_{S_\pm(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)})} Z^{(m-1)}[\varphi(\lambda), \psi(\mu)] \\
&\quad = \int_{\partial S_\pm(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)})} \Omega^{(m-2)}[\varphi(\lambda), \psi(\mu)] \\
&= \int_{\sigma_x^{(m-2)}} \Omega^{(m-2)}[\varphi(\lambda), \psi(\mu)] - \int_{\sigma_{x_0}^{(m-2)}} \Omega^{(m-2)}[\varphi(\lambda), \psi(\mu)] \\
&:= \Omega_x(\lambda, \mu) - \Omega_{x_0}(\lambda, \mu),
\end{aligned}$$

$$\begin{aligned}
&\int_{S_\pm(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)})} \bar{Z}^{(m-1),\intercal}[\varphi(\lambda), \psi(\mu)] \\
&\quad = \int_{\partial S_\pm(\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)})} \bar{\Omega}^{(m-2),\intercal}[\varphi(\lambda), \psi(\mu)] \\
&= \int_{\sigma_x^{(m-2)}} \bar{\Omega}^{(m-2),\intercal}[\varphi(\lambda), \psi(\mu)] - \int_{\sigma_{x_0}^{(m-2)}} \bar{\Omega}^{(m-2),\intercal}[\varphi(\lambda), \psi(\mu)] \\
&:= \Omega_x^{\circledast}(\lambda, \mu) - \Omega_{x_0}^{\circledast}(\lambda, \mu)
\end{aligned}$$

for the set of functions $(\varphi(\lambda),\psi(\mu))\in\mathcal{H}_0^*\times\mathcal{H}_0,\ \lambda,\mu\in\Sigma$, with operator kernels $\Omega_x(\lambda,\mu)$, $\Omega_x^{\circledast}(\lambda,\mu)$ and $\Omega_{x_0}(\lambda,\mu)$, $\Omega_x^{\circledast}(\lambda,\mu)$, $\lambda,\mu\in\Sigma$, acting naturally on the Hilbert space $L^2_{(\rho)}(\Sigma;\mathbb{C})$. Moreover, these kernels are assumed to be nondegenerate in $L^2_{(\rho)}(\Sigma;\mathbb{C})$ and satisfying the homotopy conditions

$$\lim_{x\to x_0}\Omega_x(\lambda,\mu)=\Omega_{x_0}(\lambda,\mu),\quad \lim_{z\to x_0}\Omega_x^{\circledast}(\lambda,\mu)=\Omega_{x_0}^{\circledast}(\lambda,\mu).$$

Define now actions of the following two linear Delsarte permutation operators $\mathbf{\Omega}_\pm:\mathcal{H}\to\mathcal{H}$ and $\mathbf{\Omega}_\pm^{\circledast}:\mathcal{H}^*\to\mathcal{H}^*$, still upon a fixed set of functions $(\varphi(\lambda),\psi(\mu))\in\mathcal{H}_0^*\times\mathcal{H}_0,\ \lambda,\mu\in\Sigma$:

$$\begin{aligned}\tilde{\psi}(\lambda)&=\mathbf{\Omega}_\pm(\psi(\lambda)):=\int_\Sigma d\rho(\eta)\int_\Sigma d\rho(\mu)\psi(\eta)\Omega_x^{-1}(\eta,\mu)\Omega_{x_0}(\mu,\lambda),\\ \tilde{\varphi}(\lambda)&=\mathbf{\Omega}_\pm^{\circledast}(\varphi(\lambda)):=\int_\Sigma d\rho(\eta)\int_\Sigma d\rho(\mu)\varphi(\eta)\Omega_x^{\circledast,-1}(\mu,\eta)\Omega_{x_0}^{\circledast}(\lambda,\mu).\end{aligned}\tag{10.87}$$

Making use of the expressions (10.87), owing to the arbitrariness of the chosen set of functions $(\varphi(\lambda),\psi(\mu))\in\mathcal{H}_0^*\times\mathcal{H}_0,\ \lambda,\mu\in\Sigma$, we can easily find the operator expressions for $\mathbf{\Omega}_\pm$ and $\mathbf{\Omega}_\pm^{\circledast}:\mathcal{H}_-\to\mathcal{H}_-$, forcing the kernels $\Omega_{x_0}(\lambda,\mu)$ and $\Omega_{x_0}^{\circledast}(\lambda,\mu)$, $\lambda,\mu\in\Sigma$, to vary, namely

$$\begin{aligned}\tilde{\psi}(\lambda)&=\int_\Sigma d\rho(\eta)\int_\Sigma d\rho(\mu)\psi(\eta)\Omega_x(\eta,\mu)\Omega_x^{-1}(\mu,\lambda)\\ &-\int_\Sigma d\rho(\eta)\int_\Sigma d\rho(\mu)\psi(\eta)\Omega_x^{-1}(\eta,\mu)]\\ &\times\int_{S_\pm(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})}Z^{(m-1)}[\varphi(\mu),\psi(\lambda)])\\ &=\psi(\lambda)-\int_\Sigma d\rho(\eta)\int_\Sigma d\rho(\mu)\int_\Sigma d\rho(\nu)\int_\Sigma d\rho(\xi)\psi(\eta)\Omega_x^{-1}(\eta,\nu)\\ &\times\Omega_{x_0}(\nu,\xi)]\Omega_{x_0}^{-1}(\xi,\mu)\int_{S_\pm(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})}Z^{(m-1)}[\varphi(\mu),\psi(\lambda)]\\ &=\psi(\lambda)-\int_\Sigma d\rho(\eta)\int_\Sigma d\rho(\mu)\tilde{\psi}(\eta)\Omega_{x_0}^{-1}(\eta,\mu)]\\ &\times\int_{S_\pm(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})}Z^{(m-1)}[\varphi(\mu),\psi(\lambda)]\\ &=(\mathbf{1}-\int_\Sigma d\rho(\eta)\int_\Sigma d\rho(\mu)\tilde{\psi}(\eta)\Omega_{x_0}^{-1}(\eta,\mu)\\ &\times\int_{S_\pm(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})}Z^{(m-1)}[\varphi(\mu),(\cdot)])\ \psi(\lambda)\\ &:=\mathbf{\Omega}_\pm\cdot\psi(\lambda);\end{aligned}$$

$$
\begin{aligned}
\tilde{\varphi}(\lambda) &= \int_\Sigma d\rho(\eta) \int_\Sigma d\rho(\mu)\varphi(\eta)\Omega_x^{\circledast,-1}(\mu,\eta)\Omega_x^{\circledast}(\lambda,\mu) \\
&- \int_\Sigma d\rho(\eta) \int_\Sigma d\rho(\mu)\varphi(\eta)\Omega_x^{\circledast,-1}(\mu,\eta) \\
&\times \int_{S_\pm(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} \bar{Z}^{(m-1),\intercal}[\varphi(\lambda),\psi(\mu)] \\
&= \varphi(\lambda) - \int_\Sigma d\rho(\eta) \int_\Sigma d\rho(\nu) \int_\Sigma d\rho(\xi) \int_\Sigma d\rho(\mu)\varphi(\eta)\Omega_x^{\circledast,-1}(\xi,\eta) \\
&\times \Omega_{x_0}^{\circledast}(\nu,\xi)\Omega_{x_0}^{\circledast,-1}(\mu,\nu) \int_{S_\pm(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} \bar{Z}^{(m-1),\intercal}[\varphi(\lambda),\psi(\mu)] \\
&= (\mathbf{1} - \int_\Sigma d\rho(\eta) \int_\Sigma d\rho(\mu)\tilde{\varphi}(\eta)\Omega_{x_0}^{\circledast,-1}(\mu,\eta)\times \\
&\times \int_{S_\pm(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} \bar{Z}^{(m-1),\intercal}[(\cdot),\psi(\mu)]) \ \varphi(\lambda) \\
&:= \mathbf{\Omega}_\pm^{\circledast} \cdot \varphi(\lambda),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{\Omega}_\pm &:= \mathbf{1} - \int_\Sigma d\rho(\eta) \int_\Sigma d\rho(\mu)\tilde{\psi}(\eta)\Omega_{x_0}^{-1}(\eta,\mu) \\
&\quad \times \int_{S_\pm(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} Z^{(m-1)}[\varphi(\mu),(\cdot)], \\
\mathbf{\Omega}_\pm^{\circledast} &:= \mathbf{1} - \int_\Sigma d\rho(\eta) \int_\Sigma d\rho(\mu)\tilde{\varphi}(\eta)\Omega_{x_0}^{\circledast,-1}(\mu,\eta) \\
&\quad \times \int_{S_\pm(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} \bar{Z}^{(m-1),\intercal}[(\cdot),\psi(\mu)]
\end{aligned} \tag{10.88}
$$

are Volterra multi-dimensional integral operators. Note that now the elements $(\varphi(\lambda),\psi(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}_0$ and $(\tilde{\varphi}(\lambda),\tilde{\psi}(\mu)) \in \tilde{\mathcal{H}}_0^* \times \tilde{\mathcal{H}}_0$, $\lambda,\mu \in \Sigma$, inside the operator expressions (10.88) are not arbitrary - they are fixed. Therefore, the operators (10.88) realize an extension of their actions (10.87) on a fixed pair of functions $(\varphi(\lambda),\psi(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}_0$, $\lambda,\mu \in \Sigma$, on the whole functional space $\mathcal{H}^* \times \mathcal{H}$.

Due to the symmetry of expressions (10.87) and (10.88) with respect to two sets of functions $(\varphi(\lambda),\psi(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}_0$ and $(\tilde{\varphi}(\lambda),\tilde{\psi}(\mu)) \in \tilde{\mathcal{H}}_0^* \times \tilde{\mathcal{H}}_0$, $\lambda,\mu \in \Sigma$, it is very easy to prove the following result.

Lemma 10.2. *The operators (10.88) are bounded and invertible Volterra* *type expressions in $\mathcal{H}^* \times \mathcal{H}$ whose inverse are given as follows:*

$$\begin{aligned}\mathbf{\Omega}_{\pm}^{-1} &:= \mathbf{1} - \int_{\Sigma} d\rho(\eta) \int_{\Sigma} d\rho(\mu)\psi(\eta)\tilde{\Omega}_{x_0}^{-1}(\eta,\mu) \\ &\times \int_{S_{\pm}(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} Z^{(m-1)}[\tilde{\varphi}(\mu),(\cdot)] \\ \mathbf{\Omega}_{\pm}^{\circledast,-1} &:= \mathbf{1} - \int_{\Sigma} d\rho(\eta) \int_{\Sigma} d\rho(\mu)\varphi(\eta)\Omega_{x_0}^{\circledast,-1}(\mu,\eta) \\ &\times \int_{S_{\pm}(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)})} \bar{Z}^{(m-1),\intercal}[(\cdot),\tilde{\psi}(\mu)] \end{aligned} \tag{10.89}$$

where the two sets of functions $(\varphi(\lambda),\psi(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}_0$ and $(\tilde{\varphi}(\lambda),\tilde{\psi}(\mu)) \in \tilde{\mathcal{H}}_0^* \times \tilde{\mathcal{H}}_0$, $\lambda,\mu \in \Sigma$, are taken as arbitrary but fixed.

For the expressions (10.89) to be compatible with mappings (10.87), the following must hold:

$$\begin{aligned}\psi(\lambda) = \mathbf{\Omega}_{\pm}^{-1} \cdot \tilde{\psi}(\lambda) = \int_{\Sigma} d\rho(\eta) \int_{\Sigma} d\rho(\mu)\tilde{\psi}(\eta)\tilde{\Omega}_x^{-1}(\eta,\mu)]\tilde{\Omega}_{x_0}(\mu,\lambda), \\ \varphi(\lambda) = \mathbf{\Omega}_{\pm}^{\circledast,-1} \cdot \tilde{\varphi}(\lambda) = \int_{\Sigma} d\rho(\eta) \int_{\Sigma} d\rho(\mu)\tilde{\varphi}(\eta)\tilde{\Omega}_x^{\circledast,-1}(\mu,\eta)\tilde{\Omega}_{x_0}^{\circledast}(\lambda,\mu),\end{aligned} \tag{10.90}$$

where for any two sets of functions $(\varphi(\lambda),\psi(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}_0$ and $(\tilde{\varphi}(\lambda),\tilde{\psi}(\mu)) \in \tilde{\mathcal{H}}_0^* \times \tilde{\mathcal{H}}_0$, $\lambda,\mu \in \Sigma$, one has

$$(< \tilde{\mathrm{L}}^*\tilde{\varphi}(\lambda),\tilde{\psi}(\mu) > - < \tilde{\varphi}(\lambda),\tilde{\mathrm{L}}\tilde{\psi}(\mu) >)dx = d(\tilde{Z}^{(m-1)}[\tilde{\varphi}(\lambda),\tilde{\psi}(\mu)]),$$

$$\tilde{Z}^{(m-1)}[\tilde{\varphi}(\lambda),\tilde{\psi}(\mu)] = d\tilde{\Omega}^{(m-2)}[\tilde{\varphi}(\lambda),\tilde{\psi}(\mu)] \tag{10.91}$$

when

$$\tilde{\mathrm{L}} := \mathbf{\Omega}_{\pm}\mathrm{L}\mathbf{\Omega}_{\pm}^{-1}, \quad \tilde{\mathrm{L}}^* := \mathbf{\Omega}_{\pm}^{\circledast}\mathrm{L}^*\mathbf{\Omega}_{\pm}^{\circledast,-1}. \tag{10.92}$$

Moreover, the expressions above for $\tilde{\mathrm{L}} : \mathcal{H} \to \mathcal{H}$ and $\tilde{\mathrm{L}}^* : \mathcal{H}^* \to \mathcal{H}^*$ do not depend on the choice of the indices of the operators $\mathbf{\Omega}_+$ or $\mathbf{\Omega}_-$ and are consequently differential. Since the last condition determines the Delsarte transmutation operators (10.89), we need to verify the following result.

Theorem 10.7. *The pair $(\tilde{\mathrm{L}}, \tilde{\mathrm{L}}^*)$ of operator expressions $\tilde{\mathrm{L}} := \mathbf{\Omega}_{\pm}\mathrm{L}\mathbf{\Omega}_{\pm}^{-1}$ and $\tilde{\mathrm{L}}^* := \mathbf{\Omega}_{\pm}^{\circledast}\mathrm{L}^*\mathbf{\Omega}_{\pm}^{\circledast,-1}$ acting in the space $\mathcal{H} \times \mathcal{H}^*$ is purely differential for any suitably chosen hypersurfaces $S_{\pm}(\sigma_x^{(m-2)},\sigma_{x_0}^{(m-2)}) \in C_{m-1}(M;\mathbb{C})$ from the homology group $C_{m-1}(M;\mathbb{C})$.*

Proof. To prove the theorem, it is necessary to show that the formal pseudo-differential expressions corresponding to operators $\tilde{\mathrm{L}}$ and $\tilde{\mathrm{L}}^*$ contain no integral elements. Making use of an idea in [284, 365], one can prove this using the following lemma.

Lemma 10.3. *A pseudo-differential operator* $\mathrm{L} : \mathcal{H} \to \mathcal{H}$ *is purely differential if and only if the following equality*

$$(h, (\mathrm{L}\frac{\partial^{|\alpha|}}{\partial x^\alpha})_+ f) = (h, \mathrm{L}_+ \frac{\partial^{|\alpha|}}{\partial x^\alpha} f) \tag{10.93}$$

holds for any $|\alpha| \in \mathbb{Z}_+$ *and all* $(h, f) \in \mathcal{H}^* \times \mathcal{H}$, *that is the condition (10.93) is equivalent to the equality* $\mathrm{L}_+ = \mathrm{L}$, *where, as usual, the symbol* $(\cdots)$ *means the purely differential part of the corresponding expression inside the parentheses.*

Using this lemma and exact expressions of operators (10.88), similar to the calculations done in [365], one shows right away that operators $\tilde{\mathrm{L}}$ and $\tilde{\mathrm{L}}^*$, depending respectively only both on the homological cycles $\sigma_x^{(m-2)}, \sigma_{x_0}^{(m-2)} \in C_{m-2}(M; \mathbb{C})$ from a simplicial chain complex $\mathcal{K}(M)$, marked by points $x, x_0 \in \mathbb{R}^m$, and on two sets of functions $(\varphi(\lambda), \psi(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}_0$ and $(\tilde{\varphi}(\lambda), \tilde{\psi}(\mu)) \in \tilde{\mathcal{H}}_0^* \times \tilde{\mathcal{H}}_0$, $\lambda, \mu \in \Sigma$, are purely differential, thereby completing the proof. $\square$

The differential-geometric construction suggested above can be nontrivially generalized for the case of $m \in \mathbb{Z}_+$ commuting differential operators in a Hilbert space $\mathcal{H}$ giving rise to a new look at the theory of Delsarte transmutation operators based on differential-geometric and topological de Rham–Hodge techniques. These aspects will be discussed in detail in the next two sections.

10.5 The general differential-geometric and topological structure of Delsarte transmutation operators: A generalized de Rham–Hodge theory

Here we explain the corresponding differential-geometric and topological nature of the spectral related results obtained above and generalize them to a set $\mathcal{L}$ of commuting differential operators that are Delsarte related with another commuting set $\tilde{\mathcal{L}}$ of differential operators in $\mathcal{H}$. These results are based on the generalized De Rham–Hodge theory [97, 96, 377, 386, 393] of special differential complexes, which gives rise to effective analytical expressions for the corresponding Delsarte transmutation Volterra type operators in a given Hilbert space $\mathcal{H}$. As a by-product, one obtains the integral operator structure of Delsarte transmutation operators for polynomial pencils of differential operators in $\mathcal{H}$, which has many applications both in the spectral theory of multi-dimensional operator pencils [152, 340, 342] and in soliton theory [121, 247, 284, 326, 406] for multi-dimensional integrable dynamical systems on functional manifolds - both of which are very important for diverse applications in modern mathematical physics.

Let $M := \bar{\mathbb{R}}^m$ denote as before a suitably compactified metric space of dimension $m = \dim M \in \mathbb{Z}_+$ (without boundary) and define a finite set $\mathcal{L}$ of smooth commuting linear differential operators

$$\mathrm{L}_j(x|\partial) := \sum_{|\alpha|=0}^{n(L_j)} a_\alpha^{(j)}(x)\partial^{|\alpha|}/\partial x^\alpha \tag{10.94}$$

with respect to $x \in M$, with sufficiently smooth coefficients $a_\alpha^{(j)} \in \mathcal{S}(M; End\,\mathbb{C}^N)$, $|\alpha| = 0, \ldots, n(\mathrm{L}_j)$, $n(\mathrm{L}_j) \in \mathbb{Z}_+$, $1 \le j \le m$, and acting in the Hilbert space $\mathcal{H} := L^2(M;\mathbb{C}^N)$. It is also assumed that the domains $D(\mathrm{L}_j) := D(\mathcal{L}) \subset \mathcal{H}, 1 \le j \le m$, are dense in $\mathcal{H}$.

Consider now a generalized external anti-derivation

$$d_{\mathcal{L}} : \Lambda(M;\mathcal{H}) \rightarrow \Lambda(M;\mathcal{H})$$

acting in the Grassmann algebra $\Lambda(M;\mathcal{H})$ as follows: for any $\beta^{(k)} \in \Lambda^k(M;\mathcal{H})$, $0 \le k \le m$,

$$d_{\mathcal{L}}\beta^{(k)} := \sum_{j=1}^{m} dx_j \wedge \mathrm{L}_j(x|\partial)\beta^{(k)} \in \Lambda^{k+1}(M;\mathcal{H}). \tag{10.95}$$

It is easy to see that the (10.95) in the case $\mathrm{L}_j(x|\partial) := \partial/\partial x_j$, $j = 1, \ldots, m$, coincides with the standard exterior differentiation $d = \sum_{j=1}^m dx_j \wedge \partial/\partial x_j$ on the Grassmann algebra $\Lambda(M;\mathcal{H})$. Making use of the operation (10.95) on $\Lambda(M;\mathcal{H})$, one can construct the following generalized de Rham cochain complex

$$\mathcal{H} \rightarrow \Lambda^0(M;\mathcal{H}) \xrightarrow{d_{\mathcal{L}}} \Lambda^1(M;\mathcal{H}) \xrightarrow{d_{\mathcal{L}}} \ldots \xrightarrow{d_{\mathcal{L}}} \Lambda^m(M;\mathcal{H}) \xrightarrow{d_{\mathcal{L}}} 0. \tag{10.96}$$

The following important property concerning the complex (10.96) follows easily from the equality $d_{\mathcal{L}}d_{\mathcal{L}} = 0$ implied by the commutation of the operators (10.94).

Lemma 10.4. *The cochain sequence (10.96) is a complex.*

Next, we follow the ideas developed in [96, 377]. A differential form $\beta \in \Lambda(M;\mathcal{H})$ is $d_{\mathcal{L}}$-*closed* if $d_{\mathcal{L}}\beta = 0$, and a form $\gamma \in \Lambda(M;\mathcal{H})$ will be called $d_{\mathcal{L}}$-*homologous to zero* if there exists on M a form $\omega \in \Lambda(M;\mathcal{H})$ such that $\gamma = d_{\mathcal{L}}\omega$.

We shall use the standard algebraic Hodge star-operation

$$\star : \Lambda^k(M;\mathcal{H}) \to \Lambda^{m-k}(M;\mathcal{H}), \tag{10.97}$$

$0 \leq k \leq m$, as follows [86, 96, 97, 393]: if $\beta \in \Lambda^k(M;\mathcal{H})$, then the form $\star\beta \in \Lambda^{m-k}(M;\mathcal{H})$ satisfies:

i) the $(m-k)$-dimensional volume $|\star\beta|$ of the form $\star\beta$ equals the k-dimensional volume $|\beta|$ of the form β;

ii) the m-dimensional measure $\bar{\beta}^\intercal \wedge \star\beta > 0$ for a given orientation on M.

Define also on the space $\Lambda(M;\mathcal{H})$ the following natural scalar product: for any $\beta, \gamma \in \Lambda^k(M;\mathcal{H})$, $k = 0, \ldots, m$,

$$(\beta, \gamma) := \int_M \bar{\beta}^\intercal \wedge \star\gamma. \tag{10.98}$$

Using the scalar product (10.98), we have the corresponding Hilbert space

$$\mathcal{H}_\Lambda(M) := \bigoplus_{k=0}^{m} \mathcal{H}_\Lambda^k(M).$$

Notice also here that the Hodge star $\star$-operation satisfies the following easily verifiable property: for any $\beta, \gamma \in \mathcal{H}_\Lambda^k(M)$, $0 \leq k \leq m$,

$$(\beta, \gamma) = (\star\beta, \star\gamma), \tag{10.99}$$

that is, the Hodge operation $\star : \mathcal{H}_\Lambda(M) \to \mathcal{H}_\Lambda(M)$ is an isometry and its standard adjoint with respect to the scalar product (10.98) operation satisfies the condition $(\star)' = (\star)^{-1}$.

Let $d'_\mathcal{L}$ denote the formal adjoint of the external weak differential operation $d_\mathcal{L} : \mathcal{H}_\Lambda(M) \to \mathcal{H}_\Lambda(M)$. Making use of the operations $d'_\mathcal{L}$ and $d_\mathcal{L}$ in $\mathcal{H}_\Lambda(M)$, one can define [86, 96, 393] the generalized Laplace–Hodge operator $\Delta_\mathcal{L} : \mathcal{H}_\Lambda(M) \to \mathcal{H}_\Lambda(M)$ as

$$\Delta_\mathcal{L} := d'_\mathcal{L} d_\mathcal{L} + d_\mathcal{L} d'_\mathcal{L}. \tag{10.100}$$

A form $\beta \in \mathcal{H}_\Lambda(M)$ satisfying the equality

$$\Delta_\mathcal{L}\beta = 0 \tag{10.101}$$

is called *harmonic*. One can readily verify that a harmonic form $\beta \in \mathcal{H}_\Lambda(M)$ also satisfies the following two adjoint conditions:

$$d'_\mathcal{L}\beta = 0, \quad d_\mathcal{L}\beta = 0, \tag{10.102}$$

implied by (10.100) and (10.102).

It is not hard to check that the following differential operation in $\mathcal{H}_\Lambda(M)$

$$d^*_\mathcal{L} := \star d'_\mathcal{L}(\star)^{-1} \tag{10.103}$$

also determines the usual [97, 144, 386] exterior anti-differential operation in $\mathcal{H}_\Lambda(M)$. The corresponding dual to (10.96) the cochain complex

$$\mathcal{H} \to \Lambda^0(M;\mathcal{H}) \overset{d^*_{\mathcal{L}}}{\to} \Lambda^1(M;\mathcal{H}) \overset{d^*_{\mathcal{L}}}{\to} ... \overset{d^*_{\mathcal{L}}}{\to} \Lambda^m(M;\mathcal{H}) \overset{d^*_{\mathcal{L}}}{\to} 0 \tag{10.104}$$

is evidently exact too, as $d^*_{\mathcal{L}} d^*_{\mathcal{L}} = 0$ owing to (10.103).

Let $\mathcal{H}^k_{\Lambda(\mathcal{L})}(M)$ denote the cohomology groups of $d_{\mathcal{L}}$-closed and $\mathcal{H}^k_{\Lambda(\mathcal{L}^*)}(M)$ the cohomology groups of $d^*_{\mathcal{L}}$-closed differential forms, respectively, and $\mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L})}(M)$ be the abelian groups of harmonic differential forms from the Hilbert subspaces $\mathcal{H}^k_\Lambda(M)$, $0 \le k \le m$. We also define the standard Hilbert–Schmidt rigged chain [28] of positive and negative Hilbert spaces of differential forms

$$\mathcal{H}^k_{\Lambda,+}(M) \subset \mathcal{H}^k_\Lambda(M) \subset \mathcal{H}^k_{\Lambda,-}(M), \tag{10.105}$$

the corresponding rigged chains of Hilbert subspaces for harmonic forms

$$\mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L}),+}(M) \subset \mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L})}(M) \subset \mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L}),-}(M) \tag{10.106}$$

and cohomology groups:

$$\begin{array}{l} \mathcal{H}^k_{\Lambda(\mathcal{L}),+}(M) \subset \mathcal{H}^k_{\Lambda(\mathcal{L})}(M) \subset \mathcal{H}^k_{\Lambda(\mathcal{L}),-}(M), \\ \mathcal{H}^k_{\Lambda(\mathcal{L}^*),+}(M) \subset \mathcal{H}^k_{\Lambda(\mathcal{L}^*)}(M) \subset \mathcal{H}^k_{\Lambda(\mathcal{L}^*),-}(M) \end{array} \tag{10.107}$$

for any $0 \le k \ge m$. Assume also that the Laplace–Hodge operator (10.100) is elliptic in $\mathcal{H}^0_\Lambda(M)$. Now reasoning as in [86, 97, 386, 393], one can prove the following generalized de Rham–Hodge theorem.

Theorem 10.8. *The groups of harmonic forms $\mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L}),-}(M)$ are, respectively, isomorphic to the cohomology groups $(H^k(M;\mathbb{C}))^{|\Sigma|}$, $0 \le k \le m$, where $H^k(M;\mathbb{C})$ is the k-th cohomology group of the manifold M with complex coefficients, $\Sigma \subset \mathbb{C}^p$, $p \in \mathbb{Z}_+$, $|\Sigma| :=$ card Σ, is a set of suitable "spectral" parameters marking the linear space of independent $d^*_{\mathcal{L}}$-closed 0-forms from $\mathcal{H}^0_{\Lambda(\mathcal{L}),-}(M)$ and the direct sum decompositions*

$$\begin{aligned} \mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L}),-}(M) \oplus \Delta\mathcal{H}^k_{\Lambda,-}(M) &= \mathcal{H}^k_{\Lambda,-}(M) \\ &= \mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L}),-}(M) \oplus d_{\mathcal{L}}\mathcal{H}^{k-1}_{\Lambda,-}(M) \oplus d'_{\mathcal{L}}\mathcal{H}^{k+1}_{\Lambda,-}(M) \end{aligned}$$

hold for any $k = 0, \ldots, m$.

Another variant of the above result was formulated in [377–380] and reads as the following generalized de Rham–Hodge theorem.

Theorem 10.9 (Skrypnik [377]). *The generalized cohomology groups $\mathcal{H}^k_{\Lambda(\mathcal{L}),-}(M)$ are isomorphic, respectively, to the cohomology groups $(H^k(M;\mathbb{C}))^{|\Sigma|}$, $0 \le k \le m$.*

A proof of this theorem is based on a special sequence [377] of differential Lagrange type identities. Define the following closed subspace

$$\mathcal{H}_0^* := \{\varphi^{(0)}(\lambda) \in \mathcal{H}^0_{\Lambda(\mathcal{L}^*),-}(M) : d^*_{\mathcal{L}}\varphi^{(0)}(\lambda) = 0,\ \varphi^{(0)}(\lambda)|_\Gamma = 0,\ \lambda \in \Sigma\} \tag{10.108}$$

for a smooth $(m-1)$-dimensional hypersurface $\Gamma \subset M$ and $\Sigma \subset (\sigma(\mathcal{L}) \cap \bar{\sigma}(\mathcal{L}^*)) \times \Sigma_\sigma \subset \mathbb{C}^p$, where $\mathcal{H}^0_{\Lambda(\mathcal{L}^*),-}(M)$ is, as above, a suitable Hilbert–Schmidt rigged [28, 32, 376] zeroth-order cohomology group space from the cochain given by (10.107), and $\sigma(\mathcal{L})$ and $\sigma(\mathcal{L}^*)$ are, respectively, mutual spectra of the sets of commuting operators $\mathcal{L}$ and $\mathcal{L}^*$ in $\mathcal{H}$. Thus, the dimension $\dim \mathcal{H}_0^* = card\ \Sigma$ is known. The next lemma is fundamental for a proof of the above theorem.

Lemma 10.5 (Skrypnik [377–380]**).** *There exists a set of differential* $(k+1)$*-forms* $Z^{(k+1)}[\varphi^{(0)}(\lambda), d_{\mathcal{L}}\psi^{(k)}] \in \Lambda^{k+1}(M;\mathbb{C})$ *and a set of* k*-forms* $Z^{(k)}[\varphi^{(0)}(\lambda), \psi^{(k)}] \in \Lambda^k(M;\mathbb{C})$, $0 \le k \le m$, *parametrized by a set* $\Sigma \ni \lambda$ *and semilinear in* $(\varphi^{(0)}(\lambda), \psi^{(k)}) \in \mathcal{H}_0^* \times \mathcal{H}^k_{\Lambda,-}(M)$, *such that*

$$Z^{(k+1)}[\varphi^{(0)}(\lambda), d_{\mathcal{L}}\psi^{(k)}] = dZ^{(k)}[\varphi^{(0)}(\lambda), \psi^{(k)}] \tag{10.109}$$

for all $k = 0, \ldots, m$ *and* $\lambda \in \Sigma$.

Proof. Consider the following generalized Lagrange type identity holding for any pair $(\varphi^{(0)}(\lambda), \psi^{(k)}) \in \mathcal{H}_0^* \times \mathcal{H}^k_{\Lambda,-}(M)$:

$$\begin{aligned}
0 &=< d^*_{\mathcal{L}}\varphi^{(0)}(\lambda), \star(\psi^{(k)} \wedge \bar{\gamma}) > \\
&:=< \star d'_{\mathcal{L}}(\star)^{-1}\varphi^{(0)}(\lambda), \star(\psi^{(k)} \wedge \bar{\gamma}) > \\
&=< d'_{\mathcal{L}}(\star)^{-1}\varphi^{(0)}(\lambda), \psi^{(k)} \wedge \bar{\gamma} >=< (\star)^{-1}\varphi^{(0)}(\lambda), d_{\mathcal{L}}\psi^{(k)} \wedge \bar{\gamma} > \\
&+ Z^{(k+1)}[\varphi^{(0)}(\lambda), d_{\mathcal{L}}\psi^{(k)}] \wedge \bar{\gamma} \\
&=< (\star)^{-1}\varphi^{(0)}(\lambda), d_{\mathcal{L}}\psi^{(k)} \wedge \bar{\gamma} > + dZ^{(k)}[\varphi^{(0)}(\lambda), \psi^{(k)}] \wedge \bar{\gamma},
\end{aligned} \tag{10.110}$$

where

$$Z^{(k+1)}[\varphi^{(0)}(\lambda), d_{\mathcal{L}}\psi^{(k)}] \in \Lambda^{k+1}(M;\mathbb{C}),\ k = 0, \ldots, m,$$

and

$$Z^{(k)}[\varphi^{(0)}(\lambda), \psi^{(k)}] \in \Lambda^{k-1}(M;\mathbb{C}),\ k = 0, \ldots, m,$$

are semilinear differential forms parametrized by a parameter $\lambda \in \Sigma$, and $\bar{\gamma} \in \Lambda^{m-k-1}(M;\mathbb{C})$ is an arbitrary constant $(m-k-1)$-form. Now the semilinear differential $(k+1)$-forms $Z^{(k+1)}[\varphi^{(0)}(\lambda), d_{\mathcal{L}}\psi^{(k)}] \in \Lambda^{k+1}(M;\mathbb{C})$ and k-forms $Z^{(k)}[\varphi^{(0)}(\lambda), \psi^{(k)}] \in \Lambda^k(M;\mathbb{C})$, $0 \le k \le m$, $\lambda \in \Sigma$, are forms of the desired type, so the proof is complete. $\square$

Using this, we can construct the cohomology group isomorphism in Theorem 10.3. Namely, following [239, 377–380], let us take a singular simplicial [97, 179, 386, 393] complex $\mathcal{K}(M)$ of the manifold M and introduce linear mappings $B_\lambda^{(k)} : \mathcal{H}_{\Lambda,-}^k(M) \to C_k(M;\mathbb{C})$, $\lambda \in \Sigma$, where $C_k(M;\mathbb{C})$, $0 \leq k \leq m$, are as before free abelian groups over the field $\mathbb{C}$ generated, respectively, by all k-chains of simplexes $S^{(k)} \in C_k(M;\mathbb{C})$, $k = 0, \ldots, m$, from the singular simplicial complex $\mathcal{K}(M)$, as follows:

$$B_\lambda^{(k)}(\psi^{(k)}) := \sum_{S^{(k)} \in C_k(M;\mathbb{C}))} S^{(k)} \int_{S^{(k)}} Z^{(k)}[\varphi^{(0)}(\lambda), \psi^{(k)}] \tag{10.111}$$

with $\psi^{(k)} \in \mathcal{H}_{\Lambda,-}^k(M)$, $0 \leq k \leq m$. Then, owing to the maps (10.111), we have the following result.

Theorem 10.10 (Skrypnik [377]). *The operations (10.111) parametrized by $\lambda \in \Sigma$ realize the cohomology group isomorphism formulated in Theorem 10.9.*

Proof. To verify this result, we pass over in (10.111) to the corresponding cohomology $\mathcal{H}_{\Lambda(\mathcal{L}),-}^k(M)$ and homology $H_k(M;\mathbb{C})$ groups of M. If we take an element $\psi^{(k)} := \psi^{(k)}(\mu) \in \mathcal{H}_{\Lambda(\mathcal{L}),-}^k(M)$, $0 \leq k \leq m$, solving the equation $d_{\mathcal{L}}\psi^{(k)}(\mu) = 0$ with $\mu \in \Sigma_k$ a set of the related "spectral" parameters marking elements of $\mathcal{H}_{\Lambda(\mathcal{L}),-}^k(M)$, then it follows directly from (10.111) and the identity (10.110) that

$$dZ^{(k)}[\varphi^{(0)}(\lambda), \psi^{(k)}(\mu)] = 0 \tag{10.112}$$

for all pairs $(\lambda, \mu) \in \Sigma \times \Sigma_k$, $k = 0, \ldots, m$. Hence, owing to the Poincaré lemma [97, 144, 386, 393], there exist differential $(k-1)$-forms $\Omega^{(k-1)}[\varphi^{(0)}(\lambda), \psi^{(k)}(\mu)] \in \Lambda^{k-1}(M;\mathbb{C})$ such that

$$Z^{(k)}[\varphi^{(0)}(\lambda), \psi^{(k)}(\mu)] = d\Omega^{(k-1)}[\varphi^{(0)}(\lambda), \psi^{(k)}(\mu)] \tag{10.113}$$

for all pairs $(\varphi^{(0)}(\lambda), \psi^{(k)}(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}_{\Lambda(\mathcal{L}),-}^k(M)$ parametrized by $(\lambda, \mu) \in \Sigma \times \Sigma_k$, $0 \leq k \leq m$. As a result of passing on the right-hand side of (10.111) to the homology groups $H_k(M;\mathbb{C})$, it follows from Stokes' theorem [97, 144, 386, 393] that the maps

$$B_\lambda^{(k)} : \mathcal{H}_{\Lambda(\mathcal{L}),-}^k(M) \rightleftarrows H_k(M;\mathbb{C}) \tag{10.114}$$

are isomorphisms for every $\lambda \in \Sigma$ and $\lambda \in \Sigma$. Making use of Poincaré duality [97, 179, 386, 393] between the homology groups $H_k(M;\mathbb{C})$ and the cohomology groups $H^k(M;\mathbb{C})$, $0 \leq k \leq m$, one obtains finally the desired, that is $\mathcal{H}_{\Lambda(\mathcal{L}),-}^k(M) \simeq (H^k(M;\mathbb{C}))^{|\Sigma|}$. □

Assume that $M := \mathrm{T}^r \times \bar{\mathbb{R}}^s$, $\dim M = s + r \in \mathbb{Z}_+$, and $\mathcal{H} := L^2(\mathrm{T}^r; L^2(\mathbb{R}^s; \mathbb{C}^N))$, where $\mathrm{T}^r := \prod_{j=1}^r \mathrm{T}_j$, $\mathrm{T}_j := [0, T_j) \subset \mathbb{R}_+$, and set

$$d_{\mathcal{L}} = \sum_{j=1}^{r} dt_j \wedge \mathrm{L}_j(t; x|\partial), \ \mathrm{L}_j(t; x|\partial) := \partial/\partial t_j - L_j(t; x|\partial), \tag{10.115}$$

with

$$L_j(t; x|\partial) = \sum_{|\alpha|=0}^{n(L_j)} a_\alpha^{(j)}(t; x)\partial^{|\alpha|}/\partial x^\alpha, \tag{10.116}$$

being differential operations depending parametrically on $t \in \mathrm{T}^r$ and defined on dense subspaces $D(\mathrm{L}_j) = D(\mathcal{L}) \subset L^2(\mathbb{R}^s; \mathbb{C}^N)$, $j = 1, \ldots, r$. It is assumed also that operators $\mathrm{L}_j : \mathcal{H} \to \mathcal{H}$ commute with one another.

Take now such a fixed pair $(\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx) \in \mathcal{H}_0^* \times \mathcal{H}^s_{\Lambda(\mathcal{L}),-}(M)$, parametrized by elements $(\lambda, \mu) \in \Sigma \times \Sigma$, for which, owing to both Theorem 10.10 and Stokes' theorem [97, 144, 179, 386, 393], the following equality

$$B_\lambda^{(s)}(\psi^{(0)}(\mu)dx) = S^{(s)}_{(t;x)} \int_{\partial S^{(s)}_{(t;x)}} \Omega^{(s-1)}[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx] \tag{10.117}$$

holds, where $S^{(s)}_{(t;x)} \in C_s(M; \mathbb{C})$ is a fixed element parametrized by an arbitrarily chosen point $(t; x) \in M \cap S^{(s)}_{(t;x)}$. Consider next the expressions

$$\begin{aligned} &\Omega_{(t;x)}(\lambda, \mu) := \textstyle\int_{\partial S^{(s)}_{(t;x)}} \Omega^{(s-1)}[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx], \\ &\Omega_{(t_0;x_0)}(\lambda, \mu) := \textstyle\int_{\partial S^{(s)}_{(t_0;x_0)}} \Omega^{(s-1)}[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx], \end{aligned} \tag{10.118}$$

with a point $(t_0; x_0) \in M \cap S^{(s)}_{(t_0;x_0)}$ fixed, $\lambda, \mu \in \Sigma$, and interpret them as the corresponding kernels [28] of the integral invertible operators of Hilbert–Schmidt type $\Omega_{(t;x)}, \Omega_{(t_0;x_0)} : L^2_{(\rho)}(\Sigma; \mathbb{C}) \to L^2_{(\rho)}(\Sigma; \mathbb{C})$, where ρ is a finite Borel measure on the parameter set Σ. It is also assumed that the boundaries $\partial S^{(s)}_{(t;x)} := \sigma^{(s-1)}_{(t;x)}$ and $\partial S^{(s)}_{(t_0;x_0)} := \sigma^{(s-1)}_{(t_0;x_0)}$ are homologous to each other as $(t; x) \to (t_0; x_0) \in M$. Now, motivated by the expression (10.117), define

$$\mathbf{\Omega}_\pm : \psi^{(0)}(\eta) \to \tilde{\psi}^{(0)}(\eta) \tag{10.119}$$

for $\psi^{(0)}(\eta)dx \in \mathcal{H}^s_{\Lambda(\mathcal{L}),-}(M)$, $\eta \in \Sigma$, and some $\tilde{\psi}^{(0)}(\eta)dx \in \mathcal{H}^s_{\Lambda,-}(M)$, where for any $\eta \in \Sigma$,

$$\begin{aligned} \tilde{\psi}^{(0)}(\eta) &:= \psi^{(0)}(\eta) \cdot \Omega^{-1}_{(t;x)} \Omega_{(t_0;x_0)} \\ &= \int_\Sigma d\rho(\mu) \int_\Sigma d\rho(\xi) \psi^{(0)}(\mu) \Omega^{-1}_{(t;x)}(\mu, \xi) \Omega_{(t_0;x_0)}(\xi, \eta). \end{aligned} \tag{10.120}$$

Suppose that the elements (10.120) are related to another Delsarte transformed cohomology group $\mathcal{H}^s_{\Lambda(\tilde{\mathcal{L}}),-}(M)$, so that the condition

$$d_{\tilde{\mathcal{L}}}\tilde{\psi}^{(0)}(\eta)dx = 0 \tag{10.121}$$

for $\tilde{\psi}^{(0)}(\eta)dx \in \mathcal{H}^s_{\Lambda(\tilde{\mathcal{L}}),-}(M)$, $\eta \in \Sigma$, and some new exterior anti-derivation operation in $\mathcal{H}_{\Lambda,-}(M)$

$$d_{\tilde{\mathcal{L}}} := \sum_{j=1}^{m} dx_j \wedge \tilde{\mathrm{L}}_j(t;x|\partial),\ \tilde{\mathrm{L}}_j(t;x|\partial) := \partial/\partial t_j - \tilde{L}_j(t;x|\partial) \tag{10.122}$$

hold, where

$$\tilde{L}_j(t;x|\partial) = \sum_{|\alpha|=0}^{n(L_j)} \tilde{a}^{(j)}_{\alpha}(t;x)\partial^{|\alpha|}/\partial x^{\alpha}, \tag{10.123}$$

$0 \le j \le r$, are differential operators in $L^2(\mathbb{R}^s;\mathbb{C}^N)$ depending parametrically on $t \in \mathrm{T}^r$.

Set

$$\tilde{\mathrm{L}}_j := \mathbf{\Omega}_{\pm}\mathrm{L}_j\mathbf{\Omega}_{\pm}^{-1} \tag{10.124}$$

for each $0 \le j \le r$, where $\mathbf{\Omega}_{\pm} : \mathcal{H} \to \mathcal{H}$ are the corresponding Delsarte transmutation operators related to elements $S_{\pm}(\sigma^{(s-1)}_{(x;t)}, \sigma^{(s-1)}_{(x_0;t_0)}) \in C_s(M;\mathbb{C})$, which in turn are related to the mutually homologous boundaries $\partial S^{(s)}_{(x;t)} = \sigma^{(s-1)}_{(x;t)}$ and $\partial S^{(s)}_{(x_0;t_0)} = \sigma^{(s-1)}_{(x_0;t_0)}$. Since all of the operators $\mathrm{L}_j : \mathcal{H} \to \mathcal{H}$, $0 \le j \le r$, commute, the same is true for the transformed operators (10.124), that is $[\tilde{\mathrm{L}}_j, \tilde{\mathrm{L}}_k] = 0$, $k, j = 0, \ldots, m$, which is obviously equivalent, owing to (10.124), to the general expression

$$d_{\tilde{\mathcal{L}}} = \mathbf{\Omega}_{\pm} d_{\mathcal{L}} \mathbf{\Omega}_{\pm}^{-1}. \tag{10.125}$$

In order to satisfy conditions (10.125) and (10.121), let us consider the following expressions corresponding to (10.117)

$$\tilde{B}^{(s)}_{\lambda}(\tilde{\psi}^{(0)}(\eta)dx) = S^{(s)}_{(t;x)}\tilde{\Omega}_{(t;x)}(\lambda,\eta), \tag{10.126}$$

related to the associated exterior differentiation (10.125), where $S^{(s)}_{(t;x)} \in C_s(M;\mathbb{C})$ and $(\lambda,\eta) \in \Sigma \times \Sigma$. Assume further that there are mappings

$$\mathbf{\Omega}^{\circledast}_{\pm} : \varphi^{(0)}(\lambda) \to \tilde{\varphi}^{(0)}(\lambda) \tag{10.127}$$

with $\mathbf{\Omega}^{\circledast}_{\pm} : \mathcal{H}^* \to \mathcal{H}^*$ being operators related (but not necessary adjoint) to the Delsarte transmutation operators $\mathbf{\Omega}_{\pm} : \mathcal{H} \to \mathcal{H}$ and satisfying the standard relationships $\tilde{\mathrm{L}}^*_j := \mathbf{\Omega}^{\circledast}_{\pm}\mathrm{L}^*_j\mathbf{\Omega}^{\circledast,-1}_{\pm}$, $0 \le j \le r$. The proper Delsarte type operators $\mathbf{\Omega}_\pm : \mathcal{H}^0_{\Lambda(\mathcal{L}),-}(M) \to \mathcal{H}^0_{\Lambda(\tilde{\mathcal{L}}),-}(M)$ are related with two different realizations of the action (10.120) under the necessary conditions

$$d_{\tilde{\mathcal{L}}}\tilde{\psi}^{(0)}(\eta)dx = 0, \quad d^*_{\tilde{\mathcal{L}}}\tilde{\varphi}^{(0)}(\lambda) = 0; \tag{10.128}$$

meaning that the embeddings $\tilde{\varphi}^{(0)}(\lambda) \in \mathcal{H}^0_{\Lambda(\tilde{\mathcal{L}}^*),-}(M)$, $\lambda \in \Sigma$, and $\tilde{\psi}^{(0)}(\eta)dx \in \mathcal{H}^s_{\Lambda(\tilde{\mathcal{L}}),-}(M)$, $\eta \in \Sigma$, are satisfied. Now we need to formulate a result that is being important for the validity of conditions (10.128).

Lemma 10.6. *The following invariance property*

$$\tilde{Z}^{(s)} = \Omega_{(t_0;x_0)}\Omega^{-1}_{(t;x)}Z^{(s)}\Omega^{-1}_{(t;x)}\Omega_{(t_0;x_0)} \tag{10.129}$$

holds for any $(t;x)$ and $(t_0;x_0) \in M$.

As a result of (10.129) and the symmetry invariance between cohomology spaces $\mathcal{H}^0_{\Lambda(\mathcal{L}),-}(M)$ and $\mathcal{H}^0_{\Lambda(\tilde{\mathcal{L}}),-}(M)$, one obtains the following pairs of related mappings:

$$\begin{aligned} \psi^{(0)} &= \tilde{\psi}^{(0)}\tilde{\Omega}^{-1}_{(t;x)}\tilde{\Omega}_{(t_0;x_0)}, \quad \varphi^{(0)} = \tilde{\varphi}^{(0)}\tilde{\Omega}^{\circledast,-1}_{(t;x)}\tilde{\Omega}^{\circledast}_{(t_0;x_0)}, \\ \tilde{\psi}^{(0)} &= \psi^{(0)}\Omega^{-1}_{(t;x)}\Omega_{(t_0;x_0)}, \quad \tilde{\varphi}^{(0)} = \varphi^{(0)}\Omega^{\circledast,-1}_{(t;x)}\Omega^{\circledast}_{(t_0;x_0)}, \end{aligned} \tag{10.130}$$

where the integral operator kernels from $L^2_{(\rho)}(\Sigma;\mathbb{C})\otimes L^2_{(\rho)}(\Sigma;\mathbb{C})$ are defined as

$$\begin{aligned} \tilde{\Omega}_{(t;x)}(\lambda,\mu) &:= \int_{\sigma^{(s)}_{(t;x)}} \tilde{\Omega}^{(s-2)}[\tilde{\varphi}^{(0)}(\lambda), \tilde{\psi}^{(0)}(\mu)dx], \\ \tilde{\Omega}^{\circledast}_{(t;x)}(\lambda,\mu) &:= \int_{\sigma^{(s)}_{(t;x)}} \bar{\tilde{\Omega}}^{(s-2),\intercal}[\tilde{\varphi}^{(0)}(\lambda), \tilde{\psi}^{(0)}(\mu)dx] \end{aligned} \tag{10.131}$$

for all $(\lambda,\mu) \in \Sigma\times\Sigma$, giving rise to finding proper Delsarte transmutation operators that ensure the pure differential nature of the transformed expressions (10.124).

Note also that by virtue of (10.129) and (10.130), the operator property

$$\Omega_{(t_0;x_0)}\Omega^{-1}_{(t;x)}\Omega_{(t_0;x_0)} + \tilde{\Omega}_{(t_0;x_0)}\Omega^{-1}_{(t;x)}\Omega_{(t_0;x_0)} = 0 \tag{10.132}$$

holds for any $(t_0;x_0)$ and $(t;x) \in M$; meaning that $\tilde{\Omega}_{(t_0;x_0)} = -\Omega_{(t_0;x_0)}$.

One can now define, analogously to (10.108), the following three subspaces that are closed and dense in $\mathcal{H}^0_{\Lambda,-}(M)$:

$$\begin{aligned} \mathcal{H}_0 &:= \{\psi^{(0)}(\mu) \in \mathcal{H}^0_{\Lambda(\mathcal{L}),-}(M) : d_{\mathcal{L}}\psi^{(0)}(\mu) = 0, \quad \psi^{(0)}(\mu)|_\Gamma = 0, \ \mu\in\Sigma\}, \\ \tilde{\mathcal{H}}_0 &:= \{\tilde{\psi}^{(0)}(\mu) \in \mathcal{H}^0_{\Lambda(\tilde{\mathcal{L}}),-}(M) : d_{\tilde{\mathcal{L}}}\tilde{\psi}^{(0)}(\mu) = 0, \quad \tilde{\psi}^{(0)}(\mu)|_{\tilde{\Gamma}} = 0, \ \mu\in\Sigma\}, \\ \tilde{\mathcal{H}}^*_0 &:= \{\tilde{\varphi}^{(0)}(\eta) \in \mathcal{H}^0_{\Lambda(\tilde{\mathcal{L}}^*),-}(M) : d^*_{\tilde{\mathcal{L}}}\tilde{\psi}^{(0)}(\eta) = 0, \quad \tilde{\varphi}^{(0)}(\eta)|_{\tilde{\Gamma}} = 0, \ \eta\in\Sigma\}, \end{aligned} \tag{10.133}$$

where Γ and $\tilde{\Gamma} \subset M$ some smooth $(s-1)$-dimensional manifolds, and also construct the actions

$$\begin{aligned} \mathbf{\Omega}_{\pm} &: \psi^{(0)} \to \tilde{\psi}^{(0)} := \psi^{(0)}\Omega^{-1}_{(t;x)}\Omega_{(t;x)}, \\ \mathbf{\Omega}^{\circledast}_{\pm} &: \varphi^{(0)} \to \tilde{\varphi}^{(0)} := \varphi^{(0)}\Omega^{\circledast,-1}_{(t;x)}\Omega^{\circledast}_{(t_0;x_0)} \end{aligned} \tag{10.134}$$

on arbitrary but fixed pairs of elements $(\varphi^{(0)}(\lambda), \psi^{(0}(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}_0$, parametrized by the set Σ, where one needs to have all the pairs $(\tilde{\varphi}^{(0)}(\lambda), \tilde{\psi}^{(0)}(\mu)dx)$, $\lambda, \mu \in \Sigma$, belong to $\mathcal{H}^0_{\Lambda(\tilde{\mathcal{L}}^*),-}(M) \times \mathcal{H}^s_{\Lambda(\tilde{\mathcal{L}}),-}(M)$. Note also that the related operator property (10.132) can be concisely written as

$$\tilde{\Omega}_{(t;x)} = \tilde{\Omega}_{(t_0;x_0)}\Omega^{-1}_{(t;x)}\Omega_{(t_0;x_0)} = -\Omega_{(t_0;x_0)}\Omega^{-1}_{(t;x)}\Omega_{(t_0;x_0)}. \tag{10.135}$$

From the expressions (10.134), we now construct the following operator kernels from the Hilbert space $L^2_{(\rho)}(\Sigma;\mathbb{C}) \otimes L^2_{(\rho)}(\Sigma;\mathbb{C})$:

$$\begin{aligned} \Omega_{(t;x)}(\lambda,\mu) - \Omega_{(t_0;x_0)}(\lambda,\mu) &= \int_{\partial S^{(s)}_{(t;x)}} \Omega^{(s-1)}[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx] \\ &\quad - \int_{\partial S^{(s)}_{(t_0;x_0)}} \Omega^{(s-1)}[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx] \\ &= \int_{S^{(s)}_{\pm}(\sigma^{(s-1)}_{(t;x)}, \sigma^{(s-1)}_{(t_0;x_0)})} d\Omega^{(s-1)}[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx] \\ &= \int_{S^{(s)}_{\pm}(\sigma^{(s-1)}_{(t;x)}, \sigma^{(s-1)}_{(t_0;x_0)})} Z^{(s)}[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx], \end{aligned} \tag{10.136}$$

and, similarly,

$$\begin{aligned} \Omega^{\circledast}_{(t;x)}(\lambda,\mu) - \Omega^{\circledast}_{(t_0;x_0)}(\lambda,\mu) &= \int_{\partial S^{(s)}_{(t;x)}} \bar{\Omega}^{(s-1),\intercal}[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx] \\ &\quad - \int_{\partial S^{(s)}_{(t_0;x_0)}} \bar{\Omega}^{(s-1),\intercal}[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx] \\ &= \int_{S^{(s)}_{\pm}(\sigma^{(s-1)}_{(t;x)}, \sigma^{(s-1)}_{(t_0;x_0)})} d\bar{\Omega}^{(s-1),\intercal}[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx] \\ &= \int_{S^{(s)}_{\pm}(\sigma^{(s-1)}_{(t;x)}, \sigma^{(s-1)}_{(t_0;x_0)})} \bar{Z}^{(s-1),\intercal}[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx], \end{aligned} \tag{10.137}$$

where $\lambda, \mu \in \Sigma$, and the s-dimensional manifolds $S^{(s)}_{+}(\sigma^{(s-1)}_{(t;x)}, \sigma^{(s-1)}_{(t_0;x_0)})$ and $S^{(s)}_{-}(\sigma^{(s-1)}_{(t;x)}, \sigma^{(s-1)}_{(t_0;x_0)}) \subset C_{s-1}(M)$ are spanned smoothly without self-intersection between two homologous cycles $\sigma^{(s-1)}_{(t;x)} = \partial S^{(s)}_{(t;x)}$ and $\sigma^{(s-1)}_{(t_0;x_0)} :=$

$\partial S^{(s)}_{(t_0;x_0)} \in C_{s-1}(M;\mathbb{C})$ such that the boundary $\partial(S^{(s)}_{+}(\sigma^{(s-1)}_{(t;x)},\sigma^{(s-1)}_{(t_0;x_0)}) \cup S^{(s)}_{-}(\sigma^{(s-1)}_{(t;x)},\sigma^{(s-1)}_{(t_0;x_0)})) = \varnothing$. Since the integral operators $\Omega_{(t_0;x_0)},\Omega^{\circledast}_{(t_0;x_0)}: L^2_{(\rho)}(\Sigma;\mathbb{C}) \to L^2_{(\rho)}(\Sigma;\mathbb{C})$ are at a fixed point $(t_0;x_0) \in M$, obviously constant and assumed to be invertible, to extend the actions (10.134) to all of $\mathcal{H}\times\mathcal{H}^*$, one can apply to them the classical variation of constants approach, making use of the expression (10.137). As a result, we easily obtain the following Delsarte transmutation integral operator expressions

$$
\begin{aligned}
\mathbf{\Omega}_{\pm} &= \mathbf{1} - \int_{\Sigma\times\Sigma} d\rho(\xi)d\rho(\eta)\tilde{\psi}(x;\xi)\Omega^{-1}_{(t_0;x_0)}(\xi,\eta) \\
&\quad \times \int_{S^{(s)}_{\pm}(\sigma^{(s-1)}_{(t;x)},\sigma^{(s-1)}_{(t_0;x_0)})} Z^{(s)}[\varphi^{(0)}(\eta),\cdot], \\
\mathbf{\Omega}^{\circledast}_{\pm} &= \mathbf{1} - \int_{\Sigma\times\Sigma} d\rho(\xi)d\rho(\eta)\tilde{\varphi}(x;\eta)\Omega^{\circledast,-1}_{(t_0;x_0)}(\xi,\eta) \\
&\quad \times \int_{S^{(s)}_{\pm}(\sigma^{(s-1)}_{(t;x)},\sigma^{(s-1)}_{(t_0;x_0)})} \bar{Z}^{(s),\intercal}[\cdot,\psi^{(0)}(\xi)dx]
\end{aligned}
\tag{10.138}
$$

for fixed pairs $(\varphi^{(0)}(\xi),\psi^{(0)}(\eta)) \in \mathcal{H}^*_0\times\mathcal{H}_0$ and $(\tilde{\varphi}^{(0)}(\lambda),\tilde{\psi}^{(0)}(\mu)) \in \tilde{\mathcal{H}}^*_0\times\tilde{\mathcal{H}}_0$, $\lambda,\mu\in\Sigma$, which are bounded invertible integral operators of Volterra type on the whole space $\mathcal{H}\times\mathcal{H}^*$. Applying the same arguments as in the first section, one also can show that the transformed sets of operators $\tilde{\mathrm{L}}_j := \mathbf{\Omega}_{\pm}\mathrm{L}_j\mathbf{\Omega}^{-1}_{\pm}$ and $\tilde{\mathrm{L}}^*_k := \mathbf{\Omega}^{\circledast}_{\pm}\mathrm{L}^*_k\mathbf{\Omega}^{\circledast,-1}_{\pm}$, $1\le k\le r$, are purely differential. Thus, we have essentially proved the following important result.

Theorem 10.11. *The expressions (10.138) are bounded invertible Delsarte transmutation integral operators of Volterra type on $\mathcal{H}\times\mathcal{H}^*$, transforming, respectively, given commuting sets of operators L_j and their formally adjoint ones L^*_k into the pure differential sets of operators $\tilde{\mathrm{L}}_j := \mathbf{\Omega}_{\pm}\mathrm{L}_j\mathbf{\Omega}^{-1}_{\pm}$ and $\tilde{\mathrm{L}}^*_k := \mathbf{\Omega}^{\circledast}_{\pm}\mathrm{L}^*_k\mathbf{\Omega}^{\circledast,-1}_{\pm}$, $1\le j,k\le r$. Moreover, there are closed subspaces $\mathcal{H}_0\subset\mathcal{H}$ and $\tilde{\mathcal{H}}_0\subset\mathcal{H}$ such that the operator $\mathbf{\Omega}\in\mathcal{L}(\mathcal{H})$ depend strongly on the topological structure of the generalized cohomology groups $\mathcal{H}^0_{\Lambda(\mathcal{L}),-}(M)$ and $\mathcal{H}^0_{\Lambda(\tilde{\mathcal{L}}),-}(M)$, and they are parametrized by elements $S^{(s)}_{\pm}(\sigma^{(s-1)}_{(t;x)},\sigma^{(s-1)}_{(t_0;x_0)}) \in C_s(M;\mathbb{C})$.*

Suppose now that the differential operators $L_j := L_j(x|\partial)$, $1\le j\le r$, considered above are independent of $t\in\mathrm{T}^r\subset\mathbb{R}^r_+$. Then, evidently, one can take

$$
\begin{aligned}
\mathcal{H}_0 := \{&\psi^{(0)}_{\mu}(\xi)\in L_{2,-}(\mathbb{R}^s;\mathbb{C}^N) : L_j\psi^{(0)}_{\mu}(\xi)=\mu_j\psi^{(0)}_{\mu}(\xi),\quad 1\le j\le r,\\
&\psi^{(0)}_{\mu}(\xi)|_{\Gamma}=0,\ \mu:=(\mu_1,...,\mu_r)\in\sigma(\tilde{\mathcal{L}})\cap\bar{\sigma}(\mathcal{L}^*),\ \xi\in\Sigma_{\sigma}\},
\end{aligned}
$$

$$\tilde{\mathcal{H}}_0 := \{\tilde{\psi}_\mu^{(0)}(\xi) \in L_{2,-}(\mathbb{R}^s;\mathbb{C}^N) : \tilde{L}_j\tilde{\psi}_\mu^{(0)}(\xi) = \mu_j\tilde{\psi}_\mu^{(0)}(\xi), \quad 1 \le j \le r,$$
$$\tilde{\psi}_\mu^{(0)}(\xi)|_{\tilde{\Gamma}} = 0, \ \mu := (\mu_1,...,\mu_r) \in \sigma(\tilde{\mathcal{L}}) \cap \bar{\sigma}(\mathcal{L}^*), \ \xi \in \Sigma_\sigma\},$$

$$\mathcal{H}_0^* := \{\varphi_\lambda^{(0)}(\eta) \in L_{2,-}(\mathbb{R}^s;\mathbb{C}^N) : L_j^*\varphi_\lambda^{(0)}(\eta) = \bar{\lambda}_j\varphi_\lambda^{(0)}(\eta), \quad 1 \le j \le r,$$
$$\varphi_\lambda^{(0)}(\eta)|_{\Gamma} = 0, \ \lambda := (\lambda_1,...,\lambda_r) \in \sigma(\tilde{\mathcal{L}}) \cap \bar{\sigma}(\mathcal{L}^*), \ \eta \in \Sigma_\sigma\},$$

$$\tilde{\mathcal{H}}_0^* := \{\tilde{\varphi}_\lambda^{(0)}(\eta) \in L_{2,-}(\mathbb{R}^s;\mathbb{C}^N) : \tilde{L}_j^*\tilde{\varphi}_\lambda^{(0)}(\eta) = \bar{\lambda}_j\tilde{\varphi}_\lambda^{(0)}(\eta), \quad 1 \le j \le r,$$
$$\tilde{\varphi}_\lambda^{(0)}(\eta)|_{\tilde{\Gamma}} = 0, \ \lambda := (\lambda_1,...,\lambda_r) \in \sigma(\tilde{\mathcal{L}}) \cap \bar{\sigma}(\mathcal{L}^*), \ \eta \in \Sigma_\sigma\}$$

and construct the corresponding Delsarte transmutation operators

$$\begin{aligned}\mathbf{\Omega}_\pm = \mathbf{1} - &\int_{\sigma(\tilde{\mathcal{L}})\cap\bar{\sigma}(\mathcal{L}^*)} d\rho_\sigma(\lambda) \int_{\Sigma_\sigma\times\Sigma_\sigma} d\rho_{\Sigma_\sigma}(\xi)d\rho_{\Sigma_\sigma}(\eta) \\ &\times \int_{S_\pm^{(s)}(\sigma_x^{(s-1)},\sigma_{x_0}^{(s-1)})} dx\tilde{\psi}_\lambda^{(0)}(\xi)\Omega_{(x_0)}^{-1}(\lambda;\xi,\eta)\bar{\varphi}_\lambda^{(0),\intercal}(\eta)(\cdot)\end{aligned} \tag{10.139}$$

and

$$\begin{aligned}\mathbf{\Omega}_\pm^\circledast = \mathbf{1} - &\int_{\sigma(\tilde{\mathcal{L}})\cap\bar{\sigma}(\mathcal{L}^*)} d\rho_\sigma(\lambda) \int_{\Sigma} d\rho_{\Sigma_\sigma}(\xi)d\rho_{\Sigma_\sigma}(\eta) \\ &\times \int_{S_\pm^{(s)}(\sigma_x^{(s-1)},\sigma_{x_0}^{(s-1)})} dx\tilde{\varphi}_\lambda^{(0)}(\xi)\bar{\Omega}_{(x_0)}^{\intercal,-1}(\lambda;\xi,\eta) \times \bar{\psi}_\lambda^{(0),\intercal}(\eta)(\cdot),\end{aligned} \tag{10.140}$$

acting on the Hilbert space $L^2(\mathbb{R}^s;\mathbb{C}^N)$, where for any $(\lambda;\xi,\eta) \in (\sigma(\tilde{\mathcal{L}}) \cap \bar{\sigma}(\mathcal{L}^*)) \times \Sigma_\sigma^2$ the kernels

$$\begin{aligned}\Omega_{(x_0)}(\lambda;\xi,\eta) &:= \textstyle\int_{\sigma_{x_0}^{(s-1)}} \Omega^{(s-1)}[\varphi_\lambda^{(0)}(\xi), \psi_\lambda^{(0)}(\eta)dx], \\ \Omega_{(x_0)}^\circledast(\lambda;\xi,\eta) &:= \textstyle\int_{\sigma_{x_0}^{(s-1)}} \bar{\Omega}^{(s-1),\intercal}[\varphi_\lambda^{(0)}(\xi), \psi_\lambda^{(0)}(\eta)dx]\end{aligned} \tag{10.141}$$

belong to $L^2_{(\rho)}(\Sigma_\sigma;\mathbb{C}) \times L^2_{(\rho)}(\Sigma_\sigma;\mathbb{C})$ for every $\lambda \in \sigma(\tilde{\mathcal{L}}) \cap \bar{\sigma}(\mathcal{L}^*)$ considered as a parameter. Moreover, as $\partial\mathbf{\Omega}_\pm/\partial t_j = 0$, $j = 1,\dots,r$, one easily obtains the set of differential expressions

$$\mathcal{R}(\tilde{\mathcal{L}}) := \{\tilde{L}_j(x|\partial) := \mathbf{\Omega}_\pm L_j(x|\partial)\mathbf{\Omega}_\pm^{-1} : 1 \le j \le r\}, \tag{10.142}$$

which is a ring of mutually commuting differential operators in $L^2(\mathbb{R}^s;\mathbb{C}^N)$ generated by the corresponding initial ring $\mathcal{R}(\mathcal{L})$.

This problem in the one-dimensional case was treated and effectively solved in [213, 406] by means of algebraic-geometric and inverse spectral transform techniques. Our approach gives another look at this problem in multi-dimensions and is of special interest due to its transparent dependence on the dimension of the differential operators.

10.6 A special case: Relations with Lax systems

Consider our de Rham–Hodge theory of a commuting set $\mathcal{L}$ of two differential operators in a Hilbert space $\mathcal{H} := L^2(\mathrm{T}^2; H)$, $H := L^2(\mathbb{R}^s; \mathbb{C}^N)$, for the special case when $M := \mathrm{T}^2 \times \bar{\mathbb{R}}^s$ and

$$\mathcal{L} := \{\mathrm{L}_j := \partial/\partial t_j - L_j(t; x|\partial) : t_j \in \ \mathrm{T}_j := [0, T_j) \subset \mathbb{R}_+, \ j = 1, 2\},$$

where $\mathrm{T}^2 := \mathrm{T}_1 \times \mathrm{T}_2$,

$$L_j(t; x|\partial) := \sum_{|\alpha|=0}^{n(L_j)} a_\alpha^{(j)}(t; x) \partial^{|\alpha|}/\partial x^\alpha \tag{10.143}$$

with coefficients $a_\alpha^{(j)} \in C^1(\mathrm{T}^2; S(\mathbb{R}^s; End\mathbb{C}^N))$, $\alpha \in \mathbb{Z}_+^s$, and $|\alpha| = 0, \ldots, n(L_j)$, $j = 1, 2$. The corresponding scalar product is

$$(\varphi, \psi) := \int_{\mathrm{T}^2} dt \int_{\mathbb{R}^s} dx < \varphi, \psi > \tag{10.144}$$

for any pair $(\varphi, \psi) \in \mathcal{H} \times \mathcal{H}$ and the generalized exterior derivative is

$$d_{\mathcal{L}} := \sum_{j=1}^{2} dt_j \wedge \mathrm{L}_j, \tag{10.145}$$

where one assumes that for all $t \in \mathrm{T}^2$ and $x \in \mathbb{R}^s$ the commutator satisfies

$$[\mathrm{L}_1, \mathrm{L}_2] = 0. \tag{10.146}$$

Obviously this means that the corresponding generalized de Rham–Hodge cochain complexes

$$\begin{aligned} &\mathcal{H} \to \Lambda^0(M; \mathcal{H}) \stackrel{d_{\mathcal{L}}}{\to} \Lambda^1(M; \mathcal{H}) \stackrel{d_{\mathcal{L}}}{\to} \ldots \stackrel{d_{\mathcal{L}}}{\to} \Lambda^m(M; \mathcal{H}) \stackrel{d_{\mathcal{L}}}{\to} 0, \\ &\mathcal{H} \to \Lambda^0(M; \mathcal{H}) \stackrel{d_{\mathcal{L}}^*}{\to} \Lambda^1(M; \mathcal{H}) \stackrel{d_{\mathcal{L}}^*}{\to} \ldots \stackrel{d_{\mathcal{L}}^*}{\to} \Lambda^m(M; \mathcal{H}) \stackrel{d_{\mathcal{L}}^*}{\to} 0 \end{aligned} \tag{10.147}$$

are exact. Now define (using (10.108) and (10.133)) the closed subspaces $\mathcal{H}_0$ and $\mathcal{H}_0 \subset \mathcal{H}_-$ as follows: $\mathcal{H}_0$ is the set of $\psi^{(0)}(\lambda; \eta) \in \mathcal{H}^0_{\Lambda(\mathcal{L}),-}(M)$ such that

$$\begin{gathered} \partial \psi^{(0)}(\lambda; \eta)/\partial t_j = L_j(t; x|\partial)\psi^{(0)}(\lambda; \eta), \ j = 1, 2, \\ \psi^{(0)}(\lambda; \eta)|_{t=t_0} = \psi_\lambda(\eta) \in H_-, \ \psi^{(0)}(\lambda; \eta)|_\Gamma = 0, \\ (\lambda; \eta) \in \Sigma \subset (\sigma(\mathcal{L}) \cap \bar{\sigma}(\mathcal{L}^*)) \times \Sigma_\sigma, \end{gathered}$$

and $\mathcal{H}_0^*$ is the set of $\varphi^{(0)}(\lambda; \eta) \in \mathcal{H}^0_{\Lambda(\mathcal{L}),-}(M)$ such that

$$\begin{gathered} -\partial \varphi^{(0)}(\lambda; \eta)/\partial t_j = L_j^*(t; x|\partial)\varphi^{(0)}(\lambda; \eta), \ j = 1, 2, \\ \varphi^{(0)}(\lambda; \eta)|_{t=t_0} = \varphi_\lambda(\eta) \in H_-, \ \varphi^{(0)}(\lambda; \eta)|_\Gamma = 0, \\ (\lambda; \eta) \in \Sigma \subset (\sigma(\mathcal{L}) \cap \bar{\sigma}(\mathcal{L}^*)) \times \Sigma_\sigma, \end{gathered}$$

for some hypersurface $\Gamma \subset M$ and a "spectral" degenerate set $\Sigma_\sigma \in \mathbb{C}^{p-1}$. Using the subspaces $\mathcal{H}_0$ and $\mathcal{H}_0^*$, one can now proceed to the construction of the Delsarte transmutation operators $\mathbf{\Omega}_\pm : H \leftrightarrows H$ in the general form similar to (10.140) with kernels $\Omega_{(t_0;x_0)}(\lambda;\xi,\eta) \in L^2_{(\rho)}(\Sigma_\sigma;\mathbb{C}) \otimes L^2_{(\rho)}(\Sigma_\sigma;\mathbb{C})$ for every $\lambda \in \sigma(\mathcal{L}) \cap \bar{\sigma}(\mathcal{L}^*)$, defined as

$$\begin{aligned} \Omega_{(t_0;x_0)}(\lambda;\xi,\eta) &:= \int_{\sigma^{(s-1)}_{(t_0;x_0)}} \Omega^{(s-1)}[\varphi^{(0)}(\lambda;\xi),\psi^{(0)}(\lambda;\eta)dx], \\ \Omega^{\circledast}_{(t_0;x_0)}(\lambda;\xi,\eta) &:= \int_{\sigma^{(s-1)}_{(t_0;x_0)}} \bar{\Omega}^{(s-1),\intercal}[\varphi^{(0)}(\lambda;\xi),\psi^{(0)}(\lambda;\eta)dx] \end{aligned} \tag{10.148}$$

for all $(\lambda;\xi,\eta) \in (\sigma(\mathcal{L}) \cap \bar{\sigma}(\mathcal{L}^*)) \times \Sigma_\sigma^2$. As a result, one obtains for the corresponding product $\rho := \rho_\sigma \odot \rho_{\Sigma_\sigma^2}$ the integral expressions:

$$\begin{aligned} \mathbf{\Omega}_\pm = \mathbf{1} - &\int_{\sigma(\mathcal{L})\cap\bar{\sigma}(\mathcal{L}^*)} d\rho_\sigma(\lambda) \int_{\Sigma_\sigma\times\Sigma_\sigma} d\rho_{\Sigma_\sigma}(\xi)d\rho_{\Sigma_\sigma}(\eta) \\ &\times \int_{S_\pm^{(s)}(\sigma^{(s-1)}_{(t_0;x)},\sigma^{(s-1)}_{(t_0;x_0)})} dx\tilde{\psi}^{(0)}(\lambda;\xi)\Omega^{-1}_{(t_0;x_0)}(\lambda;\xi,\eta)\bar{\varphi}^{(0),\intercal}(\lambda;\eta)(\cdot), \end{aligned} \tag{10.149}$$

$$\begin{aligned} \mathbf{\Omega}^{\circledast}_\pm = \mathbf{1} - &\int_{\sigma(\mathcal{L})\cap\bar{\sigma}(\mathcal{L}^*)} d\rho_\sigma(\lambda) \int_{\Sigma_\sigma\times\Sigma_\sigma} d\rho_{\Sigma_\sigma}(\xi)d\rho_{\Sigma_\sigma}(\eta) \\ &\times \int_{S_\pm^{(s)}(\sigma^{(s-1)}_{(t_0;x)},\sigma^{(s-1)}_{(t_0;x_0)})} dx\tilde{\varphi}^{(0)}_\lambda(\xi)\bar{\Omega}^{\intercal,-1}_{(t_0;x_0)}(\lambda;\xi,\eta) \times \bar{\psi}^{(0),\intercal}(\lambda;\eta)(\cdot), \end{aligned} \tag{10.150}$$

where $S_+^{(s)}(\sigma^{(s-1)}_{(t_0;x)},\sigma^{(s-1)}_{(t_0;x_0)}) \in C_s(M;\mathbb{C})$ is a smooth s-dimensional manifold spanned between two homologous cycles $\sigma^{(s-1)}_{(t_0;x)}$ and $\sigma^{(s-1)}_{(t_0;x_0)} \in \mathcal{K}(M)$ and $S_-^{(s)}(\sigma^{(s-1)}_{(t_0;x)},\sigma^{(s-1)}_{(t_0;x_0)}) \in C_s(M;\mathbb{C})$ is its smooth counterpart such that

$$\partial(S_+^{(s)}(\sigma^{(s-1)}_{(t_0;x)},\sigma^{(s-1)}_{(t_0;x_0)}) \cup S_-^{(s)}(\sigma^{(s-1)}_{(t_0;x)},\sigma^{(s-1)}_{(t_0;x_0)})) = \varnothing.$$

Employing results in the preceding sections, one can construct from (10.149) and (10.150) the corresponding factorized Fredholm operators $\mathbf{\Omega}$ and $\mathbf{\Omega}^{\circledast}$: $H \to H$, $H = L^2(\mathbb{R};\mathbb{C}^N)$, as follows:

$$\mathbf{\Omega} := \mathbf{\Omega}_+^{-1}\mathbf{\Omega}_-, \qquad \mathbf{\Omega}^{\circledast} := \mathbf{\Omega}_+^{\circledast-1}\mathbf{\Omega}_-^{\circledast}. \tag{10.151}$$

It is important to observe here that kernels $\hat{K}_\pm(\mathbf{\Omega})$ and $\hat{K}_\pm(\mathbf{\Omega}^{\circledast}) \in H_-\otimes H_-$ satisfy the generalized [28] determining equations in the tensor form

$$\begin{aligned} (\tilde{\mathcal{L}}_{ext} \otimes \mathbf{1})\hat{K}_\pm(\mathbf{\Omega}) &= (\mathbf{1}\otimes\mathcal{L}^*_{ext})\hat{K}_\pm(\mathbf{\Omega}), \\ (\tilde{\mathcal{L}}^*_{ext} \otimes \mathbf{1})\hat{K}_\pm(\mathbf{\Omega}^{\circledast}) &= (\mathbf{1}\otimes\mathcal{L}_{ext})\hat{K}_\pm(\mathbf{\Omega}^{\circledast}). \end{aligned} \tag{10.152}$$

Since it is clear that $\mathrm{supp}\hat{K}_+(\mathbf{\Omega}) \cap \mathrm{supp}\hat{K}_-(\mathbf{\Omega}) = \varnothing$ and $\mathrm{supp}\hat{K}_+(\mathbf{\Omega}^{\circledast}) \cap \mathrm{supp}\hat{K}_-(\mathbf{\Omega}^{\circledast}) = \oslash$, it follows from [152, 272, 340] that the corresponding Gelfand–Levitan–Marchenko equations

$$\begin{aligned} \hat{K}_+(\mathbf{\Omega}) + \hat{\Phi}(\mathbf{\Omega}) + \hat{K}_+(\mathbf{\Omega}) * \hat{\Phi}(\mathbf{\Omega}) &= \hat{K}_-(\mathbf{\Omega}), \\ \hat{K}_+(\mathbf{\Omega}^{\circledast}) + \hat{\Phi}(\mathbf{\Omega}^{\circledast}) + \hat{K}_+(\mathbf{\Omega}^{\circledast}) * \hat{\Phi}(\mathbf{\Omega}^{\circledast}) &= \hat{K}_-(\mathbf{\Omega}^{\circledast}), \end{aligned} \tag{10.154}$$

where $\mathbf{\Omega} : = \mathbf{1} + \hat{\Phi}(\mathbf{\Omega})$, $\mathbf{\Omega}^{\circledast} := 1 + \hat{\Phi}(\mathbf{\Omega}^{\circledast})$, can be solved [145, 272] in the space $\mathcal{B}_\infty^{\pm}(H)$ for kernels $\hat{K}_\pm(\mathbf{\Omega})$ and $\hat{K}_\pm(\mathbf{\Omega}^{\circledast}) \in H_- \otimes H_-$ depending parametrically on $t \in \mathrm{T}^2$. Consequently, the Delsarte transformed differential operators $\tilde{\mathrm{L}}_j : \mathcal{H} \to \mathcal{H}$, $j = 1, 2$, commute and satisfying

$$\tilde{\mathrm{L}}_j = \partial/\partial t_j - \mathbf{\Omega}_\pm L_j \mathbf{\Omega}_\pm^{-1} - (\partial \mathbf{\Omega}_\pm/\partial t_j)\mathbf{\Omega}_\pm^{-1} := \partial/\partial t_j - \tilde{L}_j, \tag{10.155}$$

where the operator expressions for $\tilde{L}_j \in \mathcal{L}(H)$, $j = 1, 2$, prove to be purely differential. This property makes it possible to construct nonlinear partial differential equations on coefficients of differential operators (10.155) and solve them by means of either inverse spectral methods [232, 233, 247, 406] or the Darboux–Bäcklund [253, 342, 366] transforms, producing a wide class of exact soliton-like solutions. Another rather complicated and very interesting aspects of the approach devised in this chapter concerns regular algorithms for treating differential operator expressions depending on a "spectral" parameter $\lambda \in \mathbb{C}$, which were recently discussed in [152, 340]. Some results on the structure of the Delsarte–Lions transmutation operators can be easily adapted for constructing effective transformations of Cartan type connections for multi-dimensional integrable Davey–Stewartson type nonlinear differential systems on a Riemannian manifold M, vanishing on a three-dimensional integral submanifold $M_\alpha \subset M$. The results can be used to investigate a wide class of exact special solutions to such differential systems, which having diverse applications [108, 121, 173, 266, 262, 365, 393] in solving problems of modern differential topology and mathematical physics.

10.7 Geometric and spectral theory aspects of Delsarte–Darboux binary transformations

The differential-geometric analysis of Delsarte–Darboux transformations above for differential operator expressions, acting on a functional space $\mathcal{H} = L^1(\mathrm{T};H)$, where $\mathrm{T} = \mathbb{R}^2$ and $H := L^2(\mathbb{R}^2;\mathbb{C}^2)$, appears to have a deep relationship with classical generalized de Rham–Hodge theory [377–380].

Concerning our problem of describing the spectral structure of Delsarte–Darboux type transmutations acting in $\mathcal{H}$, we now begin by considering some basics of the generalized de Rham–Hodge differential complex theory devised for studying transformations of differential operators. Let a smooth metric space M be a suitably compactified form of the space $\mathbb{R}^m$, $m \in \mathbb{Z}_+$. Then one can define on $M_{\mathrm{T}} := \mathrm{T} \times M$ the standard Grassmann algebra $\Lambda(M_{\mathrm{T}};\mathcal{H})$ of differential forms on $\mathrm{T}\times M$ and consider a generalized external Skrypnik [377, 378] anti-derivation operator $d_{\mathcal{L}} : \Lambda(M_{\mathrm{T}};\mathcal{H}) \to \Lambda(M_{\mathrm{T}};\mathcal{H})$ acting as follows: for any $\beta^{(k)} \in \Lambda^k(M_{\mathrm{T}};\mathcal{H})$, $0 \le k \le m$,

$$d_{\mathcal{L}}\beta^{(k)} := \sum_{j=1}^{2} dt_j \wedge \mathrm{L}_j(t;x|\partial)\beta^{(k)} \in \Lambda^{k+1}(M_{\mathrm{T}};\mathcal{H}), \tag{10.155}$$

where

$$\mathrm{L}_j(t;x|\partial) := \partial/\partial t_j - L_j(t;x|\partial) \tag{10.156}$$

$j = 1,2$, are suitably defined commuting linear differential operators in $\mathcal{H}$, that is

$$[\mathrm{L}_1, \mathrm{L}_2] = 0. \tag{10.157}$$

We shall assume that the differential expressions

$$L_j(t;x|\partial) := \sum_{|\alpha|=0}^{n_j(L)} a_\alpha^{(j)}(t;x)\frac{\partial^{|\alpha|}}{\partial x^\alpha}, \tag{10.158}$$

with coefficients $a_\alpha^{(j)} \in C^1(\mathrm{T}; C^\infty(M; End\mathbb{C}^N))$, $0 \le |\alpha| \le n_j(L)$, $n_j^\alpha \in \mathbb{Z}_+$, $j = 0,1$, are closed normal densely defined operators in the Hilbert space H for any $t \in \mathrm{T}$. It is easy to see that the anti-derivation of $d_{\mathcal{L}}$ defined by (10.155) is a generalization of the usual exterior derivative

$$d = \sum_{j=1}^{m} dx_j \wedge \frac{\partial}{\partial x_j} + \sum_{s=1}^{2} dt_s \wedge \frac{\partial}{\partial t_s} \tag{10.159}$$

for which it is obvious that

$$[\frac{\partial}{\partial x_j};\frac{\partial}{\partial x_k}] = 0, \quad [\frac{\partial}{\partial t_s};\frac{\partial}{\partial t_l}] = 0, \; [\frac{\partial}{\partial x_j};\frac{\partial}{\partial t_s}] = 0 \tag{10.160}$$

hold for all $j,k = 1,\ldots,m$ and $s,l = 1,2$. Now, making the substitutions (10.159) $\partial/\partial x_j \longrightarrow \mathrm{A}_j$, $\partial/\partial t_s \longrightarrow \mathrm{L}_s$, $1 \le j \le m$, $s = 1,2$, one obtains the linear anti-derivation homomorphism

$$d_{\mathcal{A}} := \sum_{j=1}^{m} dx_j \wedge \mathrm{A}_j(t;x|\partial) + \sum_{j=1}^{2} dt_s \wedge \mathrm{L}_s(t;x|\partial), \tag{10.161}$$

where the differential expressions $\mathrm{A}_j, \mathrm{L}_S : \mathcal{H} \longrightarrow \mathcal{H}$ for all $j, k = 1, \ldots, m$ and $s, l = 1, 2$, satisfy the commutation conditions $[\mathrm{A}_j, \mathrm{A}_k] = 0$, $[\mathrm{L}_s, \mathrm{L}_s] = 0$, $[\mathrm{A}_j, \mathrm{L}_s] = 0$. Whence, the operation (10.161) defines on $\Lambda(M_\mathrm{T}; \mathcal{H})$ an anti-derivation homomorphism with respect to which the cochain sequence

$$\mathcal{H} \longrightarrow \Lambda^0(M_\mathrm{T}; \mathcal{H}) \xrightarrow{d_\mathcal{A}} \Lambda^1(M_\mathrm{T}; \mathcal{H}) \xrightarrow{d_\mathcal{A}} \ldots \xrightarrow{d_\mathcal{A}} \Lambda^{\mathrm{m}+2}(M_\mathrm{T}; \mathcal{H}) \xrightarrow{d_\mathcal{A}} 0 \tag{10.162}$$

is a complex, as $d_\mathcal{A} d_\mathcal{A} \equiv 0$. Since the linear anti-derivation map (10.155) is a particular case of (10.161), it follows that the corresponding cochain sequence (10.162) is also a complex.

Next, we employ ideas developed in [97, 377–380]. A differential form $\beta \in \Lambda(M_\mathrm{T}; \mathcal{H})$ will be called *$d_\mathcal{A}$-closed* if $d_\mathcal{A}\beta = 0$ and a form $\gamma \in \Lambda(M_\mathrm{T}; \mathcal{H})$ will be called *exact* or *$d_\mathcal{A}$-homologous to zero* if there exists on M_T a form $\omega \in \Lambda(M_\mathrm{T}; \mathcal{H})$ such that $\gamma = d_\mathcal{A}\omega$. Recall the standard [86, 97, 386, 393] algebraic Hodge star-operation

$$* : \Lambda^{\mathrm{k}}(M_\mathrm{T}; \mathcal{H}) \longrightarrow \Lambda^{\mathrm{m}+2-\mathrm{k}}(M_\mathrm{T}; \mathcal{H}), \tag{10.163}$$

$0 \leq k \leq m + 2$, defined as follows: if $\beta \in \Lambda^{\mathrm{k}}(M_\mathrm{T}; \mathcal{H})$, then the form $*\beta \in \Lambda^{\mathrm{m}+2-\mathrm{k}}(M_\mathrm{T}; \mathcal{H})$ is such that:

- the $(m - k + 2)$-dimensional volume $| * \beta|$ of the form $*\beta$ equals k-dimensional volume $|\beta|$ of the form β;
- the $(m + 2)$-dimensional measure $\bar{\beta}^\intercal \wedge *\beta > 0$ for a fixed orientation on M_T.

Define also on the space $\Lambda(M_\mathrm{T}; \mathcal{H})$ the following natural scalar product: for any $\beta, \gamma \in \Lambda^{\mathrm{k}}(M_\mathrm{T}; \mathcal{H})$, $0 \leq k \leq m$,

$$(\beta, \gamma) := \int_{M_\mathrm{T}} \bar{\beta}^\intercal * \gamma. \tag{10.164}$$

This makes

$$\mathcal{H}_\Lambda(M_\mathrm{T}) := \bigoplus_{k=0}^{m+2} \mathcal{H}_\Lambda^k(M_\mathrm{T}) \tag{10.165}$$

into a Hilbert space that is well suited for analysis. Note also that the Hodge star $*$-operation satisfies the following easily verified property: for any $\beta, \gamma \in \mathcal{H}_\Lambda^k(M_\mathrm{T})$, $k = 0, \ldots, m$,

$$(\beta, \gamma) = (*\beta, *\gamma), \tag{10.166}$$

that is the Hodge operation $* : \mathcal{H}_\Lambda(M_\mathrm{T}) \to \mathcal{H}_\Lambda(M_\mathrm{T})$ is unitary and its standard adjoint with respect to the scalar product (10.164) operation satisfies the condition $(*)' = (*)^{-1}$.

Let $d_{\mathcal{L}}^{'}$ be the formal adjoint expression to the weak differential operation (10.155). Using the operations $d_{\mathcal{L}}'$ and $d_{\mathcal{L}}$ in the $\mathcal{H}_{\Lambda}(M_{\mathrm{T}})$ one can naturally define [86, 97, 377, 386, 393] the generalized Laplace–Hodge operator $\Delta_{\mathcal{L}} : \mathcal{H}_1(M_{\mathrm{T}}) \longrightarrow \mathcal{H}_1(M_{\mathrm{T}})$ as

$$\Delta_{\mathcal{L}} = d_{\mathcal{L}}' d_{\mathcal{L}} + d_{\mathcal{L}}' d_{\mathcal{L}} \ . \tag{10.167}$$

A form $\beta \in \mathcal{H}_{\Lambda}(M_{\mathrm{T}})$ satisfying

$$\Delta_{\mathcal{L}} \beta = 0 \tag{10.168}$$

is said to be *harmonic* [86, 97, 377, 393]. It is easy to see that a harmonic form $\beta \in \mathcal{H}_{\Lambda}(M_{\mathrm{T}})$ satisfies

$$d_{\mathcal{L}}' \beta = 0, \quad d_{\mathcal{L}} \beta = 0, \tag{10.169}$$

which come from (10.167) and (10.168).

One can readily check that the following differential operators in $\mathcal{H}_{\Lambda}(M_{\mathrm{T}})$

$$d_{\mathcal{L}}^* := * d_{\mathcal{L}}' (*)^{-1} \tag{10.170}$$

define a new exterior anti-derivation operation in $\mathcal{H}_{\Lambda}(M_{\mathrm{T}})$, and we immediately obtain the following result from the fact that $d_{\mathcal{L}}^* d_{\mathcal{L}}^* = 0$.

Lemma 10.7. *The dual cochain sequence*

$$\mathcal{H} \longrightarrow \Lambda^0(M_{\mathrm{T}}; \mathcal{H}) \xrightarrow{d_{\mathcal{L}}^*} \Lambda^1(M_{\mathrm{T}}; \mathcal{H}) \xrightarrow{d_{\mathcal{L}}^*} \ldots \xrightarrow{d_{\mathcal{L}}^*} \Lambda^{m+2}(M_{\mathrm{T}}; \mathcal{H}) \xrightarrow{d_{\mathcal{L}}^*} 0 \tag{10.171}$$

corresponding to (10.162) is a complex.

Denote by $\mathcal{H}^k_{\Lambda(\mathcal{L})}(M_{\mathrm{T}})$ the cohomology groups of $d_{\mathcal{L}}$-closed and by $\mathcal{H}^k_{\Lambda(\mathcal{L}^*)}(M_{\mathrm{T}})$ the cohomology groups of $d_{\mathcal{L}}^*$-closed differential forms, respectively, and by $\mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L})}(M_{\mathrm{T}})$ the abelian groups of harmonic differential forms from the Hilbert subspaces $\mathcal{H}^k_{\Lambda}(M_{\mathrm{T}})$, $0 \leq k \leq m+2$. Before formulating our next results, we define the standard Hilbert–Schmidt rigged chain [28, 32] of positive and negative Hilbert spaces of differential forms

$$\mathcal{H}^k_{\Lambda,+}(M_{\mathrm{T}}) \subset \mathcal{H}^k_{\Lambda}(M_{\mathrm{T}}) \subset \mathcal{H}^k_{\Lambda,-}(M_{\mathrm{T}}), \tag{10.172}$$

the corresponding hereditary rigged chains of harmonic forms:

$$\mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L}),+}(M_{\mathrm{T}}) \subset \mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L})}(M_{\mathrm{T}}) \subset \mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L}),-}(M_{\mathrm{T}}) \tag{10.173}$$

and chains of cohomology group:

$$\begin{gathered} \mathcal{H}^k_{\Lambda(\mathcal{L}),+}(M_{\mathrm{T}}) \subset \mathcal{H}^k_{\Lambda(\mathcal{L})}(M_{\mathrm{T}}) \subset \mathcal{H}^k_{\Lambda(\mathcal{L}),-}(M_{\mathrm{T}}), \\ \mathcal{H}^k_{\Lambda(\mathcal{L}^*),+}(M_{\mathrm{T}}) \subset \mathcal{H}^k_{\Lambda(\mathcal{L}^*)}(M_{\mathrm{T}}) \subset \mathcal{H}^k_{\Lambda(\mathcal{L}^*),-}(M_{\mathrm{T}}) \end{gathered} \tag{10.174}$$

for all $0 \leq k \leq m+2$. Assume also that the Laplace–Hodge operator (10.167) is reduced upon the space $\mathcal{H}^0_\Lambda(M)$. Now by reasoning analogous to that in [86, 97, 393], one can prove the following generalized [97, 378–380] de Rham–Hodge theorem along the same lines as the analogous result in Section 10.5.

Theorem 10.12. *The groups of harmonic forms $\mathcal{H}^k_{\Lambda,+}(M_\mathrm{T})$ are, respectively, isomorphic to the homology groups $(H^k(M_\mathrm{T};\mathbb{C}))^{|\Sigma|}$, where $H^k(M_\mathrm{T};\mathbb{C})$ is the k-th cohomology group of the manifold M_T with complex coefficients, a set $\Sigma \subset \mathbb{C}^p$, $p \in \mathbb{Z}_+$, is the set of suitable "spectral" parameters marking the linear space of independent $d^*_\mathcal{L}$-closed 0-form from $\mathcal{H}^0_{\Lambda(\mathcal{L}),-}(M_\mathrm{T})$ and, moreover, the following direct sum decompositions*

$$\mathcal{H}^k_{\Lambda,+}(M_\mathrm{T}) = \mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L}),+}(M_\mathrm{T}) \oplus \Delta_\mathcal{L}\mathcal{H}^k_{\Lambda,+}(M_\mathrm{T}) \tag{10.175}$$

$$= \mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L}),+}(M_\mathrm{T}) \oplus d_\mathcal{L}\mathcal{H}^{k-1}_{\Lambda,+}(M_\mathrm{T}) \oplus d'_\mathcal{L}\mathcal{H}^{k+1}_{\Lambda,+}(M_\mathrm{T})$$

hold for any $k = 0,\ldots,m+2$.

Another variant of the above theorem was proved in [377, 378] and reads as the following generalized de Rham–Hodge theorem. It can be proved using a special sequence [234, 377–380] of differential Lagrange type identities (as in Section 10.5).

Theorem 10.13. *The generalized cohomology groups $\mathcal{H}^k_{\Lambda(\mathcal{L}),+}(M_\mathrm{T})$ are isomorphic, respectively, to the cohomology groups $(H^k(M_\mathrm{T};\mathbb{C}))^{|\Sigma|}$, $0 \leq k \leq m+2$.*

Define the following closed subspace

$$\mathcal{H}^*_0 := \{\varphi^{(0)}(\eta) \in \mathcal{H}^0_{\Lambda(\mathcal{L}^*),-}(M_\mathrm{T}) : d^*_\mathcal{L}\varphi^{(0)}(\eta) = 0,\ \varphi^{(0)}(\eta)|_\Gamma,\ \eta \in \Sigma\} \tag{10.176}$$

for some smooth $(m+1)$-dimensional hypersurface $\Gamma \subset M_\mathrm{T}$ and $\Sigma \subset (\sigma(L)\cap \bar{\sigma}(L)) \times \Sigma_\sigma \subset \mathbb{C}^p$, where $\mathcal{H}^0_{\Lambda(\mathcal{L}^*),-}(M_\mathrm{T})$ is, as above, a suitable Hilbert–Schmidt rigged [28, 32] zeroth-order cohomology group Hilbert space from the co-chain given by (10.174), $\sigma(L)$ and $\sigma(L^*)$ are, respectively, mutual generalized spectra of the sets of differential operators L and L^* in H at $t = 0 \in \mathrm{T}$. Thereby, the dimension $\dim\mathcal{H}^*_0 = card\ \Sigma := |\Sigma|$ is assumed to be known. The next lemma first stated in [377–379] is important for a proof of Theorem 10.12 and can be proved in the same way as the corresponding result in Section 10.5.

Lemma 10.8. *There are sets of $(k+1)$-forms $Z^{(k+1)}[\varphi^{(0)}(\eta), d_{\mathcal{L}}\psi^{(k)}]$ in $\Lambda^{k+1}(M_{\mathrm{T}};\mathbb{C})$ and k-forms $Z^{(k)}[\varphi^{(0)}(\eta),\psi^{(k)}]$ in $\Lambda^k(M_{\mathrm{T}};\mathbb{C})$ parametrized by the set $\Sigma \ni \eta$, which are semilinear in $(\varphi^{(0)}(\eta),\psi^{(k)}) \in \mathcal{H}_0^* \times \mathcal{H}^k_{\Lambda,+}(M_{\mathrm{T}})$, and such that*

$$Z^{(k+1)}[\varphi^{(0)}(\eta), d_{\mathcal{L}}\psi^{(k)}] = dZ^k[\phi^{(0)}(\eta\ ),\psi^{(k)}] \tag{10.177}$$

for all $k = 0,\dots,m+2$ and $\eta \in \Sigma$.

Using Lemma 10.8, one can construct the cohomology group isomorphism claimed in Theorem 10.12 (cf. Section 10.5). In particular, following [377, 378], let us take some singular simplicial [96, 97, 386, 393] complex $\mathcal{K}(M_{\mathrm{T}})$ of the compact metric space M_{T} and introduce a set of linear mappings $B_\lambda^{(k)} : \mathcal{H}^k_{\Lambda,+}M_{\mathrm{T}} \longrightarrow C_k(M_{\mathrm{T}};\mathbb{C}),\ \lambda \in \Sigma,$ where $C_k(M_{\mathrm{T}};\mathbb{C})$, $0 \le k \le m+2$ are free abelian groups over the field $\mathbb{C}$ generated, respectively, by all k-chains of singular simplices $S^{(k)} \subset M_{\mathrm{T}}$, $0 \le k \le m+2$, from the simplicial complex $\mathcal{K}(M_{\mathrm{T}})$, as follows:

$$B_\lambda^{(k)}(\psi^{(k)}) := \sum_{S^{(k)} \in C_k(M_{\mathrm{T}};\mathbb{C}))} S^{(k)} \int_{S^{(k)}} Z^{(k)}[\varphi^{(0)}(\lambda),\psi^{(k)}] \tag{10.178}$$

with $\psi^{(k)} \in \mathcal{H}^k_{\Lambda,+}(M_{\mathrm{T}})$, $0 \le k \le m+2$. Then, the following theorem [377, 378] based on mappings (10.178) can be proved just as its analog in Section 10.5.

Theorem 10.14. *The set of operators (10.178) parametrized by $\lambda \in \Sigma$ realizes the cohomology group isomorphism formulated in Theorem 10.12.*

10.8 The spectral structure of Delsarte–Darboux transmutation operators in multi-dimensions

We shall exploit, as in Section 10.5, the fact that the differential operators $\mathrm{L}_j : \mathcal{H} \to \mathcal{H}$, $j = 1,2$, are of the special form (10.156). Assume that differential expressions (10.158) are normal closed operators defined on a dense subspace $D(L) \subset L^2(M;\mathbb{C}^N)$.

It follows from Theorem 10.14 that there is a pair $(\varphi^{(0)}(\lambda),\psi^{(k)}(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}^k_{\Lambda(\mathcal{L}),+}(M_{\mathrm{T}})$ parametrized by elements $(\lambda,\mu) \in \Sigma \times \Sigma_k$, for which the equality

$$B_\lambda^{(m)}(\psi^{(0)}(\mu)dx) = S^{(m)}_{(t;x)} \int_{\partial S^{(m)}_{(t;x)}} \Omega^{(m-1)}[\varphi^{(0)}(\lambda),\psi^{(0)}(\mu)dx] \tag{10.179}$$

holds, where $S^{(m)}_{(t;x)} \in H_m(M_{\mathrm{T}};\mathbb{C})$ is some arbitrary but fixed element parametrized by any chosen point $(t;x) \in M_{\mathrm{T}} \cap \partial S^{(m)}_{(t;x)}$. Consider the integral expressions

$$\Omega_{(t;x)}(\lambda,\mu) := \int_{\sigma^{(m-1)}_{(t;x)}} \Omega^{(m-1)}[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx], \tag{10.180}$$
$$\Omega_{(t_0;x_0)}(\lambda,\mu) := \int_{\sigma^{(m-1)}_{(t_0;x_0)}} \Omega^{(m-1)}[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx],$$

with $(t_0;x_0) \in M_{\mathrm{T}} \cap \partial S^{(m)}_{(t_0;x_0)}$ with fixed boundaries $\sigma^{(m-1)}_{(t;x)} := \partial S^{(m)}_{t;x}$, $\sigma^{(m-1)}_{(t_0;x_0)} := \partial S^{(m)}_{t_0;x_0}$ assumed to be homologous as $(t;x_0) \longrightarrow (t;x) \in M_{\mathrm{T}}$, $(\lambda,\mu) \in \Sigma \times \Sigma_k$, and interpret them as the kernels [28, 32, 115] of the corresponding invertible integral operators of Hilbert–Schmidt type $\Omega_{(t;x)}, \Omega_{(t_0;x_0)} : L^2_{(\rho)}(\Sigma;\mathbb{C}) \longrightarrow L^2_{(\rho)}(\Sigma;\mathbb{C})$, where ρ is some finite Borel measure on the parameter set Σ. Define the invertible operators

$$\mathbf{\Omega}_{\pm} : \psi^{(0)}(\mu) \longrightarrow \tilde{\psi}^{(0)}(\mu) \tag{10.181}$$

for $\psi^{(0)}(\mu)dx \in \mathcal{H}^m_{\Lambda(\mathcal{L}),+}(M_{\mathrm{T}})$ and some $\tilde{\psi}^{(0)}(\mu)dx \in \mathcal{H}^m_{\Lambda(\mathcal{L}),+}(M_{\mathrm{T}})$, $\mu \in \Sigma$, where for any $\eta \in \Sigma$

$$\begin{aligned}\tilde{\psi}^{(0)}(\eta) &:= \psi^{(0)}(\eta) \cdot \Omega^{-1}_{(t;x)} \cdot \Omega_{(t_0;x_0)} \\ &= \int_\Sigma d\rho(\mu) \int_\Sigma d\rho(\xi) \psi^{(0)}(\mu) \Omega^{-1}_{(t;x)}(\mu,\xi) \Omega_{(t_0;x_0)}(\xi,\eta),\end{aligned} \tag{10.182}$$

which is motivated by the expression (10.179). More precisely, consider the diagram

$$\begin{array}{ccc} \mathcal{H}^m_{\Lambda(\mathcal{L}),+}(M_{\mathrm{T}}) & \xrightarrow{\Omega_\pm} & \mathcal{H}^m_{\Lambda(\tilde{\mathcal{L}}),+}(M_{\mathrm{T}}), \\ B^{(m)}_\lambda \downarrow & \swarrow \tilde{B}^{(m)}_\lambda & \\ H_m(M_{\mathrm{T}};\mathbb{C}) & & \end{array} \tag{10.183}$$

which is assumed to be commutative for another cochain complex

$$\mathcal{H} \longrightarrow \Lambda^0(M_{\mathrm{T}};\mathcal{H}) \xrightarrow{d_{\tilde{\mathcal{L}}}} \Lambda^1(M_{\mathrm{T}};\mathcal{H}) \xrightarrow{d_{\tilde{\mathcal{L}}}} ... \xrightarrow{d_{\tilde{\mathcal{L}}}} \Lambda^{m+2}(M_{\mathrm{T}};\mathcal{H}) \xrightarrow{d_{\tilde{\mathcal{L}}}} 0. \tag{10.184}$$

Here the generalized anti-derivation

$$d_{\tilde{\mathcal{L}}} := \sum_{j=1}^{2} dt_j \wedge \tilde{\mathrm{L}}_j(t;x|\partial) \tag{10.185}$$

with

$$\tilde{\mathrm{L}}_j = \partial/\partial t_j - \tilde{L}_j(t;x|\partial), \tag{10.186}$$

$$\tilde{L}_j(t;x|\partial) := \sum_{|\alpha|=0}^{n_j(\tilde{L})} \tilde{a}_\alpha^{(j)}(t;x)\frac{\partial^{|\alpha|}}{\partial x^\alpha},$$

where coefficients $\tilde{a}_\alpha^{(j)} \in C^1(\mathrm{T}; C^\infty(M; \mathrm{End}\,\mathbb{C}^N),\ 0 \leq |\alpha| \leq n_j(L)$, and $n_j(\tilde{L}) := n_j(L) \in \mathbb{Z}_+,\ j = 1,2$. The corresponding isomorphisms $\tilde{B}_\lambda^{(m)}$: $\mathcal{H}^m_{\Lambda(\mathcal{L}),+}(M_\mathrm{T}) \longrightarrow H_m(M_\mathrm{T};\mathbb{C})$, $\lambda \in \Sigma$, act, by definition, as follows:

$$\tilde{B}_\lambda^{(m)}(\tilde{\psi}^{(0)}(\mu)dx) = S_{(t;x)}^{(m)} \int_{\partial S_{(t;x)}^{(m)}} \tilde{\Omega}^{(m-1)}[\tilde{\varphi}^{(0)}(\lambda), \tilde{\psi}^{(0)}(\mu)dx], \tag{10.187}$$

where $\tilde{\varphi}^{(0)}(\lambda) \in \tilde{\mathcal{H}}_0^* \subset \mathcal{H}^0_{\Lambda(\mathcal{L}^*),-}(M_\mathrm{T}),\ \ \lambda \in (\sigma(\tilde{L}) \cap \bar{\sigma}(\tilde{L}^*)) \times \Sigma_\sigma$, and

$$\tilde{\mathcal{H}}_0^* := \{\tilde{\varphi}^{(0)}(\lambda) \in \mathcal{H}^m_{\Lambda(\mathcal{L}^*),-}(M_\mathrm{T}) : d^*_{\tilde{\mathcal{L}}}\tilde{\varphi}^{(0)}(x) = 0, \tilde{\varphi}^{(0)}(\lambda)|_{\tilde{\Gamma}} = 0, \lambda \in \Sigma\} \tag{10.188}$$

for some hypersurface $\tilde{\Gamma} \subset\ M_\mathrm{T}$. Define the following closed subspace

$$\tilde{\mathcal{H}}_0 := \{\tilde{\psi}^{(0)}(\mu) \in \mathcal{H}^0_{\Lambda(\mathcal{L}^*),-}(M_\mathrm{T}) : d^*_{\tilde{\mathcal{L}}}\tilde{\psi}^{(0)}(\lambda) = 0, \tilde{\psi}^{(0)}(\mu)|_{\tilde{\Gamma}} = 0, \mu \in \Sigma\} \tag{10.189}$$

for the hyperspace $\tilde{\Gamma} \subset\ M_\mathrm{T}$, introduced above.

Suppose now that the elements (10.182) belong to the closed subspace (10.189), that is

$$d_{\tilde{\mathcal{L}}}\tilde{\psi}^{(0)}(\mu) = 0. \tag{10.190}$$

Define as in (10.189) a closed subspace $\tilde{\mathcal{H}}_0^* \subset \mathcal{H}^m_{\Lambda(\mathcal{L}^*),-}(M_\mathrm{T})$ as

$$\mathcal{H}_0 := \{\psi^{(0)}(\lambda) \in \mathcal{H}^0_{\Lambda(\mathcal{L}^*),-}(M_\mathrm{T}) : d_{\mathcal{L}}\psi^{(0)}(\lambda) = 0, \psi^{(0)}(\lambda)|_{\Gamma} = 0, \lambda \in \Sigma\} \tag{10.191}$$

for all $\mu \in \Sigma$. Owing to the commutativity of the diagram (10.183), there exist two invertible maps

$$\mathbf{\Omega}_\pm : \mathcal{H}_0 \to \tilde{\mathcal{H}}_0, \tag{10.192}$$

which depend on how they are extended over the whole Hilbert space $\mathcal{H}^m_{\Lambda,-}(M_\mathrm{T})$. Extend the operators (10.192) to all of $\mathcal{H}^m_{\Lambda,-}(M_\mathrm{T})$ by means of the standard method [340, 365] of variation of constants, taking into account that the kernels $\Omega_{(t;x)}(\lambda,\mu), \Omega_{(t_0;x_0)}(\lambda,\mu) \in L^2_{(\rho)}(\Sigma;\mathbb{C}) \otimes L^2_{(\rho)}(\Sigma;\mathbb{C})$,

$\lambda, \mu \in \Sigma$, satisfy

$$\begin{aligned}
&\Omega_{(t;x)}(\lambda,\mu) - \Omega_{(t_0;x_0)}(\lambda,\mu) \qquad (10.193)\\
&= \int_{\partial S^{(m)}_{(t;x)}} \Omega^{(m-1)}[\varphi^{(0)}(x), \psi^{(0)}(\mu)dx] - \int_{\partial S^{(m)}_{(t_0;x_0)}} \Omega^{(m-1)}[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx]\\
&= \int_{S^{(m)}_{\pm}(\sigma^{(m-1)}_{(t;x)}, \sigma^{(m-1)}_{(t_0;x_0)})} d\Omega^{(m-1)}[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx]\\
&= \int_{S^{(m)}_{\pm}(\sigma^{(m-1)}_{(t;x)}, \sigma^{(m-1)}_{(t_0;x_0)})} Z^{(m)}[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx],
\end{aligned}$$

where the m-dimensional open manifolds $S^{(m)}_{\pm}(\sigma^{(m-1)}_{(t;x)}, \sigma^{(m-1)}_{(t_0;x_0)}) \subset M_{\mathrm{T}}$ are spanned smoothly without self-intersection between two homologous cycles $\sigma^{(m-1)}_{(t;x)} = \partial S^{(m)}_{(t;x)}$ and $\sigma^{(m-1)}_{(t_0;x_0)} = \partial S^{(m)}_{(t_0;x_0)} \in C_{m-1}(M_{\mathrm{T}};\mathbb{C})$ such that the boundary $\partial(S^{(m)}_{+}(\sigma^{(m-1)}_{(t_0;x_0)}, \sigma^{(m-1)}_{(t_0;x_0)}) \cup S^{(m)}_{-}(\sigma^{(m-1)}_{(t;x)}, \sigma^{(m-1)}_{(t_0;x_0)})) = \varnothing$.

Making use of the relationship (10.193), it is easy to obtain the following integral operator expressions in $\mathcal{H}_{-}$:

$$\begin{aligned}
\mathbf{\Omega}_{\pm} = \mathbf{1} - \int_{\Sigma} d\rho(\eta)\tilde{\psi}^{(0)}(\xi)\Omega^{-1}_{(t_0;x_0)}(\xi,\eta)\\
\times \int_{S^{(m)}_{\pm}(\sigma^{(m-1)}_{(t;x)}, \sigma^{(m-1)}_{(t_0;x_0)})} Z^{(m)}[\varphi^{(0)}(\eta), (\cdot)dx]
\end{aligned} \qquad (10.194)$$

defined for fixed pairs $(\varphi^{(0)}(\xi), \psi^{(0)}(\eta)) \in \mathcal{H}^{*}_{0} \times \mathcal{H}_{0}$ and $(\tilde{\varphi}^{(0)}(\xi), \tilde{\psi}^{(0)}(\mu)) \in \tilde{\mathcal{H}}^{*}_{0} \times \tilde{\mathcal{H}}_{0}$, $\lambda, \mu \in \Sigma$. these operators are bounded, invertible and of Volterra type [78, 115, 145, 272] on the whole Hilbert space $\mathcal{H}$. Moreover, for the differential operators $\tilde{\mathrm{L}}_j : \mathcal{H} \longrightarrow \mathcal{H}$ one readily finds that

$$\tilde{\mathrm{L}}_j = \mathbf{\Omega}_{\pm}\mathrm{L}_j\mathbf{\Omega}^{-1}_{\pm} \qquad (10.195)$$

for $j = 1, 2$, where the left-hand side of (10.195) does not depend on signs "$\pm$" of the right-hand sides. Whence, the Volterra integral operators (10.194) are Delsarte–Darboux transmutation operators mapping a given set $\mathcal{L}$ of differential operators into a new set $\tilde{\mathcal{L}}$ of differential operators transformed via the Delsarte expressions (10.195).

Suppose now that all of the differential operators $L_j(t;x|\partial)$, $j = 1, 2$, considered above do not depend on the variable $t \in \mathrm{T}$. Then, one can take

$$
\begin{aligned}
&\mathcal{H}_0 := \{\psi_\mu^{(0)}(\xi) \in L^2_{-}(M;\mathbb{C}^N) : L_j\psi_\mu^{(0)}(\xi) = \mu_j\psi_\mu^{(0)}(\xi),\\
&\quad j=1,2,\ \psi_\mu^{(0)}(\xi)|_{\tilde{\Gamma}} = 0, \mu=(\mu_1,\mu_2)\in\sigma(\tilde{L})\cap\overline{\sigma}(L^*), \xi\in\Sigma_\sigma\}\\
&\tilde{\mathcal{H}}_0 := \{\tilde{\psi}_\mu^{(0)}(\xi) \in L^2_{\cdot-}(M;\mathbb{C}^N) : \tilde{L}_j\tilde{\psi}_\mu^{(0)}(\xi) = \mu_j\tilde{\psi}_\mu^{(0)}(\xi),\\
&\quad j=1,2,\ \tilde{\psi}_\mu^{(0)}(\xi)|_{\tilde{\Gamma}} = 0, \mu=(\mu_1,\mu_2)\in\sigma(\tilde{L})\cap\overline{\sigma}(L^*), \xi\in\Sigma_\sigma\}\\
&\mathcal{H}_0^* := \{\varphi_\lambda^{(0)}(\eta) \in L^2_{\cdot-}(M;\mathbb{C}^N) : L_j^*\varphi_\lambda^{(0)}(\eta) = \bar{\lambda}_j\varphi_\lambda^{(0)}(\eta), \qquad (10.196)\\
&\quad j=1,2, \varphi_\lambda^{(0)}(\eta)|_{\tilde{\Gamma}} = 0, \lambda=(\lambda_1,\lambda_2)\in\sigma(\tilde{L})\cap\overline{\sigma}(L^*), \eta\in\Sigma_\sigma\}\\
&\tilde{\mathcal{H}}_0^* := \{\tilde{\varphi}_\lambda^{(0)}(\eta) \in L^2_{\cdot-}(M;\mathbb{C}^N) : \tilde{L}_j^*\tilde{\varphi}_\lambda^{(0)}(\eta) = \bar{\lambda}_j\varphi_\lambda^{(0)}(\eta),\\
&\quad j=1,2,\ \tilde{\varphi}_\lambda^{(0)}(\eta)|_{\tilde{\Gamma}} = 0, \lambda=(\lambda_1,\lambda_2)\in\sigma(\tilde{L})\cap\overline{\sigma}(L^*), \eta\in\Sigma_\sigma\}
\end{aligned}
$$

and construct the corresponding Delsarte–Darboux transmutation operators

$$
\begin{aligned}
\mathbf{\Omega}_\pm = 1 - &\int_{\sigma(\tilde{L})\cap\overline{\sigma}(L^*)} d\rho_\sigma(\lambda)\int_{\Sigma_\sigma\times\Sigma_\sigma} d\rho_{\Sigma_\sigma}(\xi)d\rho_{\Sigma_\sigma}(\eta) \qquad (10.197)\\
&\times\int_{S_\pm^{(m)}\sigma^{(m-1)}_{(t_0;x_0)},\sigma^{(m-1)}_{(t_0;x_0)}} dx\tilde{\psi}_\lambda^{(0)}(\xi)\Omega_{x_0}^{-1}(\lambda;\xi;\eta)\bar{\varphi}_\lambda^{(0),\intercal}(\eta)(\cdot)
\end{aligned}
$$

acting in the Hilbert space $L^2_+(M;\mathbb{C}^N)$, where for any $(\lambda;\xi,\eta)\in(\sigma(\tilde{L})\cap\overline{\sigma}(L^*))\times\Sigma_\sigma^2$, the kernels

$$
\Omega_{(x_0)}(\lambda;\xi,\eta) := \int_{\sigma^{(m-1)}_{x_0}} \Omega^{(m-1)}[\varphi_\lambda^{(0)}(\xi),\psi_\lambda^{(0)}(\eta)dx] \qquad (10.198)
$$

for $(\xi,\eta)\in\Sigma_\sigma^2$ and every $\lambda\in\sigma(\tilde{L})\ \cap\overline{\sigma}(L^*)$ belong to $L^2_{(\rho)}(\Sigma_\sigma;\mathbb{C})\otimes L^2_{(\rho)}(\Sigma_\sigma;\mathbb{C})$. Moreover, as $\partial\mathbf{\Omega}_\pm/\partial t_j = 0$, $j=1,2$, one can readily obtain that the differential expressions

$$
\tilde{L}_j(x|\partial) := \mathbf{\Omega}_\pm L_j(x|\partial)\mathbf{\Omega}_\pm^{-1} \qquad (10.199)
$$

$j=1,2$, which obviously commute.

The Volterra operators (10.197) possess some additional properties. For example, the following Fredholm integral operator in H :

$$
\mathbf{\Omega} := \mathbf{\Omega}_+^{-1}\mathbf{\Omega}_-, \qquad (10.200)
$$

which can be written in the form

$$
\mathbf{\Omega}=\mathbf{1}+\Phi(\mathbf{\Omega}), \qquad (10.201)
$$

where the operator $\Phi(\mathbf{\Omega})\in\mathcal{B}_\infty(H)$, is compact. Moreover, owing to the relationships (10.199), it is easy to show that the commutator conditions

$$
[\mathbf{\Omega}, L_j] = 0 \qquad (10.202)
$$

hold for $j = 1, 2$.

Denote now by $\hat{\Phi}(\mathbf{\Omega}) \in H_- \otimes H_-$ and $\hat{K}_+(\mathbf{\Omega})$, $\hat{K}_-(\mathbf{\Omega}) \in H_- \otimes H_-$ the kernels corresponding [28, 32] to operators $\Phi(\mathbf{\Omega}) \in \mathcal{B}_\infty(H)$ and $\mathbf{\Omega}_\pm - \mathbf{1} \in \mathcal{B}_\infty(H)$. Then, since the supports supp $K_+(\mathbf{\Omega}) \cap \mathrm{supp} K_-(\mathbf{\Omega}) = \sigma_x^{(m-1)} \cup \sigma_{x_0}^{(m-1)}$, (10.200) and (10.201) imply the well-known Gelfand–Levitan–Marchenko linear integral equation

$$\hat{K}_+(\mathbf{\Omega}) + \Phi(\hat{\mathbf{\Omega}}) + \hat{K}_+(\mathbf{\Omega})_+ * \hat{\Phi}(\mathbf{\Omega}) = \hat{K}_-(\mathbf{\Omega}), \tag{10.203}$$

allowing to find the factorizing Fredholm operator (10.200) kernel $\hat{K}_+(\mathbf{\Omega})(x; y) \in H_- \otimes H_-$ for all $y \in \mathrm{supp} K_+(\mathbf{\Omega})$. The conditions (10.202) can be rewritten as

$$(L_{j,ext} \otimes \mathbf{1})\hat{\Phi}(\mathbf{\Omega}) = (1 \otimes L_{j,ext}^*)\hat{\Phi}(\mathbf{\Omega}) \tag{10.204}$$

for $j = 1, 2$, where $L_{j,ext} \in \mathcal{L}(H_-)$ and their adjoint $L_{j,ext}^* \in \mathcal{L}(H_-)$ are the corresponding extensions [28, 32, 342] of the differential operators L_j and $L_j^* \in \mathcal{L}(H)$.

For the relationships (10.199) one has [28, 342] kernel conditions similar to (10.204):

$$(\tilde{L}_{j,ext} \otimes \mathbf{1})\hat{K}_\pm(\mathbf{\Omega}) = (\mathbf{1} \otimes L_{j,ext}^*)\hat{K}_\pm(\mathbf{\Omega}), \tag{10.205}$$

where as above, $\tilde{L}_{j,ext} \in \mathcal{L}(H_-)$, $j = 1, 2$, are the corresponding rigging extensions of the differential operators $\tilde{L}_j \in \mathcal{L}(H)$.

We now turn to the question about the general differential and spectral structure of the transformed operator expression (10.195). It is clear that the conditions (10.203) and (10.204) on the kernels $\hat{K}_\pm(\mathbf{\Omega}) \in \mathcal{H}_- \otimes \mathcal{H}_-$ of Delsarte–Darboux transmutation operators are necessary for the operator expressions (10.195) to exist and be differential. Are these conditions also sufficient? A partially affirmative answer will be given in the sequel.

To study this question let us consider the Volterra operators (10.194) and (10.197) with kernels satisfying the conditions (10.203) and (10.204), assuming that suitable oriented manifolds $S_\pm^{(m)}(\sigma_{(t;x)^{(m-1)}}, \sigma_{(t_0;x_0)^{(m-1)}}) \in C_m(M_\mathrm{T}; \mathbb{C})$ are given as

$$\begin{aligned} S_+^{(m)}(\sigma_{(t;x)^{(m-1)}}, \sigma_{(t_0;x_0)^{(m-1)}}) &= \{(t'; x') \in M_\mathrm{T} : \\ &\qquad t' = P(t; x|x'),\ t \in \mathrm{T}\}, \\ S_-^{(m)}(\sigma_{(t;x)^{(m-1)}}, \sigma_{(t_0;x_0)^{(m-1)}}) &= \{(t'; x') \in M_\mathrm{T} : \\ &\qquad t' = P(t; x|x') \in \mathrm{T} \backslash [t_0, t]\}, \end{aligned} \tag{10.206}$$

where $P \in C^\infty(M_\mathrm{T} \times M; \mathrm{T})$ is smooth and such that the boundaries $\partial S_\pm^{(m)}(\sigma_{(t;x)}^{(m-1)}, \sigma_{(t_0;x_0)}^{(m-1)}) = \pm(\sigma_{(t;x)}^{(m-1)} - \sigma_{(t_0;x_0)}^{(m-1)})$ with cycles $\sigma_{(t;x)}^{(m-1)}$ and

$\sigma^{(m-1)}_{(t_0;x_0)} \in \mathcal{K}(M_{\mathrm{T}})$ are homologous to each other for any choice of points $(t_0;x_0)$ and $(t;x) \in M_{\mathrm{T}}$. Then simple but cumbersome calculations, based on considerations from [141] and [117], show that the resulting expressions on the right-hand side of

$$\tilde{\mathrm{L}} = \mathrm{L} + [\mathrm{K}_{\pm}(\boldsymbol{\Omega}), \mathrm{L}] \cdot \boldsymbol{\Omega}_{\pm}^{-1} \tag{10.207}$$

are exactly equal and differential if there is such an expression for an operator $\mathrm{L} \in \mathcal{L}(\mathcal{H})$.

As for the inverse operators $\boldsymbol{\Omega}_{\pm}^{-1} \in \mathcal{L}(\mathcal{H})$ in (10.207), it is clear owing to the functional symmetry between closed subspaces $\mathcal{H}_0$ and $\tilde{\mathcal{H}}_0 \subset \tilde{\mathcal{H}}_-$ that the defining relationships (10.192) and (10.182) are reversible, that is there exist inverse operators $\boldsymbol{\Omega}_{\pm}^{-1} : \tilde{\mathcal{H}}_0 \to \mathcal{H}_0$, such that

$$\boldsymbol{\Omega}_{\pm}^{-1} : \tilde{\psi}^{(0)}(\lambda) \longrightarrow \psi^{(0)}(\lambda) := \tilde{\psi}^{(0)}(\lambda) \cdot \tilde{\Omega}^{-1}_{(t;x)} \tilde{\Omega}_{(t;x)} \tag{10.208}$$

for suitable kernels $\tilde{\Omega}_{(t;x)}(\lambda,\mu)$ and $\tilde{\Omega}_{(t_0;x_0)}(\lambda,\mu) \in L^2_{(\rho)}(\Sigma;\mathbb{C}) \otimes L^2_{(\rho)}(\Sigma;\mathbb{C})$, related naturally with the transformed differential expression $\tilde{\mathrm{L}} \in \mathcal{L}(\mathcal{H})$. Consequently, it follows from the expressions (10.208) that one has, similar to (10.197), the inverse integral operators

$$\begin{aligned}\boldsymbol{\Omega}_{\pm}^{-1} = \mathbf{1} - \int_{\Sigma} d\rho(\xi) \int_{\Sigma} d\rho(\eta) \psi^{(0)}(\xi) \tilde{\Omega}^{-1}_{t_0;x_0}(\xi,\eta) \\ \times \int_{S^{(m)}_{\pm}(\sigma^{(m-1)}_{(t;x)}, \sigma^{(m-1)}_{(t_0;x_0)})} \tilde{Z}^{(m)}[\tilde{\varphi}^{(0)}(\eta), (\cdot) dx]\end{aligned} \tag{10.209}$$

defined for fixed pairs $(\tilde{\varphi}^{(0)}(\xi), \tilde{\psi}^{(0)}(\eta)) \in \tilde{\mathcal{H}}_0^* \times \tilde{\mathcal{H}}_0$ and $(\varphi^{(0)}(\xi), \psi^{(0)}(\eta)) \in \mathcal{H}_0^* \times \mathcal{H}_0$, $\xi, \eta \in \Sigma$, which are bounded invertible operators of Volterra type on $\mathcal{H}$. In particular, the compatibility conditions $\boldsymbol{\Omega}_{\pm}\boldsymbol{\Omega}_{\pm}^{-1} = \mathbf{1} = \boldsymbol{\Omega}_{\pm}^{-1}\boldsymbol{\Omega}_{\pm}$ must be fulfilled in $\mathcal{H}$, involving some restrictions identifying measures ρ and Σ and possible asymptotic conditions of coefficient functions of the differential expression $\mathrm{L} \in \mathcal{L}$. Such restrictions have been mentioned in [201, 403–405], where relationships with the local and nonlocal Riemann problems were discussed.

Within the above framework, one can give a natural interpretation of so-called Bäcklund transformations for the coefficient functions of a given differential operator $\mathrm{L} \in \mathcal{L}(\mathcal{H})$. In particular, following the symbolic considerations in [230], we reinterpret the approach devised there for constructing the Bäcklund transformations by making use of the techniques from the theory of Delsarte transmutation operators. Let us define two different Delsarte–Darboux transformed differential operators

$$\mathrm{L}_1 = \boldsymbol{\Omega}_{1,\pm} \mathrm{L} \boldsymbol{\Omega}_{1,\pm}^{-1}, \qquad \mathrm{L}_2 = \boldsymbol{\Omega}_{2,\pm} \mathrm{L} \boldsymbol{\Omega}_{2,\pm}^{-1}, \tag{10.210}$$

where $\mathbf{\Omega}_{1,+}, \mathbf{\Omega}_{2,-} \in \mathcal{L}(\mathcal{H})$ are Delsarte transmutation Volterra operators in $\mathcal{H}$ with Borel spectral measures ρ_1 and ρ_2 on Σ, such that the following conditions

$$\mathbf{\Omega}_{1,+}^{-1}\mathbf{\Omega}_{1,-} = \mathbf{\Omega} = \mathbf{\Omega}_{2,+}^{-1}\mathbf{\Omega}_{2,-} \tag{10.211}$$

are satisfied. Now it follows readily from the conditions (10.210) and relationships (10.211) that the operator $\mathrm{B} := \mathbf{\Omega}_{2,-}\mathbf{\Omega}_{1,+}^{-1} \in \mathcal{L}(\mathcal{H})$ satisfies

$$\mathrm{L}_2\mathrm{B} = \mathrm{B}\mathrm{L}_1, \quad \mathbf{\Omega}_{2,\pm}\mathrm{B} = \mathrm{B}\mathbf{\Omega}_{1,\pm}, \tag{10.212}$$

which motivates the following definition.

Definition 10.6. An invertible symbolic mapping $\mathrm{B} : \mathcal{L}(\mathcal{H}) \longrightarrow \mathcal{L}\,(\mathcal{H})$ will be called a Darboux–Bäcklund transformation of an operator $\mathrm{L}_1 \in \mathcal{L}(\mathcal{H})$ into the operator $\mathrm{L}_2 \in \mathcal{L}(\mathcal{H})$ if

$$[\mathrm{QB}, \mathrm{L}_1] = 0 \tag{10.213}$$

for some linear differential operator $\mathrm{Q} \in \mathcal{L}(\mathcal{H})$.

The condition (10.213) can be realized as follows: Take any differential expression $\mathrm{q} \in \mathcal{L}(\mathcal{H})$ satisfying the symbolic equation

$$[\mathrm{qB}, \mathrm{L}] = 0. \tag{10.214}$$

Then, using a transformation like (10.210), from (10.211) one finds that

$$[\mathrm{QB}, \mathrm{L}_1] = 0, \tag{10.215}$$

where owing to (10.212)

$$\mathrm{QB} := \mathbf{\Omega}_{1,+} q \mathrm{B} \mathbf{\Omega}_{1,+}^{-1} = \mathbf{\Omega}_{1,+} \mathrm{q} \mathbf{\Omega}_{2,+}^{-1}\mathrm{B}. \tag{10.216}$$

Therefore, the expression $\mathrm{Q} = \mathbf{\Omega}_{1,+} q \mathbf{\Omega}_{2,+}^{-1}$ is also seen to be differential, owing to the conditions (10.212).

The considerations above for a symbolic mapping $\mathrm{B} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ give rise to an effective tool for constructing self-Bäcklund transformations for coefficients of differential operator expressions $\mathrm{L}_1, \mathrm{L}_2 \in \mathcal{L}(\mathcal{H})$, which have many applications [92, 253, 326, 365, 406] in spectral and soliton theories.

Let us investigate the structure of Delsarte–Darboux transformations for a polynomial differential operators pencil

$$\mathrm{L}(\lambda; x|\partial) := \sum_{j=0}^{n(L)} L_j(x|\partial)\lambda^j, \tag{10.217}$$

where $n(L) \in \mathbb{Z}_+$ and $\lambda \in \mathbb{C}$ is a complex parameter. We seek the Delsarte–Darboux transformations $\mathbf{\Omega}_{\lambda,\pm} \in \mathcal{L}(H)$, $\lambda \in \mathbb{C}$, corresponding to (10.217), such that for some polynomial differential operators pencil $\tilde{\mathrm{L}}(\lambda;x|\partial)\in\mathcal{L}(\mathcal{H})$ the following Delsarte–Lions [95] transmutation condition

$$\tilde{\mathrm{L}}\boldsymbol{\Omega}_{\lambda,\pm}=\boldsymbol{\Omega}_{\lambda,\pm}\mathrm{L} \tag{10.218}$$

holds for almost all $\lambda\in\mathbb{C}$. To find such transformations $\boldsymbol{\Omega}_{\lambda\pm}\in\mathcal{L}(H)$, we consider a parameter ($\tau\in\mathbb{R}$) dependent differential operator $\mathrm{L}_\tau(x|\partial)\in\mathcal{L}(\mathcal{H}_\tau)$ of the form

$$\mathrm{L}_\tau(x|\partial):=\sum_{j=0}^{n(L)}L_j(x|\partial)\partial^j/\partial\tau^j, \tag{10.219}$$

acting in the functional space $\mathcal{H}_\tau=C^{q(L)}(\mathbb{R}_\tau;\mathcal{H})$ for some $q(L)\in\mathbb{Z}_+$. Then one can easily construct the corresponding Delsarte–Darboux transformations $\boldsymbol{\Omega}_{\tau,\pm}\in\mathcal{L}(\mathcal{H}_\tau)$ of Volterra type for a differential operator expression

$$\tilde{\mathrm{L}}_\tau(x|\partial):=\sum_{j=0}^{n(L)}\tilde{L}_j(x|\partial)\partial^j/\partial\tau^j, \tag{10.220}$$

if the following Delsarte–Lions [95] transmutation conditions

$$\tilde{\mathrm{L}}_\tau\boldsymbol{\Omega}_{\tau,\pm}=\boldsymbol{\Omega}_{\tau,\pm}\mathrm{L}_\tau \tag{10.221}$$

hold in $\mathcal{H}_\tau$. Thus, making use of the results obtained above, one has

$$\begin{aligned}\boldsymbol{\Omega}_{\tau,\pm}=\mathbf{1}-\int_\Sigma d\rho_\Sigma(\xi)\int_\Sigma d\rho_\Sigma(\eta)\tilde{\psi}_\tau^{(0)}(\lambda;\xi)\Omega^{-1}_{(\tau_0;x_0)}(\lambda;\xi,\eta)\\ \times\int_{S_\pm^{(m)}(\sigma^{(m-1)}_{(\tau;x)},\sigma^{(m-1)}_{(\tau_0;x_0)})}Z^{(m)}[\varphi_\tau^{(0)}(\lambda;\eta),(\cdot)dx]\end{aligned} \tag{10.222}$$

defined by means of the closed subspaces $\mathcal{H}_{\tau,0}\subset\mathcal{H}_{\tau,-}$ and $\mathcal{H}^*_{\tau,0}\subset\mathcal{H}^*_{\tau,-}$:

$$\begin{aligned}\mathcal{H}_{\tau,0}:=\{\psi_\tau^{(0)}(\lambda;\xi)\in\mathcal{H}_{\tau,-}:\mathrm{L}_\tau\psi_\tau^{(0)}(\lambda;\xi)=0,\\ \psi_\tau^{(0)}(\lambda;\xi)|_{\tau=0}=\psi^{(0)}(\lambda;\xi)\in\mathcal{H},\ \mathrm{L}\psi^{(0)}(\lambda;\xi)=0,\\ \psi^{(0)}(\lambda;\xi)|_\Gamma=0,\ \lambda\in\mathbb{C},\ \xi\in\Sigma\},\end{aligned}$$

$$\begin{aligned}\mathcal{H}^*_{\tau,0}:=\{\varphi_\tau^{(0)}(\lambda;\eta)\in\mathcal{H}^*_{\tau,-}:\mathrm{L}_\tau\varphi_\tau^{(0)}(\lambda;\eta)=0,\\ \varphi_\tau^{(0)}(\lambda;\eta)|_{\tau=0}=\varphi^{(0)}(\lambda;\eta)\in\mathcal{H}^*,\ \mathrm{L}\varphi^{(0)}(\lambda;\eta)=0,\\ \varphi^{(0)}(\lambda;\eta)|_\Gamma=0,\ \lambda\in\mathbb{C},\ \eta\in\Sigma\}.\end{aligned} \tag{10.223}$$

Recalling now that our operators $L_j \in \mathcal{L}(\mathcal{H})$, $0 \leq j \leq r(L)$, do not depend on the parameter $\tau \in \mathbb{R}$, one can readily infer from (10.222) that

$$\mathbf{\Omega}_{\pm} = \mathbf{1} - \int_{\Sigma} d\rho_{\Sigma}(\xi) \int_{\Sigma} d\rho_{\Sigma}(\eta) \tilde{\psi}^{(0)}(\lambda;\xi) \Omega^{-1}_{(x_0)}(\lambda;\xi,\eta) \times \int_{S^{(m)}_{\pm}(\sigma^{(m-1)}_{(x)},\sigma^{(m-1)}_{(x_0)})} Z_0^{(m)}[\varphi^{(0)}(\lambda;\eta),(\cdot)dx], \tag{10.224}$$

where $\sigma_x^{(m-1)} := \sigma^{(m-1)}_{(\tau_0;x)}$, $\sigma^{(m-1)}_{x_0} := \sigma^{(m-1)}_{(\tau_0;x_0)} \in C_{m-1}(\mathbb{R}^m;\mathbb{C})$ and

$$Z_0^{(m)}[\varphi^{(0)}(\lambda;\eta),\psi^{(0)}dx] := Z^{(m)}[\varphi^{(0)}_{\tau}(\lambda;\eta),\psi^{(0)}_{\tau}dx]|_{d\tau=0}. \tag{10.225}$$

The closed subspaces $\mathcal{H}_0 \in \mathcal{H}_-$ and $\mathcal{H}_0^* \in \mathcal{H}_-^*$ corresponding to (10.224) are given as

$$\mathcal{H}_0 := \{\psi^{(0)}(\lambda;\xi) \in \mathcal{H}_- : \mathrm{L}\psi^{(0)}(\lambda;\xi) = 0,\ \psi^{(0)}(\lambda;\xi)|_{\Gamma} = 0,\ \lambda \in \mathbb{C},\ \xi \in \Sigma\}, \tag{10.226}$$

$$\mathcal{H}^*_{\tau,0} := \{\varphi^{(0)}(\lambda;\eta) \in \mathcal{H}^*_- : \mathrm{L}\varphi^{(0)}(\lambda;\eta) = 0,\ \varphi^{(0)}(\lambda;\eta)|_{\Gamma} = 0,\ \lambda \in \mathbb{C},\ \eta \in \Sigma\}.$$

Therefore, one can use the expressions (10.224) to construct the Delsarte-Darboux transformed linear differential pencil $\tilde{\mathrm{L}} \in \mathcal{L}(\mathcal{H})$, whose coefficients are related to those of the pencil $\mathrm{L} \in \mathcal{L}(\mathcal{H})$ via Bäcklund type relationships useful for applications (see [161, 201, 342, 365, 396]) in soliton theory.

10.9 Delsarte–Darboux transmutation operators for special multi-dimensional expressions and their applications

Example 10.1. A perturbed self-adjoint Laplace operator in $\mathbb{R}^n$.

Consider the Laplace operator $-\Delta_m$ in $H := L(\mathbb{R}^m;\mathbb{C})$ perturbed by the multiplication operator on a function $q \in W_2^2(\mathbb{R}^m;\mathbb{C})$, that is the operator

$$L(x|\partial) := -\Delta_m + q(x), \tag{10.227}$$

where $x \in \mathbb{R}^m$. The operator (13.62) is self-adjoint in H. Applying the results from Section 10.7 to the differential expression (13.62) in the Hilbert space H, one obtains the following invertible Delsarte–Darboux transmutation operators:

$$\mathbf{\Omega}_{\pm} = \mathbf{1} - \int_{\sigma(L)} d\rho_{\sigma}(\xi) \int_{\sigma(L)} d\rho_{\sigma}(\xi) \int_{\Sigma_{\sigma}} d\rho_{\Sigma_{\sigma}}(\xi) \int_{\Sigma_{\sigma}} d\rho_{\Sigma_{\sigma}}(\eta) \times \tilde{\psi}^{(0)}(\lambda;\xi)\Omega^{-1}_{(x_0)}(\lambda;\xi,\eta) \int^{(0)}_{S^{(m)}_{\pm}(\sigma^{(m-1)}_{(x)},\sigma^{(m-1)}_{(x_0)})} dy \bar{\varphi}^{(0)\intercal}(\lambda;\eta),(\cdot), \tag{10.228}$$

where $\sigma_x^{(m-1)} \in \mathcal{K}(\mathbb{R}^m)$ is a closed, possibly non-compact, simplicial hypersurface in $\mathbb{R}^m$ parametrized by a running point $x \in \sigma_x^{(m-1)}$, and $\sigma_{x_0}^{(m-1)} \in \mathcal{K}(\mathbb{R}^m)$ is a suitable simplicial hypersurface in $\mathbb{R}^m$ that is homologous to $\sigma_x^{(m-1)}$ and parametrized by a point $x_0 \in \sigma_{x_0}^{(m-1)}$. There exist exactly two m-dimensional subspaces spanning them, say $S_\pm^{(m)}(\sigma_x^{(m-1)}, \sigma_{x_0}^{(m-1)}) \in \mathcal{K}(\mathbb{R}^m)$, such that $S_+^{(m)}(\sigma_x^{(m-1)}, \sigma_{x_0}^{(m-1)}) \cup S_-^{(m)}(\sigma_x^{(m-1)}, \sigma_{x_0}^{(m-1)}) = \mathbb{R}^m$. Taking into account these subspaces, one can rewrite the Delsarte–Darboux transmutation operators (13.63) for (13.62) concisely as

$$\mathbf{\Omega}_\pm = \mathbf{1} + \int_{S_\pm^{(m)}(\sigma_x^{(m-1)}, \sigma_{x_0}^{(m-1)})} dy \hat{K}_\pm(\mathbf{\Omega})(x;y)(\cdot), \tag{10.229}$$

where, as before, $x \in \sigma_x^{(m-1)}$ and the kernels $\hat{K}_\pm(\mathbf{\Omega}) \in H_- \otimes H_-$ satisfy the equations (10.206), or equivalently,

$$\begin{aligned} &-\Delta_m(x;\partial)\hat{K}_\pm(\mathbf{\Omega})(x;y) + \Delta_m(y;\partial)\hat{K}_\pm(\mathbf{\Omega})(x;y) \\ &= (q(y) - \tilde{q}(x))\hat{K}_\pm(\mathbf{\Omega})(x;y) \end{aligned} \tag{10.230}$$

for all $x, y \in \operatorname{supp}\hat{K}_\pm(\mathbf{\Omega})$. Take for simplicity, a non-compact closed simplicial hypersurface $\sigma_x^{(m-1)} = \sigma_{x,\gamma}^{(m-1)} := \{y \in \mathbb{R}^m :< x - y, \gamma >= 0\}$ and the degenerate simplicial cycle $\sigma_{x_0}^{(m-1)} := x_0 = \infty \in \mathbb{R}^m$, where $\gamma \in \mathbb{S}^{m-1}$ is an arbitrary versor, $||\gamma|| = 0$. Then, evidently

$$S_\pm^{(m)}(\sigma_{x,\gamma}^{(m-1)}, \sigma_\infty^{(m-1)}) := S_{\pm\gamma,x}^{(m)} = \{y \in \mathbb{R}^m :< x - y, \pm\gamma > \ \geq 0\} \tag{10.231}$$

and our transmutation operators (13.64) take the form

$$\mathbf{\Omega}_{\pm\gamma} = \mathbf{1} + \int_{S_{\pm\gamma,x}^{(m)}} dy \hat{K}_{\pm\gamma}(\mathbf{\Omega})(x;y)(\cdot), \tag{10.232}$$

where supp $\hat{K}_\pm(\mathbf{\Omega}) = S_{\pm\gamma,x}^{(m)}$, $S_{+\gamma,x}^{(m)} \cap S_{-\gamma,x}^{(m)} = \sigma_{x,\gamma}^{(m-1)} \cup \sigma_\infty^{(m-1)}$ and $S_{+\gamma,x}^{(m)} \cup S_{-\gamma,x}^{(m)} = \mathbb{R}^m$ for any direction $\gamma \in \mathbb{S}^{m-1}$.

Invertible transmutation Volterra operators such as (13.68) were first constructed by Faddeev [117] for the self-adjoint perturbed Laplace operator (13.62) in $\mathbb{R}^3$. He called them [117] transformation operators with a Volterra direction $\gamma \in \mathbb{S}^{m-1}$. It is easy to see that Faddeev's expressions (13.68) are very special cases of the general expressions (13.64) obtained above.

Now making use of (13.64), we define the following Fredholm operator in the Hilbert space H :

$$\mathbf{\Omega} := (\mathbf{1} + K_+(\mathbf{\Omega}))^{-1}(\mathbf{1} + K_-(\mathbf{\Omega})) = \mathbf{1} + \Phi(\mathbf{\Omega}) \tag{10.233}$$

with the compact part $\Phi(\mathbf{\Omega}) \in \mathcal{B}_\infty(H)$. Then the commutation equality

$$[L, \Phi(\mathbf{\Omega})] = \mathbf{0} \tag{10.234}$$

together with the Gelfand–Levitan–Marchenko equation

$$K_+(\mathbf{\Omega}) + \hat{\Phi}(\mathbf{\Omega}) + \hat{K}_+(\mathbf{\Omega}) + \hat{\Phi}(\mathbf{\Omega}) = \hat{K}_-(\mathbf{\Omega}) \tag{10.235}$$

are satisfied for the corresponding kernels $\hat{K}_\pm(\mathbf{\Omega})$ and $\hat{\Phi}(\mathbf{\Omega}) \in H_- \otimes H_-$.

In [117] there is a thorough analysis of the spectral structure of the kernels $\hat{K}_\pm(\mathbf{\Omega}) \in H_- \otimes H_-$ in (13.68) using the analytical properties of the corresponding Green's functions of the operator (13.62). As one can see from (13.63), these properties depend strongly both on the structure of the spectral measures ρ_σ on $\sigma(L)$ and ρ_{Σ_σ} on Σ_σ and on analytical behavior of the kernel $\Omega_\infty(\lambda; \xi, \eta) \in L^2_{(\rho)}(\Sigma_\sigma; \mathbb{C}) \otimes L^2_{(\rho)}(\Sigma_\sigma; \mathbb{C})$, $\xi, \eta \in \Sigma_\sigma$, for all $\lambda \in \sigma(L)$. It was proved in [117] that for any direction $\gamma \in \mathbb{S}^{m-1}$ the dependence of kernels $\hat{K}_\pm(\mathbf{\Omega}) \in H_- \otimes H_-$ on the regularized determinant of the resolvent $R_\mu(L) \in \mathcal{B}(H)$, $\mu \in \mathbb{C} \backslash \sigma(L)$ is a regular point for the operator (13.62). This dependence can be clarified using the above approach.

Example 10.2. A two-dimensional Dirac type operator.

Let us define in $H := L^2(\mathbb{R}^2; \mathbb{C}^2)$ a two-dimensional Dirac operator

$$\tilde{\mathrm{L}}_1(x; \partial) := \begin{pmatrix} \partial/\partial x_1 & \tilde{u}_1(x) \\ \tilde{u}_2(x) & \partial/\partial x_2 \end{pmatrix}, \tag{10.236}$$

where $x := (x_1, x_2) \in \mathbb{R}^2$, and coefficients $\tilde{u}_j \in W_2^1(\mathbb{R}^2; \mathbb{C})$, $j = 1, 2$. The transformation properties of this operator were studied in [284]. In particular, a special class of the Delsarte–Darboux transmutation operators of the following form was constructed:

$$\mathbf{\Omega}_\pm = \mathbf{1} + \int_{S_\pm^{(2)}(\sigma_x^{(1)}, \sigma_\infty^{(1)})} dy \hat{K}_\pm(\mathbf{\Omega})(x; y)(\cdot), \tag{10.237}$$

where for two orthonormal versors γ_1 and $\gamma_2 \in \mathbb{S}^1$, $||\gamma_1|| = 1 = ||\gamma_2||$, and

$$\begin{aligned} S_+^{(2)}(\sigma_x^{(1)}, \sigma_\infty^{(1)}) &:= \{y \in \mathbb{R}^2 : < x - y, \gamma_1 > \geq 0\} \\ &\cap \{y \in \mathbb{R}^2 : < x - y, \gamma_2 > \geq 0\}, \\ S_-^{(2)}(\sigma_x^{(1)}, \sigma_\infty^{(1)}) &:= \{y \in \mathbb{R}^2 : < x - y, \gamma_1 > \leq 0\} \\ &\cup \{y \in \mathbb{R}^2 : < x - y, \gamma_2 > \leq 0\}. \end{aligned} \tag{10.238}$$

When $< x, \gamma_j >= x_j \in \mathbb{R}$, $j = 1, 2$, the corresponding kernel

$$\hat{K}_+(\mathbf{\Omega}) = \begin{pmatrix} K_{+,11}^{(1)} \delta_{<y-x,\gamma_1>} + K_{+,11}^{(0)}(x; y) & K_{+,12}^{(1)} \delta_{<y-x,\gamma_2>} + K_{+,12}^{(0)}) \\ K_{+,21}^{(1)} \delta_{<y-x,\gamma_1>} + K_{+,21}^{(0)}(x; y) & K_{+,22}^{(1)} \delta_{<y-x,\gamma_2>} + K_{+,22}^{(0)} \end{pmatrix} \tag{10.239}$$

is a Dirac delta-function, which is singular, and partially localized on half-lines $< y - x, \gamma_2 >= 0$ and $< y - x, \gamma_1 >= 0$, with regular all coefficients $K^{(l)}_{+,ij} \in C^1(\mathbb{R}^2 \times \mathbb{R}^2; \mathbb{C})$ for all $i, j = 1, 2$ and $l = 0, 1$. The same property of the transmutation kernels for the perturbed Laplace operator (13.62) was also observed in [117], where it was motivated by the necessary condition for the transformed operator $\tilde{L}(x; \partial) \in \mathcal{L}(H)$ to be differential. As one can check, the same reason for the existence of singularities applies in (13.79).

Let us now consider the general expression akin to (13.64) for the corresponding hypersurfaces $S^{(2)}_{\pm}(\sigma^{(1)}_x, \sigma^{(1)}_\infty)$ spanning a closed non-compact smooth cycle $\sigma^{(1)}_x \in \mathcal{K}(\mathbb{R}^2)$ and the infinite point $\sigma^{(1)}_\infty := \infty \in \mathcal{K}(\mathbb{R}^2)$. A running point $x \in \sigma^{(1)}_x$ is arbitrary but, as usual, fixed. The kernels $\hat{K}_{\pm}(\mathbf{\Omega}) \in H_- \times H_-$ in (13.76) satisfy the standard conditions (10.204) and (10.205), that is

$$(\tilde{\mathrm{L}}_{1,ext} \otimes \mathbf{1})\hat{K}_{\pm}(\mathbf{\Omega}) = (\mathbf{1} \otimes \mathrm{L}^*_{1,ext})\hat{K}_{\pm}(\mathbf{\Omega}), \qquad (10.240)$$
$$[\mathrm{L}_1, \Phi(\mathbf{\Omega})] = 0$$

for some matrix differential Dirac operator $\mathrm{L}_1 \in \mathcal{L}(H)$ of the form (13.62). Together with this Dirac operator, the matrix second order differential operator

$$\tilde{\mathrm{L}}_2(x; \partial) := \mathbf{1}\frac{\partial}{\partial t} + \begin{pmatrix} \frac{\partial^2}{\partial x_1^2} \pm \frac{\partial^2}{\partial x_2^2} - \tilde{v}_2 & -2\frac{\partial \tilde{u}_1}{\partial x_2} \\ -2\frac{\partial \tilde{u}_2}{\partial x_1} & \frac{\partial^2}{\partial x_1^2} \pm \frac{\partial^2}{\partial x_2^2} - \tilde{v}_1 \end{pmatrix} \qquad (10.241)$$

in the parametric space $\mathcal{H} := C^1(\mathbb{R}; H)$ was studied in [284, 285]. A scattering theory was developed and applied to the construction of soliton-like exact solutions to the so-called Davey–Stewartson nonlinear dynamical system. A key element of that analysis is the fact that the operators $\tilde{\mathrm{L}}_1$ and $\tilde{\mathrm{L}}_2 \in \mathcal{L}(H)$ commute. Namely, for the Volterra operators $\mathbf{\Omega}_{\pm} \in \mathcal{L}(\mathcal{H})$ realizing the following Delsarte–Darboux transmutations one has

$$\tilde{\mathrm{L}}_1\mathbf{\Omega}_{\pm} = \mathbf{\Omega}_{\pm}\mathrm{L}_1, \quad \tilde{\mathrm{L}}_2\mathbf{\Omega}_{\pm} = \mathbf{\Omega}_{\pm}\mathrm{L}_2. \qquad (10.242)$$

Here we set

$$\mathrm{L}_1(x; \partial) := \begin{pmatrix} \partial/\partial x_1 & 0 \\ 0 & \partial/\partial x_2 \end{pmatrix}, \qquad (10.243)$$
$$\mathrm{L}_2(x; \partial) := \mathbf{1}\frac{\partial}{\partial t} + \begin{pmatrix} \frac{\partial^2}{\partial x_1^2} \pm \frac{\partial^2}{\partial x_2^2} - \alpha_2(x_2) & 0 \\ 0 & \frac{\partial^2}{\partial x_1^2} \pm \frac{\partial^2}{\partial x_2^2} - \alpha_1(x_1) \end{pmatrix},$$

where $\alpha_j \in W_2^1(\mathbb{R};\mathbb{C})$, $j = 1,2$, are given functions. It is evident that operators (10.243) commute. Then, if the operators $\mathbf{\Omega}_\pm \in \mathcal{L}(H)$ exist and satisfy (10.242), the following commutation condition

$$[\tilde{\mathrm{L}}_1, \tilde{\mathrm{L}}_2] = 0 \tag{10.244}$$

holds, and this was effectively exploited in [284, 285].

Recall now that for the operators $\mathbf{\Omega}_\pm \in \mathcal{L}(H)$ to exist they must satisfy the kernel conditions (13.80) and

$$\begin{aligned} (\tilde{\mathrm{L}}_{2,ext} \otimes \mathbf{1})\hat{K}_\pm(\mathbf{\Omega}) &= (\mathbf{1} \otimes \mathrm{L}^*_{2,ext})\hat{K}_\pm(\mathbf{\Omega}), \\ [\mathrm{L}_2, \Phi(\mathbf{\Omega})] &= 0, \end{aligned} \tag{10.245}$$

where, as before, the operator $\Phi(\mathbf{\Omega}) \in \mathcal{B}_\infty(H)$ is defined by (13.69) as

$$\mathbf{\Omega} := \mathbf{1} + \Phi(\mathbf{\Omega}). \tag{10.246}$$

Owing to the evident commutation condition (10.244), the set of equations (13.80) and (10.245) is compatible, which gives rise to the expression of the form (13.76), where the kernel $\hat{K}_+(\mathbf{\Omega}) \in H_- \otimes H_-$ satisfies the set of differential equations generalizing those from [284, 285]:

$$\begin{gathered} \frac{\partial K_{+,11}}{\partial x_1} + \frac{\partial K_{+,11}}{\partial y_1} + \tilde{u}_1 K_{+,21} = 0, \quad \frac{\partial K_{+,12}}{\partial x_1} + \frac{\partial K_{+,12}}{\partial y_1} + \tilde{u}_1 K_{+,22} = 0, \\ \frac{\partial K_{+,21}}{\partial x_2} + \frac{\partial K_{+,21}}{\partial x_1} + \tilde{u}_2 K_{+,11} = 0, \quad \frac{\partial K_{+,22}}{\partial x_2} + \frac{\partial K_{+,22}}{\partial y_2} + \tilde{u}_2 K_{+,12} = 0, \end{gathered} \tag{10.247}$$

$$\begin{aligned} \pm\frac{\partial \tilde{u}_1}{\partial x_2} K_{+,21} &= \frac{\partial K_{+,11}}{\partial t} + [(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial y_1^2}) \pm (\frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial y_2^2})]K_{+,11} \\ &\quad + (\alpha_2(x_2) - \tilde{v}_2(x))K_{+,11}, \\ \pm\frac{\partial \tilde{u}_1}{\partial x_2} K_{+,21} &= \frac{\partial K_{+,22}}{\partial t} + [(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial y_1^2}) \pm (\frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial y_2^2})]K_{+,22} \\ &\quad + (\alpha_1(x_1) - \tilde{v}_1(x))K_{+,22}, \\ \mp 2\frac{\partial \tilde{u}_1}{\partial x_2} K_{+,22} &= \frac{\partial K_{+,12}}{\partial t} + [(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial y_1^2}) \pm (\frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial y_2^2})]K_{+,12} \\ &\quad + (\alpha_1(x_1) - \tilde{v}_2(x))K_{+,22}, \\ 2\frac{\partial \tilde{u}_2}{\partial x_1} K_{+,22} &= \frac{\partial K_{+,21}}{\partial t} + [(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial y_1^2}) \pm (\frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial y_2^2})]K_{+,21} \\ &\quad + (\alpha_2(x_2) - \tilde{v}_1(x))K_{+,11}. \end{aligned}$$

Moreover, the conditions

$$\tilde{u}_1(x) = -K^{(0)}_{+,12}|_{y=x}, \quad \tilde{u}_2(x) = -K^{(0)}_{+,21}|_{y=x}, \tag{10.248}$$
$$\tilde{v}_2(x)|_{x_1=-\infty} = \alpha_2(x_2), \quad \tilde{v}_1(x)|_{x_2=-\infty} = \alpha_1(x_1)$$

hold for all $x \in \mathbb{R}^2$ and $y \in \mathrm{supp}\hat{K}_+(\mathbf{\Omega})$, where we take into account the singular series expansion

$$\hat{K}_+(\mathbf{\Omega}) = \sum_{s=0}^{p(K_+)} K_+^{(s)} \delta^{(s-1)}_{\sigma_x^{(1)}} \tag{10.249}$$

for a finite integer $p(K_+) \in \mathbb{Z}_+$ with respect to the Dirac function $\delta_{\sigma_x^{(1)}}$: $W_2^q(\mathbb{R}^2;\mathbb{C}) \to \mathbb{R}$, $q \in \mathbb{Z}_+$, and its derivatives, having support (see [141], Chapter 3) coinciding with the closed cycle $\sigma_x^{(1)} \in \mathcal{K}(\mathbb{R}^2)$.

Remark 10.3. For the special case (13.79) discussed in [284, 285], it is easily verified that $p(K_+) = 1$ and $\sigma_x^{(1)} = \partial(\cap_{j=\overline{1,2}}\{y \in \mathbb{R}^2 :< y - x, \gamma_j >= 0\}) \subset \mathrm{supp}\hat{K}_+(\mathbf{\Omega})$. It was also shown that equations like (10.247) and (10.248) possess solutions if the Gelfand–Levitan–Marchenko equation (10.203) does.

If one uses the exact forms of operators L_1 and $\mathrm{L}_2 \in \mathcal{L}(\mathcal{H})$, it follows directly from (13.80) and (10.245) that the corresponding set of differential equations for components of the kernel $\hat{\Phi}(\mathbf{\Omega}) \in H_- \otimes H_-$ are

$$\frac{\partial \Phi_{11}}{\partial x_1} + \frac{\partial \Phi_{11}}{\partial y_1} = 0, \quad \frac{\partial \Phi_{12}}{\partial x_1} + \frac{\partial \Phi_{12}}{\partial y_1} = 0, \tag{10.250}$$
$$\frac{\partial \Phi_{21}}{\partial x_2} + \frac{\partial \Phi_{21}}{\partial y_2} = 0, \quad \frac{\partial \Phi_{22}}{\partial x_2} + \frac{\partial \Phi_{22}}{\partial y_2} = 0,$$

$$\frac{\partial \Phi_{11}}{\partial t} \pm (\frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial y_2^2})\Phi_{11} + (\alpha_2(y_2) - \alpha_2(x_2))\Phi_{11} = 0,$$
$$\frac{\partial \Phi_{12}}{\partial t} \pm (\frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial y_2^2})\Phi_{12} + (\alpha_1(y_1) - \alpha_2(x_2))\Phi_{12} = 0,$$
$$\frac{\partial \Phi_{21}}{\partial t} + (\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial y_1^2})\Phi_{21} + (\alpha_2(y_2) - \alpha_1(x_1))\Phi_{21} = 0,$$
$$\frac{\partial \Phi_{22}}{\partial t} + (\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial y_1^2})\Phi_{22} + (\alpha_1(y_1) - \alpha_1(x_1))\Phi_{22} = 0$$

for all $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2$. These generalize those found in [284, 285], which have been used to integrate the Davey–Stewartson differential equation [121, 405, 406] and thereby find soliton-like solutions. For our generalized case, the kernel (10.249) is a solution to the Gelfand–Levitan–Marchenko equations

$$K_+^{(0)}(x;y) + \Phi^{(0)}(x;y) + \int_{S_+^{(2)}(\sigma_x^{(1)},\sigma_\infty^{(1)})} K_+^{(0)}(x;\xi)\Phi^{(0)}(\xi;y)d\xi$$

$$+ \int_{\sigma_x^{(1)}} K_+^{(1)}(x;\xi)\Phi^{(0)}(\xi;y)d\sigma_x^{(1)} = 0, \tag{10.251}$$

$$K_+^{(1)}(x;y) + \Phi^{(1)}(x;y) + \int_{S_+^{(2)}(\sigma_x^{(1)},\sigma_\infty^{(1)})} K_+^{(0)}(x;\xi)\Phi^{(1)}(\xi;y)d\xi$$

$$+ \int_{\sigma_x^{(1)}} K_+^{(1)}(x;\xi)\Phi^{(1)}(\xi;y)d\sigma_x^{(1)} = 0,$$

where $y \in S_+^{(2)}(\sigma_x^{(1)},\sigma_\infty^{(1)})$ for all $x \in \mathbb{R}^2$ and

$$\hat{\Phi}(\mathbf{\Omega}) := \Phi^{(0)} + \Phi^{(1)}\delta_{\sigma_x^{(1)}} \tag{10.252}$$

is the kernel expansion corresponding to (10.249). Since the kernel (10.252) is singular, the differential equations (10.250) must be treated in the distributional sense [141].

Taking into account the exact forms of the "dressed" differential operators $\mathrm{L}_j \in \mathcal{L}(\mathcal{H})$, $j = 1, 2$, given by (13.75) and (10.241), one easily verifies that the commutativity condition (10.244) gives rise to that of $\tilde{\mathrm{L}}_j \in \mathcal{L}(\mathcal{H})$, $j = 1, 2$, which are equivalent to the Davey–Stewartson dynamical system

$$\begin{aligned} d\tilde{u}_1/dt &= -(\tilde{u}_{1,xx} + \tilde{u}_{1,yy}) + 2(\tilde{v}_1 - \tilde{v}_2), \\ d\tilde{u}_2/dt &= \tilde{u}_{2,xx} + \tilde{u}_{2,yy} + 2(\tilde{v}_2 - \tilde{v}_1), \\ \tilde{v}_{1,x} &= (\tilde{u}_1\tilde{u}_2)_y, \quad \tilde{v}_{2,x} = (\tilde{u}_1\tilde{u}_2)_x \end{aligned} \tag{10.253}$$

on an infinite-dimensional functional manifold $M_u \subset \mathcal{S}(\mathbb{R}^2;\mathbb{C})$. The exact soliton-like solutions to (10.253) are given by expressions (10.248), where the kernel $K_+^{(1)}(\mathbf{\Omega})$ solves the second linear integral equation of (10.251). On the other hand, there exists the exact expression (10.183) which solves the set of "dressed" equations

$$\tilde{\mathrm{L}}_1\tilde{\psi}^{(0)}(\eta) = 0, \quad \tilde{\mathrm{L}}_2\tilde{\psi}^{(0)}(\eta) = 0. \tag{10.254}$$

Since the kernels $\Omega(\lambda,\mu) \in L^2_{(\rho)}(\Sigma;\mathbb{C}) \otimes L^2_{(\rho)}(\Sigma;\mathbb{C})$, for $\lambda, \mu \in \Sigma$, $(t;x) \in M_{\mathrm{T}} \cap S_+^{(2)}(\sigma_x^{(1)},\sigma_\infty^{(1)})$ are given by means of exact expressions (10.181), one can find via simple calculations the corresponding analytical expression for the functions $(\tilde{u}_1,\tilde{u}_2) \in M_u$, solving the dynamical system (10.253). This procedure is often called the Darboux type transformation and was recently extensively used in [365] as a particular case of the construction above for finding soliton-like solutions to the Davey–Stewartson (10.253) and related two-dimensional modified Korteweg–de Vries flows on $M_{u.}$ Moreover, as can be observed from the technique used for constructing the Delsarte–Darboux transmutation operators $\mathbf{\Omega}_{\pm} \in \mathcal{L}(H)$, the set of solutions to (10.253) obtained by means of Darboux type transformations coincides with the corresponding set of solutions obtained by means of solving the related set of Gelfand–Levitan–Marchenko integral equations (10.250) and (10.251).

Example 10.3. An affine generalized de Rham–Hodge differential complex and related generalized self-dual Yang–Mills flows

Consider the following set of affine differential expressions in $\mathcal{H} := C^1(\mathbb{R}^{m+1}; H),\ H := L^2(\mathbb{R}^m; \mathbb{C}^N)$:

$$\mathrm{L}_i(\lambda) := \mathbf{1}\frac{\partial}{\partial p_i} - \lambda \frac{\partial}{\partial x_i} + A_i(x;p|t), \tag{10.255}$$

where $x \in \mathbb{R}^m$, $(t,p) \in \mathbb{R}^{m+1}$, $A_i \in C^1(\mathbb{R}^{m+1}; S(\mathbb{R}^m; End\mathbb{C}^N))$, $1 \leq i \leq m$, and a parameter $\lambda \in \mathbb{C}$. It is easy to construct an exact affine generalized de Rham–Hodge differential complex on $M_{\mathrm{T}} := \mathbb{R}^{m+1} \times \mathbb{R}^m$ as

$$\mathcal{H} \to \Lambda(M_{\mathrm{T}}; \mathcal{H}) \stackrel{d_{\mathcal{L}(\lambda)}}{\to} \Lambda^1(M_{\mathrm{T}}; \mathcal{H}) \to \stackrel{d_{\mathcal{L}(\lambda)}}{\cdots} \to \Lambda^{2m+1}(M_{\mathrm{T}}; \mathcal{H}) \stackrel{d_{\mathcal{L}(\lambda)}}{\to} 0, \tag{10.256}$$

where the differentiation is

$$d_{\mathcal{L}(\lambda)} := dt \wedge \mathrm{B}(\lambda) + \sum_{i=1}^{m} dp_i \wedge \mathrm{L}_i(\lambda) \tag{10.257}$$

and the affine matrix has the form

$$\mathrm{B}(\lambda) := \partial/\partial t - \sum_{s=0}^{n(B)+q} B_s(x;p|t)\lambda^{n(B)-s} \tag{10.258}$$

with matrices $B_s \in C^1(\mathbb{R}^{m+1}; S(\mathbb{R}^m; End\mathbb{C}^N))$, $0 \leq s \leq n(B)+q$, $n(B), q \in \mathbb{Z}_+$. The affine complex (10.256) is exact for all $\lambda \in \mathbb{C}$ iff the following generalized self-dual Yang–Mills equations [161]

$$\partial A_i/\partial p_j - \partial A_j/\partial p_i - [A_i, A_j] = 0, \quad \partial A_i/\partial x_j - \partial A_j/\partial x_i = 0,$$

$$\partial B_0/\partial x_i = 0,\ \partial B_{n(B)+q}/\partial p_i = 0,\ \partial B_s/\partial x_i = \partial B_{s-1}/\partial p_i + [A_i, B_{s-1}] = 0,$$

$$\partial A_i/\partial t + \partial B_{n(B)}/\partial p_i - \partial B_{n(B)+1}/\partial x_i + [A_i, B_{n(B)}] = 0 \tag{10.259}$$

hold for all $i,j=1,\dots,m$ and $(0\le s\le n(B))\vee(n(B)+q\le s\le n(B)+2)$. Assume now that the conditions (10.259) are satisfied on M_{T}. Then, making the change $\mathbb{C}\ni\lambda\to\partial/\partial\tau:\mathcal{H}\to\mathcal{H}$, $\tau\in\mathbb{R}$, one finds the following set of pure differential expressions

$$\mathrm{L}_{i(\tau)}:=\mathbf{1}\frac{\partial}{\partial p_i}-\frac{\partial^2}{\partial\tau\partial x_i}+A_i(x;p|t), \tag{10.260}$$
$$\mathrm{B}_{(\tau)}:=\partial/\partial t-\sum_{s=0}^{n(B)+q}B_s(x;p|t)(\frac{\partial}{\partial\tau})^{n(B)-s},$$

where the matrices A_i, $1\le i\le m$, and B_s, $0\le s\le n(B)+q$, do not depend on the variable $\tau\in\mathbb{R}$. Using the operator expressions (10.260), one can construct a new differential complex related to that of (10.256) given as

$$\mathcal{H}_{(\tau)}\to\Lambda(M_{\mathrm{T},\tau};\mathcal{H}_{(\tau)})\overset{d_{\mathcal{L}}}{\to}\Lambda^1(M_{\mathrm{T},\tau};\mathcal{H}_{(\tau)})\to\overset{d_{\mathcal{L}}}{\dots}\to\Lambda^{2m+2}(M_{\mathrm{T},\tau};\mathcal{H}_{(\tau)})\overset{d_{\mathcal{L}}}{\to}0, \tag{10.261}$$

where $\mathcal{H}_{(\tau)}:=C^1(\mathbb{R}^{m+1};H_{(\tau)})$, $H_{(\tau)}:=L^2(\mathbb{R}^m\times\mathbb{R}_\tau;\mathbb{C}^N)$ and

$$d_{\mathcal{L}}:=dt\wedge\mathrm{B}_{(\tau)}+\sum_{i=1}^m dp_i\wedge\mathrm{L}_{i(\tau)}. \tag{10.262}$$

The following result is a direct consequence of condition (10.259):

Lemma 10.9. *The differential complex (10.261) is exact.*

Therefore, one can construct the standard generalized de Rham–Hodge type Hilbert space decomposition

$$\mathcal{H}_\Lambda(M_{\mathrm{T},\tau}):=\overset{k=2m+2}{\underset{k=0}{\oplus}}\mathcal{H}^k_\Lambda(M_{\mathrm{T},\tau}) \tag{10.263}$$

as well the corresponding Hilbert–Schmidt rigging

$$\mathcal{H}_{\Lambda,+}(M_{\mathrm{T},\tau})\subset\mathcal{H}_\Lambda(M_{\mathrm{T},\tau})\subset\mathcal{H}_{\Lambda,-}(M_{\mathrm{T},\tau}). \tag{10.264}$$

Making use now of these results, we can define the Delsarte closed subspaces $\mathcal{H}_{0(\tau)}$ and $\tilde{\mathcal{H}}_{0(\tau)}\subset\mathcal{H}_{(\tau)-}$, related to the exact complex (10.261):

$$\mathcal{H}_{0(\tau)}:=\{\psi^{(0)}_{(\tau)}(\xi)\in\mathcal{H}^0_{\Lambda,-}(M_{\mathrm{T},\tau}):\mathrm{L}_{j(\tau)}\psi^{(0)}_{(\tau)}(\xi)=0, \tag{10.265}$$
$$\mathrm{B}_{(\tau)}\psi^{(0)}_{(\tau)}(\xi)=0,\ \psi^{(0)}_{(\tau)}(\xi)|_\Gamma=0,\ \psi^{(0)}_{(\tau)}(\xi)|_{t=0}=e^{\lambda\tau}\psi^{(0)}_\lambda(\eta)\in\mathcal{H}^0_{\Lambda,-}(M_{\mathbb{R}^m,\tau}),$$
$$\mathrm{L}_j(\lambda)\psi^{(0)}_\lambda(\eta)=0,\ \xi=(\lambda;\eta)\in\Sigma:=\mathbb{C}\times\Sigma^{(m)}_{\mathbb{C}}\},$$

$$\tilde{\mathcal{H}}_{0(\tau)} := \{\tilde{\psi}^{(0)}_{(\tau)}(\xi) \in \mathcal{H}^0_{\Lambda,-}(M_{\mathrm{T},\tau}) : \tilde{\mathrm{L}}^{(0)}_{j(\tau)}\tilde{\psi}^{(0)}_{(\tau)}(\xi) = 0,$$
$$\tilde{\mathrm{B}}_{(\tau)}\tilde{\psi}^{(0)}_{(\tau)}(\xi) = 0,\ \tilde{\psi}^{(0)}_{(\tau)}(\xi)|_{\tilde{\Gamma}} = 0,\ \tilde{\psi}^{(0)}_{(\tau)}(\xi)|_{t=0} = e^{\lambda\tau}\tilde{\psi}^{(0)}_{\lambda}(\eta) \in \mathcal{H}^0_{\Lambda,-}(M_{\mathbb{R}^m,\tau}),$$
$$\tilde{\mathrm{L}}_j(\lambda)\tilde{\psi}^{(0)}_{\lambda}(\eta) = 0,\ \xi = (\lambda;\eta) \in \Sigma : = \mathbb{C} \times \Sigma^{(m)}_{\mathbb{C}}\},$$

where Γ and $\tilde{\Gamma} \subset M_{\mathrm{T},\tau}$ are smooth hypersurfaces. Similar expressions correspond to the adjoint closed subspaces $\mathcal{H}^*_{0(\tau)}$ and $\tilde{\mathcal{H}}^*_{0(\tau)} \subset \mathcal{H}^*_{\tau,-}$, namely

$$\tilde{\mathcal{H}}_{0(\tau)} := \{\varphi^{(0)}_{(\tau)}(\xi) \in \mathcal{H}^0_{\Lambda,-}(M_{\mathrm{T},\tau}) : \mathrm{L}^*_{j(\tau)}\varphi^{(0)}_{(\tau)}(\xi) = 0, \tag{10.266}$$
$$\mathrm{B}_{(\tau)}\varphi^{(0)}_{(\tau)}(\xi) = 0,\ \varphi^{(0)}_{(\tau)}(\xi)|_{\Gamma} = 0,\ \varphi^{(0)}_{(\tau)}(\xi)|_{t=0} = e^{-\bar{\lambda}\tau}\varphi^{(0)}_{\lambda}(\eta) \in \mathcal{H}^0_{\Lambda,-}(M_{\mathbb{R}^m,\tau}),$$
$$\mathrm{L}^*_j(\lambda)\varphi^{(0)}_{\lambda}(\eta) = 0,\ \xi = (\lambda;\eta) \in \Sigma : = \mathbb{C} \times \Sigma^{(m)}_{\mathbb{C}}\},$$

$$\tilde{\mathcal{H}}_{0(\tau)} := \{\tilde{\varphi}^{(0)}_{(\tau)}(\xi) \in \mathcal{H}^0_{\Lambda,-}(M_{\mathrm{T},\tau}) : \tilde{\mathrm{L}}^*_{j(\tau)}\tilde{\varphi}^{(0)}_{(\tau)}(\xi) = 0,$$
$$\tilde{\mathrm{B}}^*_{(\tau)}\tilde{\varphi}^{(0)}_{(\tau)}(\xi) = 0,\ \tilde{\varphi}^{(0)}_{(\tau)}(\xi)|_{\tilde{\Gamma}} = 0,\ \tilde{\varphi}^{(0)}_{(\tau)}(\xi)|_{t=0} = e^{-\bar{\lambda}\tau}\tilde{\varphi}^{(0)}_{\lambda}(\eta) \in \mathcal{H}^0_{\Lambda,-}(M_{\mathbb{R}^m,\tau}),$$
$$\tilde{\mathrm{L}}^*_j(\lambda)\tilde{\varphi}^{(0)}_{\lambda}(\eta) = 0,\ \xi = (\lambda;\eta) \in \Sigma : = \mathbb{C} \times \Sigma^{(m)}_{\mathbb{C}}\}.$$

Using the closed subspaces (10.266) and (10.265), one can construct the Darboux type kernel $\tilde{\Omega}_{(t,x;\tau)}(\eta,\xi) \in L^2_{(\rho)}(\Sigma^{(m)}_{\mathbb{C}};\mathbb{C}) \otimes L^2_{(\rho)}(\Sigma^{(m)}_{\mathbb{C}};\mathbb{C})$, $\eta,\xi \in \Sigma^{(m)}_{\mathbb{C}}$, and further, the corresponding Delsarte transmutation mappings $\mathbf{\Omega}_{\pm} \in \mathcal{L}(H_{(\tau)})$. Namely, assume that the following conditions

$$\psi^{(0)}_{(\tau)}(\xi) := \tilde{\psi}^{(0)}_{(\tau)}(\xi) \cdot \tilde{\Omega}^{-1}_{(t,p;x;\tau)}\tilde{\Omega}_{(t_0,p_0,x_0;\tau)} \tag{10.267}$$

hold for any $\xi \in \mathbb{C}\times\Sigma^{(m)}_{\mathbb{C}}$, where

$$\tilde{\Omega}_{(t,x;\tau)}(\mu,\xi) := \int_{\sigma_{(t;x;\tau)}} \tilde{\Omega}^{(2m+1)}_{(\tau)}[e^{-\bar{\lambda}\tau}\tilde{\varphi}^{(0)}(\mu), e^{\lambda\tau}\tilde{\psi}^{(0)}(\eta)dx \wedge dp \wedge dt],$$

$$\tilde{Z}^{(2m+1)}_{(\tau)}[e^{-\bar{\lambda}\tau}\tilde{\varphi}^{(0)}(\mu), \sum_{i=1}^{m} e^{\lambda\tau}\tilde{\psi}^{(0)}(\xi_{(i)}) \wedge d\tau \wedge dx \overset{m}{\underset{j\neq i}{\wedge}} dp_j] \tag{10.268}$$
$$:= d\tilde{\Omega}^{(2m)}_{(\tau)}[e^{-\bar{\lambda}\tau}\tilde{\varphi}^{(0)}(\mu), \sum_{i=1}^{m} e^{\lambda\tau}\tilde{\psi}^{(0)}(\xi_{(i)}) \wedge d\tau \wedge dx \overset{m}{\underset{j\neq i}{\wedge}} dp_j],$$

and, as in (10.125), we have

$$< d^*_{\tilde{\mathcal{L}}}\tilde{\varphi}^{(0)}(\mu)e^{-\bar{\lambda}\tau}, * \sum_{i=1}^{m} e^{\lambda\tau}\tilde{\psi}^{(0)}(\xi_{(i)})dt \wedge d\tau \wedge dx \overset{m}{\underset{j\neq i}{\wedge}} dp_j > \tag{10.269}$$
$$=< (*)^{-1}\tilde{\varphi}^{(0)}(\mu)e^{-\bar{\lambda}\tau}, d_{\tilde{\mathcal{L}}}(\sum_{i=1}^{m} e^{\lambda\tau}\tilde{\psi}^{(0)}(\xi_{(i)})dt \wedge d\tau \wedge dx \overset{m}{\underset{j\neq i}{\wedge}} dp_j) >$$
$$+ d\tilde{Z}^{(2m+1)}_{(\tau)}[\tilde{\varphi}^{(0)}(\mu)e^{-\bar{\lambda}\tau}, \sum_{i=1}^{m} e^{\lambda\tau}\tilde{\psi}^{(0)}(\xi_{(i)})dt \wedge d\tau \wedge dx \overset{m}{\underset{j\neq i}{\wedge}} dp_j],$$

defining the exact $(2m+1)$-form $\tilde{Z}^{(2m+1)}_{(\tau)} \in \Lambda^{2m+1}(M_{\mathrm{T},\tau};\mathbb{C})$. Compute now the Delsarte transformed differential expressions

$$\mathrm{L}_{j(\tau)} := \hat{\mathbf{\Omega}}^{-1}_{(\tau)\pm}\tilde{\mathrm{L}}_{j(\tau)}\hat{\mathbf{\Omega}}_{(\tau)\pm}, \quad \mathrm{B}_{(\tau)} := \hat{\mathbf{\Omega}}^{-1}_{(\tau)\pm}\tilde{\mathrm{B}}_{(\tau)}\hat{\mathbf{\Omega}}_{(\tau)\pm} \tag{10.270}$$

for any $j = 1,\ldots,m$, where

$$\tilde{\mathrm{L}}_{j(\tau)} := \mathbf{1}\frac{\partial}{\partial p_j} - \frac{\partial^2}{\partial\tau\partial x_j} + \bar{A}_j, \tag{10.271}$$

$$\mathrm{B}_{(\tau)} := \partial/\partial t - \sum_{s=0}^{n(B)+q} \bar{B}_s(\frac{\partial}{\partial\tau})^{n(B)-s}$$

with all matrices $\bar{A}_j \in End\mathbb{C}^m$, $j = 1,\ldots,m$, and $\bar{B}_s \in End\,\mathbb{C}^m$, $0 \le s \le n(B)+q$, a constant. This means that, in particular, the commuting relationships

$$[\tilde{\mathrm{L}}_{j(\tau)}, \tilde{\mathrm{L}}_{i(\tau)}] = 0, \ [\tilde{\mathrm{L}}_{j(\tau)}, \tilde{\mathrm{B}}_{(\tau)}] = 0 \tag{10.272}$$

hold for all $1 \le i, j \le m$. Owing to the expressions (10.270), the induced commuting relationships

$$[\mathrm{L}_{j(\tau)}, \mathrm{L}_{i(\tau)}] = 0, \ [\mathrm{L}_{j(\tau)}, \mathrm{B}_{(\tau)}] = 0 \tag{10.273}$$

obviously hold and coincide exactly with (10.259). Moreover, reducing our differential expressions (10.270) upon functional subspaces $\mathcal{H}_{(\lambda)} := e^{\lambda\tau}\mathcal{H}$, $\lambda \in \mathbb{C}$, it is easy to obtain the set of affine differential expressions (10.255) and (10.258). We can express the reduced Delsarte transmutation operators as

$$\begin{aligned}\hat{\mathbf{\Omega}}_{\pm} = \mathbf{1} - &\int_{\Sigma^{(m)}_{\mathbb{C}}} d\rho_{\Sigma^{(m)}_{\mathbb{C}}}(\nu)\int_{\Sigma^{(m)}_{\mathbb{C}}} d\rho_{\Sigma^{(m)}_{\mathbb{C}}}(\eta)\psi^{(0)}(\lambda;\nu)\tilde{\Omega}^{-1}_{(t_0,p_0;x_0)}(\lambda;\nu,\eta)\\ &\times\int_{S^{(2m+1)}_{\pm}(\sigma^{(2m)}_{(t,p;x)},\sigma^{(2m)}_{(tt_0,p_0;x_0)})} \tilde{Z}^{(2m+1)}[\tilde{\varphi}^{(0)}(\lambda;\nu),(\cdot)\sum_{i=1}^{m} dt\wedge dx \bigwedge_{j\neq i}^{m} dp_j],\end{aligned} \tag{10.274}$$

where $\sigma^{(2m)}_{(t,p;x)}$ and $\sigma^{(2m)}_{(tt_0,p_0;x_0)} \in \mathcal{K}(M_{\mathrm{T}})$ are $2m$-dimensional closed singular simplices, and

$$\begin{aligned}&\tilde{Z}^{(2m+1)}[\tilde{\varphi}^{(0)}(\lambda;\nu),\sum_{i=1}^{m}\tilde{\psi}^{(0)}(\lambda;\eta_{(i)})dt\wedge dx \bigwedge_{j\neq i}^{m} dp_j]\\ &:= \tilde{Z}^{(2m+1)}_{(\tau)}[e^{-\bar{\lambda}\tau}\tilde{\varphi}^{(0)}(\lambda;\nu),\sum_{i=1}^{m} e^{\lambda\tau}\tilde{\psi}^{(0)}(\lambda;\eta_{(i)})d\tau\wedge dt\wedge dx \bigwedge_{j\neq i}^{m} dp_j]|_{d\tau=0},\end{aligned}$$

$$d\tilde{\Omega}_{(t,p;x)}(\lambda;\nu,\eta) := \tilde{Z}^{(2m+1)}[\tilde{\varphi}^{(0)}(\lambda;\nu), \sum_{i=1}^{m} \tilde{\psi}^{(0)}(\lambda;\eta_{(i)})dt \wedge dx \overset{m}{\underset{j\neq i}{\wedge}} dp_j], \tag{10.275}$$

since the $(2m+1)$-form (10.275) is, owing to (10.269), also exact for any $(\lambda;\nu,\eta) \in \mathbb{C} \times (\Sigma_{\mathbb{C}}^{(m)} \times \Sigma_{\mathbb{C}}^{(m)})$. Thus, the operator expression (10.274) applied to the operators (10.271) reduced upon the functional subspace $\mathcal{H}_{(\lambda)} \simeq \mathcal{H}$, $\lambda \in \mathbb{C}$, gives rise to the differential expressions

$$\mathrm{L}_j(\lambda) := \hat{\mathbf{\Omega}}_{\pm}^{-1}\tilde{\mathrm{L}}_j(\lambda)\hat{\mathbf{\Omega}}_{\pm} \quad \mathrm{B}(\lambda) := \hat{\mathbf{\Omega}}_{\pm}^{-1}\tilde{\mathrm{B}}(\lambda)\hat{\mathbf{\Omega}}_{\pm}, \tag{10.276}$$

where $\mathrm{L}_j(\lambda)\mathcal{H}_{(\lambda)} = \mathrm{L}_{j(\tau)}\mathcal{H}_{(\lambda)}$, $\mathrm{B}(\lambda)\mathcal{H}_{(\lambda)} = \mathrm{B}_{(\tau)}(\lambda)\mathcal{H}_{(\lambda)}$, $1 \leq j \leq m$, coinciding with affine differential expressions (10.255) and (10.258). To apply these results to finding exact soliton-like solutions to the self-dual Yang–Mills equations (10.259), it suffices to mention that the relationship (10.267) reduced upon the subspace $\mathcal{H}_{(\lambda)} \simeq \mathcal{H}$, $\lambda \in \mathbb{C}$, gives rise to the following map:

$$\psi^{(0)}(\lambda;\eta) := \tilde{\psi}^{(0)}(\lambda;\eta) \cdot \tilde{\Omega}_{(t,p;x)}^{-1}\tilde{\Omega}_{(t_0,p_0;x_0)}, \tag{10.277}$$

where the kernels $\tilde{\Omega}_{(t,p;x;\tau)}(\lambda;\eta,\xi) \in L^2_{(\rho)}(\Sigma_{\mathbb{C}}^{(m)};\mathbb{C}) \otimes L^2_{(\rho)}(\Sigma_{\mathbb{C}}^{(m)};\mathbb{C})$, $\eta,\xi \in \Sigma_{\mathbb{C}}^{(m)}$, for all $(t,p;x) \in M_{\mathrm{T}}$ and $\lambda \in \mathbb{C}$. Since the element $\psi^{(0)}(\lambda;\eta) \in \mathcal{H}_-$ for any $(\lambda;\xi) \in \mathbb{C}\times\Sigma_{\mathbb{C}}^{(m)}$ satisfies the set of differential equations

$$L_i(\lambda)\psi^{(0)}(\lambda;\eta) = 0, \; B(\lambda)\psi^{(0)}(\lambda;\eta) = 0, \tag{10.278}$$

for all $1 \leq i \leq m$, from (10.277) and (10.278) one finds easily exact expressions for the corresponding matrices A_j and $B_s \in C^1(\mathbb{R} \times \mathbb{R}^{m+1}; S(\mathbb{R}^m; End\,\mathbb{C}^N))$, $1 \leq j \leq m$, $0 \leq s \leq n(B)+q$, satisfying the self-dual Yang–Mills equations (10.259). Thus, we have proved the following result.

Theorem 10.15. *The integral expressions (10.274) in $\mathcal{H}$ are the Delsarte transmutation operators corresponding to the affine differential expressions (10.255), (10.259) and constant operators*

$$\tilde{\mathrm{L}}_i(\lambda) := \mathbf{1}\frac{\partial}{\partial p_i} - \lambda\frac{\partial}{\partial x_i} + \bar{A}, \quad \tilde{\mathrm{B}}(\lambda) := \partial/\partial t - \sum_{s=0}^{n(B)+q} \bar{B}_s\lambda^{n(B)-s} \tag{10.279}$$

for any $\lambda \in \mathbb{C}$. The mapping (10.277) realizes the isomorphisms between the closed subspaces

$$\begin{aligned} \mathcal{H}_0 := \{&\psi^{(0)}(\lambda;\eta) \in \mathcal{H}_- : d_{\tilde{\mathcal{L}}(\lambda)}\psi^{(0)}(\lambda;\eta) = 0, \; \psi^{(0)}(\lambda;\eta)|_{t=0} \\ &= \psi_{\lambda}^{(0)}(\eta) \in H_-, \; \psi^{(0)}(\lambda;\eta)|_{\Gamma} = 0, (\lambda;\eta) \in \mathbb{C} \times \Sigma_{\mathbb{C}}^{(m)}\} \end{aligned} \tag{10.280}$$

and

$$\begin{aligned} \tilde{\mathcal{H}}_0 := \{\tilde{\psi}^{(0)}(\lambda;\eta) \in \mathcal{H}_- : d^{(0)}_{\tilde{\mathcal{L}}(\lambda)}\tilde{\psi}(\lambda;\eta) = 0,\ \tilde{\psi}^{(0)}(\lambda;\eta)|_{t=0} \\ = \tilde{\psi}^{(0)}_\lambda(\eta) \in H_-,\ \tilde{\psi}^{(0)}(\lambda;\eta)|_{\tilde{\Gamma}} = 0, (\lambda;\eta) \in \mathbb{C}\times\Sigma^{(m)}_{\mathbb{C}}\} \end{aligned} \tag{10.281}$$

for any parameter $\lambda \in \mathbb{C}$. Moreover, the expressions (10.277) generate the standard Darboux transformations for the set of operators (10.279) and (10.255), (10.258) via the corresponding set of linear equations (10.278), thereby producing exact soliton-like solutions to the self-dual Yang–Mills equations (10.259).

As a simple consequence of Theorem 10.15 one retrieves all of the results obtained in [161], where the Delsarte–Darboux map (10.277) was chosen completely a priori without any proof and motivation in the form of an affine gauge transformation.

Results similar to the above can with minor changes also be applied to the affine differential generalized de Rham–Hodge complex (10.256) with the exterior differentiation (10.257), where

$$\begin{aligned} \mathrm{L}_i(\lambda) &:= \mathbf{1}\frac{\partial}{\partial p_i} - (\sum_{k=0}^{n_i(L)} a_{ik}\lambda^{k+1})\frac{\partial}{\partial x_i} + \sum_{k=0}^{n_i(L)} A_{ik}\lambda^k, \\ \tilde{\mathrm{B}}(\lambda) &:= \partial/\partial t - \sum_{s=0}^{n(B)+q} \bar{B}_s\lambda^{n(B)-s}, \end{aligned} \tag{10.282}$$

or

$$\begin{aligned} \mathrm{L}_i(\lambda) &:= \mathbf{1}\frac{\partial}{\partial p_i} - (\sum_{k=0}^{n_i(L)} a^{(j)}_{ik}\lambda^{k+1})\frac{\partial}{\partial x_j} + \sum_{k=0}^{n_i(L)} A_{ik}\lambda^k, \\ \tilde{\mathrm{B}}(\lambda) &:= \partial/\partial t - \sum_{s=0}^{n(B)+q} \bar{B}_s\lambda^{n(B)-s}, \end{aligned} \tag{10.283}$$

for $1 \le i \le m$, $\lambda \in \mathbb{C}$. The case (10.282) was analyzed recently in [396] using an affine gauge transformation introduced in [161]. Unfortunately, the results obtained here are generally too complicated and unwieldy. Consequently, one needs to find more mathematically motivated, clear and less cumbersome techniques for finding Delsarte–Darboux transformations and their related soliton-like exact solutions.

Chapter 11

Characteristic Classes of Chern Type and Integrability of Systems on Riemannian Manifolds

11.1 Differential-geometric problem setting

We start with a smooth m-dimensional Riemannian manifold M and two complex vector bundles over it: the tangent bundle $T(M)$ and a bundle $E(M)$ endowed with a Hermitian scalar structure $<,\cdot,\cdot>_E$ on the fibers. The related vector fields on M are denoted as $\mathcal{T}(M)$ and smooth sections of $E(M)$ as $\mathcal{E}(M)$. As usual [173, 197, 198, 266, 393], one can introduce on $\mathcal{E}(M)$ a Cartan connection Γ by means of a connection mapping

$$d_{\mathrm{A}} : \mathcal{E}(M) \to \mathcal{T}^*(M) \otimes \mathcal{E}(M), \tag{11.1}$$

which satisfies

$$d_{\mathrm{A}}(f\alpha + \beta) := df \otimes \alpha + f d_{\mathrm{A}}\alpha + d_{\mathrm{A}}\beta \tag{11.2}$$

for any smooth function $f \in \mathcal{D}(M)$ and $\alpha, \beta \in \mathcal{E}(M)$. Let $\Lambda(M) := \oplus_{p=0}^{\dim M}\Lambda^p(M)$ denote the usual [3, 14, 173, 197, 198, 393] Grassmann algebra of differential forms on M. To define the associated linear bundles $\Lambda^p(M,E) := \Lambda^p(M) \otimes \mathcal{E}(M)$ for $0 \le p \le m,$ the connection mapping (13.24) can be naturally extended on $\Lambda^p(M,E)$ as

$$d_{\mathrm{A}} : \Lambda^p(M,E) \to \Lambda^{p+1}(M,E), \tag{11.3}$$

satisfying the related Leibniz rule

$$d_{\mathrm{A}}(f^{(p)} \wedge \alpha^{(q)}) := df^{(p)} \wedge \alpha^{(q)} + (-1)^p f^{(p)} \wedge d_{\mathrm{A}}\alpha^{(q)} \tag{11.4}$$

for any $f^{(p)} \in \Lambda^p(M)$ and $\alpha^{(q)} \in \Lambda^q(M,E)$, $1 \le q,p \le m$.

The connection operation (13.26) possesses a very interesting and important property: its composition $d_{\mathrm{A}}^2 := d_{\mathrm{A}}d_{\mathrm{A}}$ is a linear mapping over the ring of smooth functions ring $\mathcal{D}(M)$:

$$d_{\mathrm{A}}^2(f\alpha^{(p)} + \beta^{(p)}) = f d_{\mathrm{A}}^2\alpha^{(p)} + d_{\mathrm{A}}^2\beta^{(p)}, \tag{11.5}$$

where $f \in \mathcal{D}(M)$ and $\alpha^{(p)}, \beta^{(p)} \in \Lambda^p(M,E)$, $p = 0,\ldots,m$, are arbitrary. The resulting linear tensor map $\Omega^{(2)} := d_{\mathrm{A}}^2 : \mathcal{E}(M) \to \Lambda^2(M,E)$ is called the *curvature tensor* and is of great importance for geometrical analysis of integrable multi-dimensional differential systems of Gromov type [159], generated by Cartan integrable [44, 85, 173, 326, 382] ideals $I(\alpha) \subset \Lambda(M, End\ E) := \Lambda(M) \otimes End\ \mathcal{E}(M)$ on the Riemannian manifold M. Moreover, one can construct the smooth integral submanifold embedding $i_\alpha : M_\alpha \to M$ for the ideal $I(\alpha) \subset \Lambda(M, End\ E)$, satisfying the following determining condition: the curvature 2-form $\Omega^{(2)} \in \Lambda^2(M,E)$ reduced upon M_α vanishes, that is $i_\alpha^*\Omega^{(2)} = 0$. This implies also that the related reduced cochain

$$E \to \Lambda^0(M_\alpha,E) \overset{d_\alpha}{\to} \Lambda^1(M_\alpha,E) \overset{d_\alpha}{\to} \ldots\Lambda^{m_\alpha}(M_\alpha,E) \overset{d_\alpha}{\to} 0 \tag{11.6}$$

is a de Rham complex, with $d_\alpha^2 = 0$, where $d_\alpha := i_\alpha^* d_{\mathrm{A}}$ and $m_\alpha := \dim M_\alpha$. Since the submanifold $M_\alpha \subset M$ also possesses the induced Riemannian structure $g_\alpha : T(M_\alpha) \times T(M_\alpha) \to \mathbb{R}$, we can construct from (13.29) a generalized de Rham–Hodge complex of Hilbert spaces $H_\Lambda^p(M_\alpha)$, $0 \le p \le m_\alpha$, whose properties, as was shown in [70, 173, 331, 366], makes it possible to describe the so-called Delsarte–Lions transmutation operators of Volterra type, which are a useful tool for constructing integrable multi-dimensional differential systems on Riemannian manifolds and finding their special exact solutions.

On the other hand, one can consider the following generalized chain of modules over M

$$E \to \Lambda^0(M,E) \overset{d_{\mathrm{A}}}{\to} \Lambda^1(M,E) \overset{d_{\mathrm{A}}}{\to} \ldots \overset{d_{\mathrm{A}}}{\to} \Lambda^m(M,E) \to 0, \tag{11.7}$$

which is not a de Rham–Hodge complex, but determines such very important [108, 260, 266, 393] geometric objects as the Chern characteristic classes and characters. Their properties and related geometric aspects of the integrability problem of multi-dimensional differential systems of Gromov type [159] on Riemannian manifolds will be treated in what follows.

11.2 The differential invariants

The connection mapping (13.26) on M can be equivalently rewritten locally on an open neighborhood $U \subset M$ as

$$d_{\mathrm{A}}|_U = d + A^{(1)}, \tag{11.8}$$

where $A^{(1)} \in \Lambda(U, End\ E)$ is a suitably determined $End\ \mathcal{E}(U)$-valued differential 1-form on $U \subset M$. In local coordinates of a point $u \in U$, we can write

$$A^{(1)} := \sum_{i=1}^{m} A_i(u)du^i, \tag{11.9}$$

where $A_i(u) \in End\ \mathcal{E}(U)$, $i = 1, \ldots, m$. Making use of the representation (13.38), one can easily obtain the following local expression for the curvature 2-form $\Omega^{(2)}|_U \in \Lambda^2(U) \otimes End\ \mathcal{E}(U)$:

$$\Omega^{(2)}|_U = dA^{(1)} + A^{(1)} \wedge A^{(1)}. \tag{11.10}$$

The expression (13.40) is convenient for locally determining the cohomology characteristic Chern type classes related to the complex (13.31)

$$ch_j(\mathrm{A})|_U := \mathrm{tr}(\frac{i}{2\pi}\Omega^{(2)}|_U)^j \in \Lambda^{2j}(U), \tag{11.11}$$

where $j \in \mathbb{Z}_+$. The expression (13.41), owing to the properties of $E(M)$, can be invariantly extended upon the whole manifold M as well defined differential forms on M, thereby determining the characteristic Chern type classes

$$ch_j(\mathrm{A}) = \mathrm{tr}(\frac{i}{2\pi}\Omega^{(2)})^j \in \Lambda^{2j}(M) \tag{11.12}$$

for $j \in \mathbb{Z}_+$. It is a standard [266, 386, 393] result that the classes (13.42) are closed and do not depend on the choice of a connection map (13.26).

As a consequence of these properties, one can define for every linear Hermitian fiber bundle $E(M)$ over M the set of corresponding characteristic Chern type classes

$$ch_j(E, M) := [ch_j(\mathrm{A})] \tag{11.13}$$

for $j \in \mathbb{Z}_+$, by means of which the Chern character $ch(E, M)$ of this linear fiber bundle $E(M)$ is determined as

$$ch(E, M) := \oplus_{j\in\mathbb{Z}_+}(j!)^{-1}ch_j(E, M) = [tr\exp(\frac{i}{2\pi}\Omega^{(2)})]. \tag{11.14}$$

The Chern character, as is well known [108, 266, 393], has many applications to modern differential topology and mathematical physics.

Concerning applications to strongly integrable multi-dimensional differential systems on Riemannian manifolds, we consider the Cartan geometric approach developed in [44, 85, 173, 326, 382]. In this approach, a nonlinear multi-dimensional differential system $\hat{\alpha}$ is represented in the form of a Cartan integrable ideal $I(\alpha) \subset \Lambda(M, End\, E)$ with coefficients in $End\mathcal{E}(M)$, where $\mathcal{E}(M)$ is a specially chosen Hermitian vector fiber bundle over a suitably chosen finite-dimensional Riemannian manifold M. The corresponding integral submanifold $M_\alpha \subset M$ of the ideal $I(\alpha)$, in general, is equivalent to the set of independent variables of our multi-dimensional integrable nonlinear differential system.

We call our multi-dimensional differential system strongly integrable if it has a connection Γ_λ parametrically dependent on $\lambda \in \mathbb{R}$, whose curvature 2-form $\Omega_\lambda^{(2)} \in \Lambda^2(M, End\, E)$ vanishes on the integral submanifold $M_\alpha \subset M$ of the ideal $I(\alpha) \subset \Lambda(M, End\, E)$. Obviously, the latter condition is equivalent to

$$\Omega_\lambda^{(2)} \in I(\alpha) \tag{11.15}$$

for all allowed values of $\lambda \in \mathbb{R}$. If the connection Γ does not depend nontrivially on the parameter $\lambda \in \mathbb{R}$, a multi-dimensional differential system is called integrable.

On the other hand, the condition (13.47) serves [173, 326] as a means for finding the corresponding connection map (13.26), assuming that it exists. The resulting search algorithm details are crucially dependent [44, 173, 326] on the related holonomy group properties of the connection Γ_λ, $\lambda \in \mathbb{R}$, on the principal fiber bundle $P(M, G)$ naturally associated with the bundle $E(M)$, where the Lie group G is the structure group of the connection Γ_λ, $\lambda \in \mathbb{R}$. Further details on the algorithm may be found in [44, 173, 326].

The condition (13.47) is important, as we can directly obtain under the conditions $H^{2j}(M, \mathbb{R}) = 0$, $j \in \mathbb{Z}_+$, that

$$ch_j(\mathrm{A}) := d\ \chi_j(\mathrm{A}) \tag{11.16}$$

for a suitably determined global and real differential $(2j-1)$-form $\chi_j(\mathrm{A}) \in \Lambda^{2j-1}(M)$, $j \in \mathbb{Z}_+$, on the manifold M. Moreover, since it is clear that all degrees $(\frac{i}{2\pi}\Omega^{(2)})^j \in I(\alpha)$ for $j \in \mathbb{Z}_+$, from the condition $i_\alpha^* I(\alpha) = 0$, where $i_\alpha : M_\alpha \to M$ is the integral submanifold imbedding mapping, one easily obtains that $i_\alpha^* ch_j(\mathrm{A}) = 0$, or equivalently

$$d\chi_j(\mathrm{A}) = 0 \tag{11.17}$$

for all $j \in \mathbb{Z}_+$, giving rise to new differential Chern type invariants on M_α. The above provides a proof of the following result.

Theorem 11.1. *If an integrable multi-dimensional nonlinear differential system $\hat{\alpha}$ is equivalent to the Cartan integrable ideal $I(\alpha) \subset \Lambda(M, End$ $E)$ on a Riemannian manifold M, satisfying the cohomology conditions $H^{2j}(M, \mathbb{R}) = 0$, $j \in \mathbb{Z}_+$, then it possesses a set of differential Chern type invariants (13.49) on a suitable integral submanifold $M_\alpha \subset M$ of the ideal*

$I(\alpha)$. If nontrivial, these differential invariants describe, in particular, a related moduli space of the linear fiber bundle $E(M)$.

It is useful to mention here that most of differential invariants (13.49) reduce to zero at higher indices $j \in \mathbb{Z}_+$. In fact, all differential invariants $\chi_j(\mathrm{A})$ for $j \geq [\dim M_\alpha/2]+1$ are identically zero. In particular, for the case of multi-dimensional integrable nonlinear differential systems, for which $\dim M_\alpha = 2$ or 3, it is easy to see that at most one differential invariant can exist. Moreover, if for such differential systems the corresponding structure Lie groups are special linear ones for which the Lie algebras are traceless, it can be readily shown that, in general, no invariant exists on M_α.

This result is quite useful for the mathematical theory of geometrically integrable multi-dimensional nonlinear differential systems that possess connection maps (13.26) on $\Lambda(M,E)$ under the additional cohomology conditions $H^{2j}(M,\mathbb{R}) = 0$, $j \in \mathbb{Z}_+$. Namely, for these connection mappings to exist on $\Lambda(M,E)$ in the higher-dimensional case $\dim M_\alpha \geq 4$, the nontrivial differential invariants can be proved to exist. Thus, in particular, this gives rise to natural topological obstructions. More precisely, a nontrivial differential invariant requires nontrivial cohomology constraints $H^s(M,\mathbb{R}) \neq 0$ for some $s \in \mathbb{Z}_+$, thereby contradicting the above zero cohomology conditions at these values $s \in \mathbb{Z}_+$. Otherwise, if $H^{2s}(M,\mathbb{R}) \neq 0$ for some $s \in \mathbb{Z}_+$, the related characteristic Chern classes $ch_s(E,M)$ for these $s \in \mathbb{Z}_+$ are, in general, strongly nontrivial, thereby defining differential $(2s-1)$-forms $\chi_s(\mathrm{A}) \in \Lambda^{2s-1}_{loc}(M)$ only locally, owing to the classical Poincaré lemma. These local differential forms, if reduced upon the integral submanifold $M_\alpha \subset M$, give rise to a set of differential multi-valued quasi-invariants $\chi_s(A) \in \Lambda^{2s-1}_{loc}(M_\alpha)$, where $d\chi_s(A) = 0$ and whose existence can mean, in particular, that the nonlinear differential system is in some sense ill-posed.

This page intentionally left blank

Chapter 12

Quantum Mathematics: Introduction and Applications

12.1 Introduction

There is a broad and inclusive view of modern mathematical physics by many mathematicians and mathematical physicists in which quantum physics and the related field of *quantum mathematics* play very important roles. During the last century, modern mathematical physics evolved within at least four components which illustrate [188] the development of the mathematics and quantum physics synergy:

1) the use of ideas from mathematics in shedding new light on the existing principles of quantum physics, either from a conceptual or from a quantitative point of view;

2) the use of ideas from mathematics in discovering new "laws of quantum physics";

3) the use of ideas from quantum physics in shedding new light on existing mathematical structures;

4) the use of ideas from quantum physics in discovering new domains in mathematics.

Each one of these topics plays some role in understanding modern mathematical physics. However, the success in directions 2) and 4) is certainly more modest than success in directions 1) and 3). In some cases it is difficult to draw a clear-cut distinction between these two sets. In fact, we are lucky when it is possible to make progress in directions 2) and 4); so much so that when we make major progress, historians like to speak of a revolution. In any case, many mathematical physicists strive to understand within their research efforts these deep and lofty goals. There are many situations however, when mathematical physicists' research efforts are directed toward one other more mundane aspect:

5) the use of ideas from quantum physics and mathematics to benefit "economic competitiveness".

Here too, one might subdivide this aspect into conceptual understanding on one hand (such as the mathematical model of Black and Sholes for pricing of derivative securities in financial markets) and invention on the other: the creation of new algorithms or materials (e.g. quantum computers) which might revolutionize technology or change our way of life. As in the first four cases, the boundaries between these domains are not sharp, and it remains open to views and interpretations. This fifth string can be characterized as "applied" mathematical physics. We shall restrict our analysis to the first four strands characterizing modern quantum physics and its mathematical aspects; it is believed that most of the more profound applied directions arise after earlier fundamental quantum physics and mathematics progress.

We have passed through an extraordinary 35-year period of development of modern fundamental mathematics and quantum physics. Much of this development has been derived from one subject to understand the other. Not only have the concepts from diverse fields been united: statistical physics, quantum field theory and functional integration; gauge theory and geometry; index theory and knot invariants, etc., but also, new phenomena have been recognized and new areas have emerged whose significance for both mathematics and modern quantum physics is only partially understood: for example, non-commutative geometry, super-analysis, mirror symmetry, new topological invariants of manifolds, and the general notion of geometric quantization.

There is no doubt that, over the past half-century, the ideas from quantum physics have led to far greater innovations in mathematics than the ideas from mathematics have in discoveries of laws of quantum physics. Recognition of this underlines the opportunities for future progress in the opposite direction: a new understanding of the quantum nature of the world is certainly an expectation among physicists and mathematicians.

Great publicity and recognition has been attached to the progress made in modern geometry, representation theory, and deformation theory due to this interaction. We shall not dwell here on the substantial progress in analysis and probability theory, which unfortunately is more difficult to understand because of its delicate dependence on subtle notions of continuity.

On the other hand, there are deep differences between pure mathematics and modern quantum physics fundamentals. They have evolved from different cultures and they each have a distinctive set of values of their own, suited for their different realms of universality. But both subjects are strongly based on *intuition*, some natural and some acquired, which form our understanding. Quantum physics describes the natural micro-world. Hence, physicists appeal to observation in order to verify the validity of a physical theory. And, although much of mathematics arises from the natural world, mathematics has no analogous testing grounds - mathematicians appeal to their own set of values, namely mathematical proof, to justify validity of a mathematical theory. In mathematical physics, when announcing results of a mathematical nature, it is necessary to claim a theorem when the proof meets the mathematical community standards for a proof; otherwise, it is necessary to make a conjecture with a detailed outline for support. Most of physics, on the other hand, has very different standards.

There is no question that the interaction between modern mathematics and quantum physics will change radically during in the years to come - if recent experience is any indicator at all future progress. One hopes that this evolution will preserve and even enhance the positive experience of being a mathematician, a physicist, or a mathematical physicist.

It is instructive to look at the beginning of the 20^{th} century and trace the way mathematics has been exerting influence on modern and classical quantum physics, and next observe the ways in which modern quantum physics is nowadays exerting so impressive an influence on modern mathematics. With this in mind, application of modern quantum mathematics to studying nonlinear dynamical systems in functional spaces and vice versa will serve as an excellent example of the synergy of which we speak. We begin with a brief history of quantum mathematics featuring transformative developments related to dynamical issues:

The beginning of the 20^{th} *century*:

- P.A.M. Dirac – first realized and used in quantum physics the fact that the commutator operation $D_a : \mathcal{A} \ni b \longrightarrow [a, b] \in \mathcal{A}$, where $a \in \mathcal{A}$ is fixed and $b \in \mathcal{A}$, is a differentiation (or derivation) of any associative algebra $\mathcal{A}$; moreover, he first constructed a spinor matrix realization of the Poincaré symmetry group $\mathcal{P}(1, 3)$ and invented the famous Dirac δ-function [100, 104] (1920-1926);

- J. von Neumann – first applied the spectral theory of self-adjoint operators in Hilbert spaces to explain the radiation spectra of atoms and the stability of the related matter, [277] (1926);

- V. Fock – first introduced the notion of many-particle Hilbert space, Fock space, and the related creation and annihilation operators acting in it, [127] (1932);

- H. Weyl – was the first to understand the fundamental role of the notion of symmetry in physics and developed a physics-oriented group theory; moreover he showed the importance of different representations of classical matrix groups for physics and studied the unitary representations of the Heisenberg–Weyl group related with creation and annihilation operators in Fock space, [397] (1931).

The end of the 20^{th} *century*:

- L. Faddeev with co-workers – quantum inverse spectral theory transform, [117, 120] (1978);
- V. Drinfeld, S. Donaldson and E. Witten – quantum groups and algebras, quantum topology, quantum super-analysis, [108, 109, 401] (1982-1994);
- Yu. Manin and R. Feynman – quantum information theory, [123, 124], [245, 246] (1980-1986);
- P. Shor, E. Deutsch, L. Grover and others – quantum computer algorithms, [99, 160, 375] (1985-1997).

As one can observe, many exciting and extraordinarily important mathematical achievements were motivated by the impressive and deep influence of ideas and perspectives from quantum physics, leading nowadays to an altogether new scientific field often called quantum mathematics.

Following this quantum mathematical way of thinking, we shall demonstrate in this chapter that a wide class of strictly nonlinear dynamical systems in functional spaces can be treated as natural objects in specially constructed Fock spaces in which the corresponding evolution flows are completely linearized. Consequently, the powerful machinery of classical mathematical tools can be applied to studying the analytical properties of exact solutions to suitably well-posed Cauchy problems.

12.2 Mathematical preliminaries: Fock space and realizations

Let Φ be a separable Hilbert space, F be a topological real linear space and $\mathcal{A} := \{A(\varphi) : \varphi \in F\}$ a family of commuting self-adjoint operators in Φ (i.e.

these operators commute in the sense of their resolutions of the identity). Consider the Gelfand rigging [28] of the Hilbert space Φ, i.e., a chain

$$\mathcal{D} \subset \Phi_+ \subset \Phi \subset \Phi_- \subset \mathcal{D}' \tag{12.1}$$

in which Φ_+ and Φ_- are also Hilbert spaces, and the inclusions are dense and continuous, i.e., Φ_+ is topologically (densely and continuously) and quasi-nucleously (the inclusion operator $i : \Phi_+ \longrightarrow \Phi$ is of the Hilbert - Schmidt type) embedded in Φ, Φ_- is the dual of Φ_+ with respect to the scalar product $< \cdot, \cdot >_\Phi$ in Φ, and $\mathcal{D}$ is a separable projective limit of Hilbert spaces, topologically embedded in Φ_+. Then, the following structural theorem [28, 30] holds:

Theorem 12.1. *Assume that the family of operators $\mathcal{A}$ satisfies the following conditions: a) $\mathcal{D} \subset DomA(\varphi)$, $\varphi \in F$, and the closure of the operator $A(\varphi) \upharpoonright \mathcal{D}$ coincides with $A(\varphi)$ for any $\varphi \in F$, that is $A(\varphi) \upharpoonright \mathcal{D} = A(\varphi)$ in Φ; b) the Range $A(\varphi) \upharpoonright \mathcal{D} \subset \Phi_+$ for any $\varphi \in F$; c) for every $f \in \mathcal{D}$ the mapping $F \ni \varphi \longrightarrow A(\varphi)f \in \Phi_+$ is linear and continuous; d) there exists a strong cyclic (vacuum) vector $|\Omega\rangle \in \bigcap_{\varphi \in F} DomA(\varphi)$, such that the set of all vectors $|\Omega\rangle$, $\prod_{j=1}^{n} A(\varphi_j)|\Omega\rangle$, $n \in \mathbb{Z}_+$, is total in Φ_+ (i.e. their linear hull is dense in Φ_+). Then there exists a probability measure μ on $(F', C_\sigma(F'))$, where F' is the dual of F and $C_\sigma(F')$ is the σ-algebra generated by cylinder sets in F' such that, for μ-almost every $\eta \in F'$ there is a generalized joint eigenvector $\omega(\eta) \in \Phi_-$ of the family $\mathcal{A}$, corresponding to the joint eigenvalue $\eta \in F'$, that is*

$$< \omega(\eta), A(\varphi)f >_\Phi = \eta(\varphi) < \omega(\eta), f >_\Phi \tag{12.2}$$

with $\eta(\varphi) \in \mathbb{R}$ denoting the pairing between F and F'. The mapping

$$\Phi_+ \ni f \longrightarrow < \omega(\eta), f >_\Phi := \hat{f}(\eta) \in \mathbb{C} \tag{12.3}$$

for any $\eta \in F'$ can be continuously extended to a unitary surjective operator $\mathcal{F} : \Phi \longrightarrow L^2_{(\mu)}(F'; \mathbb{C})$, where

$$\mathcal{F}\ f(\eta) := \hat{f}(\eta) \tag{12.4}$$

for any $\eta \in F'$ is a generalized Fourier transform, corresponding to the family $\mathcal{A}$. Moreover, the image of the operator $A(\varphi)$, $\varphi \in F'$, under the $\mathcal{F}$-mapping is the operator of multiplication by the function $F' \ni \eta \rightarrow \eta(\varphi) \in \mathbb{C}$.

We further assume that the main Hilbert space Φ possesses the standard Fock space (bose)-structure [28, 56, 334], that is

$$\Phi = \oplus_{n \in \mathbb{Z}_+} \Phi_{(s)}^{\otimes n}, \tag{12.5}$$

where the subspaces $\Phi^{\otimes n}_{(s)}$, $n \in \mathbb{Z}_+$ are the symmetrized tensor products of a $\mathcal{H} := L^2(\mathbb{R}^m;\mathbb{C})$. If a vector $g := (g_0, g_1, ..., g_n, ...) \in \Phi$, its norm is defined as

$$\|g\|_\Phi := \left(\sum_{n\in\mathbb{Z}_+} \|g_n\|_n^2 \right)^{1/2}, \tag{12.6}$$

where $g_n \in \Phi^{\otimes n}_{(s)} \simeq L^2_{(s)}((\mathbb{R}^m)^n;\mathbb{C})$ and $\|\cdot\|_n$ is the corresponding norm in $\Phi^{\otimes n}_{(s)}$ for each $n \in \mathbb{Z}_+$. Note here that for the rigging structure (12.5), there is a corresponding rigging for the Hilbert spaces $\Phi^{\otimes n}_{(s)}$, $n \in \mathbb{Z}_+$, that is

$$\mathcal{D}^n_{(s)} \subset \Phi^{\otimes n}_{(s),+} \subset \Phi^{\otimes n}_{(s)} \subset \Phi^{\otimes n}_{(s),-} \tag{12.7}$$

with suitably chosen dense and separable topological spaces of symmetric functions $\mathcal{D}^n_{(s)}$, $n \in \mathbb{Z}_+$. For the expansion (12.5), we obtain by means of projective and inductive limits [28, 30–32] the quasi-nucleous rigging of the Fock space Φ in the form (12.5):

$$\mathcal{D} \subset \Phi_+ \subset \Phi \subset \Phi_- \subset \mathcal{D}'.$$

Consider now any vector $|(\alpha)_n\rangle \in \Phi^{\otimes n}_{(s),}$ $n \in \mathbb{Z}_+$, which can be written [31, 56, 211] in the following canonical Dirac ket-form:

$$|(\alpha)_n\rangle := |\alpha_1, \alpha_2, ..., \alpha_n\rangle, \tag{12.8}$$

where

$$|\alpha_1, \alpha_2, ..., \alpha_n\rangle := \frac{1}{\sqrt{n!}} \sum_{\sigma\in S_n} |\alpha_{\sigma(1)}\rangle \otimes |\alpha_{\sigma(2)}\rangle ... |\alpha_{\sigma(n)}\rangle \tag{12.9}$$

and $|\alpha_j\rangle \in \Phi^{\otimes 1}_{(s)}(\mathbb{R}^m;\mathbb{C}) := \mathcal{H}$ for any fixed $j \in \mathbb{Z}_+$. The corresponding scalar product of base vectors with (12.9) is given as

$$\begin{aligned} &\langle(\beta)_n|(\alpha)_n\rangle := \langle\beta_n, \beta_{n-1}, ..., \beta_2, \beta_1|\alpha_1, \alpha_2, ..., \alpha_{n-1}, \alpha_n\rangle \\ &= \textstyle\sum_{\sigma\in S_n} \langle\beta_1|\alpha_{\sigma(1)}\rangle ... \langle\beta_n|\alpha_{\sigma(n)}\rangle := per\{\langle\beta_i|\alpha_j\rangle : 1 \le i, j \le n\}, \end{aligned} \tag{12.10}$$

where "*per*" denotes the permanent of the matrix and $\langle .|.\rangle$ is the corresponding product in the Hilbert space $\mathcal{H}$. Using the representation (12.8), one can define an operator $a^+(\alpha) : \Phi^{\otimes n}_{(s)} \longrightarrow \Phi^{\otimes (n+1)}_{(s)}$ for any $|\alpha\rangle \in \mathcal{H}$ as follows:

$$a^+(\alpha)|\alpha_1, \alpha_2, ..., \alpha_n\rangle := |\alpha, \alpha_1, \alpha_2, ..., \alpha_n\rangle, \tag{12.11}$$

which is called the "creation" operator in the Fock space Φ. The adjoint operator $a(\beta) := (a^+(\beta))^* : \Phi^{\otimes (n+1)}_{(s)} \longrightarrow \Phi^{\otimes n}_{(s)}$ with respect to the Fock space Φ (12.5) for any $|\beta\rangle \in \mathcal{H}$, is called the "annihilation" operator, and it acts in the form

$$a(\beta)|\alpha_1, \alpha_2, ..., \alpha_{n+1}\rangle := \sum_{\sigma \in S_n} \langle \beta, \alpha_j \rangle |\alpha_1, \alpha_2, ..., \alpha_{j-1}, \hat{\alpha}_j, \alpha_{j+1}, ..., \alpha_{n+1}\rangle, \tag{12.12}$$

where the "*hat*" over a vector means that it should be omitted from the sequence.

It is easy to check that the commutator relationship

$$[a(\alpha), a^+(\beta)] = \langle \alpha, \beta \rangle \tag{12.13}$$

holds for any vectors $|\alpha\rangle \in \mathcal{H}$ and $|\beta\rangle \in \mathcal{H}$. Expression (12.13), owing to the rigged structure (12.5), can be naturally extended to the general case, when vectors $|\alpha\rangle$ and $|\beta\rangle \in \mathcal{H}_-$, while conserving its form. In particular, taking $|\alpha\rangle := |\alpha(x)\rangle = \frac{1}{\sqrt{2\pi}} e^{i\langle \lambda, x \rangle} \in \mathcal{H}_- := L_{2,-}(\mathbb{R}^m; \mathbb{C})$ for any $x \in \mathbb{R}^m$, it follows easily from (12.13) that

$$[a(x), a^+(y)] = \delta(x - y), \tag{12.14}$$

where $a^+(x) := a^+(\alpha(x))$ and $a(y) := a(\alpha(y))$ for all $x, y \in \mathbb{R}^m$ and $\delta(\cdot)$ denotes the classical Dirac delta-function.

The construction above makes it easy to see that there exists a unique vacuum vector $|\Omega\rangle \in \mathcal{H}_+$ such that for any $x \in \mathbb{R}^m$

$$a(x)|\Omega\rangle = 0, \tag{12.15}$$

and the set of vectors

$$\left(\prod_{j=1}^{n} a^+(x_j) \right) |\Omega\rangle \in \Phi_{(s)}^{\otimes n} \tag{12.16}$$

is total in $\Phi_{(s)}^{\otimes n}$; that is, its linear integral hull over the dual functional spaces $\hat{\Phi}_{(s)}^{\otimes n}$ is dense in the Hilbert space $\Phi_{(s)}^{\otimes n}$ for every $n \in \mathbb{Z}_+$. This means that for any vector $g \in \Phi$ the representation

$$g = \oplus_{n \in \mathbb{Z}_+} \int_{(\mathbb{R}^m)^n} \hat{g}_n(x_1, ..., x_n) a^+(x_1) a^+(x_2) ... a^+(x_n) |\Omega\rangle \tag{12.17}$$

holds with the Fourier coefficients $\hat{g}_n \in \hat{\Phi}_{(s)}^{\otimes n}$ for all $n \in \mathbb{Z}_+$, with $\hat{\Phi}_{(s)}^{\otimes 1} := \mathcal{H} \simeq L^2(\mathbb{R}^m; \mathbb{C})$. This is naturally endowed with the Gelfand type quasi-nucleous rigging dual to

$$\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-, \tag{12.18}$$

allowing construction of a quasi-nucleous rigging of the dual Fock space $\hat{\Phi} := \oplus_{n\in\mathbb{Z}_+} \hat{\Phi}_{(s)}^{\otimes n}$. Whence, the chain (12.18) generates the dual Fock space quasi-nucleous rigging

$$\hat{\mathcal{D}} \subset \hat{\Phi}_+ \subset \hat{\Phi} \subset \hat{\Phi}_- \subset \hat{\mathcal{D}}' \tag{12.19}$$

with respect to the central Fock type Hilbert space $\hat{\Phi}$, where $\hat{\mathcal{D}} \simeq \mathcal{D}$, follows easily from (12.5) and (12.18).

Next, we construct the following self-adjoint operator

$$a^+(x)a(x) := \rho(x) : \Phi \to \Phi, \tag{12.20}$$

called the *density operator* at a point $x \in \mathbb{R}^m$, satisfying the commutation properties

$$\begin{aligned} [\rho(x), \rho(y)] &= 0, \\ [\rho(x), a(y)] &= -a(y)\delta(x-y), \\ [\rho(x), a^+(y)] &= a^+(y)\delta(x-y) \end{aligned} \tag{12.21}$$

for all $y \in \mathbb{R}^m$.

To construct the self-adjoint family $\mathcal{A} := \left\{\int_{\mathbb{R}^m} \rho(x)\varphi(x)dx : \varphi \in F\right\}$ of linear operators in the Fock space Φ, where $F := \mathcal{S}(\mathbb{R}^m;\mathbb{R})$ is the Schwartz functional space, one can show, using Theorem 12.1, that there exists a generalized Fourier transform (12.4) such that

$$\Phi(\mathcal{H}) = L^2_{(\mu)}(\mathcal{S}';\mathbb{C}) \simeq \int_{\mathcal{S}'}^{\oplus} \Phi_\eta d\mu(\eta) \tag{12.22}$$

for some Hilbert space sets Φ_η, $\eta \in F'$, and a suitable measure μ on $\mathcal{S}'$, with respect to which the corresponding joint eigenvector $\omega(\eta) \in \Phi_+$ for any $\eta \in F'$, generates the Fourier transformed family $\hat{\mathcal{A}} = \{\eta(\varphi) \in \mathbb{R} : \ \varphi \in \mathcal{S}\}$. Moreover, if $\dim \Phi_\eta = 1$ for all $\eta \in F$, the Fourier transformed eigenvector $\hat{\omega}(\eta) := \Omega(\eta) = 1$ for all $\eta \in F^{'}$.

Next, we consider the family of self-adjoint operators $\mathcal{A}$ as generating a unitary family $\mathcal{U} := \{U(\varphi) : \varphi \in F\} = \exp(i\mathcal{A})$, where for any $\rho(\varphi) \in \mathcal{A}$, $\varphi \in F$, the operator

$$U(\varphi) := \exp[i\rho(\varphi)] \tag{12.23}$$

is unitary and satisfies the abelian commutation condition

$$U(\varphi_1)U(\varphi_2) = U(\varphi_1 + \varphi_2) \tag{12.24}$$

for any $\varphi_1, \varphi_2 \in F$.

Since, in general, the unitary family $\mathcal{U} = \exp(i\mathcal{A})$ is defined in a Hilbert space Φ, which is not necessarily of Fock type, the important problem of describing its Hilbertian cyclic representation spaces arises, in which the factorization

$$\rho(\varphi) = \int_{\mathbb{R}^m} a^+(x)a(x)\varphi(x)dx \tag{12.25}$$

together with the relationships (12.21) hold for any $\varphi \in F$. This problem can be treated using mathematical tools devised for the representation theory of C^*-algebras [100, 104] and the Gelfand–Vilenkin [142] approach. Below we shall describe the main features of the Gelfand–Vilenkin formalism, which is better suited to providing a reasonably unified framework for constructing the corresponding representations.

Definition 12.1. Let F be a locally convex topological vector space, and $F_0 \subset F$ be a finite dimensional subspace of F. Let $F^0 \subseteq F'$ be defined by

$$F^0 := \{\xi \in F' : \ \ \xi|_{F_0} = 0\}, \tag{12.26}$$

and called the *annihilator* of F_0.

The quotient space $F'^0 := F'/F^0$ may be identified with $F_0' \subset F'$, the adjoint space of F_0.

Definition 12.2. Let $A \subseteq F'$; then the subset

$$X_{F^0}^{(A)} := \left\{\xi \in F' : \xi + F^0 \subset A\right\} \tag{12.27}$$

is called the *cylinder set* with base A and generating subspace F^0.

Definition 12.3. Let $n = \dim F_0 = \dim F_0' = \dim F'^0$. One says that a cylinder set $X^{(A)}$ has *Borel base*, if A is Borel when regarded as a subset of $\mathbb{R}^n$.

The family of cylinder sets with Borel base forms an algebra of sets.

Definition 12.4. The measurable sets in F' are the elements of the σ-algebra generated by the cylinder sets with Borel base.

Definition 12.5. A cylindrical measure in F' is a real-valued σ-pre-additive function μ defined on the algebra of cylinder sets with Borel base and satisfying the conditions $0 \leq \mu(X) \leq 1$ for any X, $\mu(F') = 1$ and $\mu\left(\coprod_{j\in\mathbb{Z}_+} X_j\right) = \sum_{j\in\mathbb{Z}_+} \mu(X_j)$, if all sets $X_j \subset F'$, $j \in \mathbb{Z}_+$, have a common generating subspace $F_0 \subset F$.

Definition 12.6. A cylindrical measure μ satisfies the commutativity condition if and only if for any bounded continuous function $\alpha : \mathbb{R}^n \longrightarrow \mathbb{R}$ of $n \in \mathbb{Z}_+$ real variables, the function

$$\alpha[\varphi_1, \varphi_2, ..., \varphi_n] := \int_{F'} \alpha(\eta(\varphi_1), \eta(\varphi_2), ..., \eta(\varphi_n)) d\mu(\eta) \tag{12.28}$$

is sequentially continuous in $\varphi_j \in F$, $1 \leq j \leq n$. (It is well known [142, 146] that in countably normalized spaces the properties of sequential and ordinary continuity are equivalent.)

Definition 12.7. A cylindrical measure μ is countably additive if and only if for any cylinder set $X = \coprod_{j \in \mathbb{Z}_+} X_j$, which is the union of countably many mutually disjoint cylinder sets $X_j \subset F'$, $j \in \mathbb{Z}_+$, $\mu(X) = \sum_{j \in \mathbb{Z}_+} \mu(X_j)$.

The following propositions can be readily proved using the concepts so far developed.

Proposition 12.1. *A countably additive cylindrical measure μ can be extended to a countably additive measure on the σ-algebra generated by the cylinder sets with Borel base. Such a measure will also be called a **cylindrical measure**.*

Proposition 12.2. *Let F be a nuclear space. Then any cylindrical measure μ on F' satisfying the continuity condition is countably additive.*

Definition 12.8. Let μ be a cylindrical measure in F'. The Fourier transform of μ is the nonlinear functional

$$\mathcal{L}(\varphi) := \int_{F'} \exp[i\eta(\varphi)] d\mu(\eta). \tag{12.29}$$

Definition 12.9. The nonlinear functional $\mathcal{L} : F \longrightarrow \mathbb{C}$ on F defined by (12.29) is called **positive-definite**, if and only if for all $f_j \in F$ and $\lambda_j \in \mathbb{C}$, $j = 1, \ldots, n$, the condition

$$\sum_{j,k=1}^{n} \bar{\lambda}_j \mathcal{L}(f_k - f_j) \lambda_k \geq 0 \tag{12.30}$$

holds for any $n \in \mathbb{Z}_+$.

Proposition 12.3. *The functional $\mathcal{L} : F \longrightarrow \mathbb{C}$ on F, defined by (12.29), is the Fourier transform of a cylindrical measure on F', if and only if it is positive-definite, sequentially continuous and satisfies the condition $\mathcal{L}(0) = 1$.*

Suppose now that we have a continuous unitary representation of the unitary family $\mathcal{U}$ in a Hilbert space Φ with a cyclic vector $|\Omega\rangle \in \Phi$. Then we can set

$$\mathcal{L}(\varphi) := \langle \Omega | U(\varphi) | \Omega \rangle \tag{12.31}$$

for any $\varphi \in F := \mathcal{S}$ - the Schwartz space on $\mathbb{R}^m$ - and observe that functional (12.31) is continuous on F owing to the continuity of the representation. Therefore, this functional is the generalized Fourier transform of a cylindrical measure μ on $\mathcal{S}'$:

$$\langle \Omega | U(\varphi) | \Omega \rangle = \int_{\mathcal{S}'} \exp[i\eta(\varphi)] d\mu(\eta). \tag{12.32}$$

From the spectral point of view based on Theorem 12.1, there is an isomorphism between the Hilbert spaces Φ and $L^2_{(\mu)}(\mathcal{S}'; \mathbb{C})$, defined by $|\Omega\rangle \longrightarrow \Omega(\eta) = 1$ and $U(\varphi)|\Omega\rangle \longrightarrow \exp[i\eta(\varphi)]$ and extended by linearity to the whole Hilbert space Φ.

In the non-cyclic case there exists a finite or denumerably infinite family of measures $\{\mu_k : k \in \mathbb{Z}_+\}$ on $\mathcal{S}'$, with $\Phi \simeq \oplus_{k \in \mathbb{Z}_+} L^2_{(\mu)}(\mathcal{S}'; \mathbb{C})$, and the unitary operator $U(\varphi) : \Phi \longrightarrow \Phi$ for any $\varphi \in \mathcal{S}'$ corresponds in all $L^2_{(\mu)}(\mathcal{S}'; \mathbb{C})$, $k \in \mathbb{Z}_+$, to $\exp[i\eta(\varphi)]$. This means that there exists a unique cylindrical measure μ on $\mathcal{S}'$ and a μ-measurable field of Hilbert spaces Φ_η on $\mathcal{S}'$ such that

$$\Phi \simeq \int_{\mathcal{S}'}^{\oplus} \Phi_\eta d\mu(\eta), \tag{12.33}$$

with $U(\varphi) : \Phi \longrightarrow \Phi$ corresponding [142] to the operator of multiplication by $\exp[i\eta(\varphi)]$ for any $\varphi \in \mathcal{S}$ and $\eta \in \mathcal{S}'$. Thus, upon constructing the nonlinear functional (12.29) in an exact analytical form, one can retrieve the representation of the unitary family $\mathcal{U}$ in the corresponding Hilbert space Φ of the Fock type, making use of the appropriate factorization (12.25) as follows: $\Phi = \oplus_{n \in \mathbb{Z}_+} \Phi_n$, where

$$\Phi_n = \underset{f_n \in L^2_s((R^m)^n; \mathbb{C})}{\operatorname{span}} \left\{ \prod_{1 \le j \le n} a^+(x_j) |\Omega\rangle \right\}, \tag{12.34}$$

for all $n \in \mathbb{Z}_+$. The cyclic vector $|\Omega\rangle \in \Phi$ can be, in particular, obtained as the ground state vector of some unbounded self-adjoint positive-definite Hamilton operator $\mathbb{H} : \Phi \longrightarrow \Phi$, commuting with the self-adjoint particles number operator

$$\mathbb{N} := \int_{\mathbb{R}^m} \rho(x) dx; \tag{12.35}$$

that is, $[\mathbb{H}, \mathbb{N}] = 0$. Moreover, the conditions

$$\mathbb{H}|\Omega\rangle = 0 \tag{12.36}$$

and

$$\inf_{g \in dom\mathbb{H}} \langle g, \mathbb{H}g\rangle = \langle\Omega|\mathbb{H}|\Omega\rangle = 0 \tag{12.37}$$

hold for the operator $\mathbb{H} : \Phi \longrightarrow \Phi$, where $dom\mathbb{H}$ denotes its domain of definition.

To find the functional (12.31), which is called the *generating Bogolubov functional* for moment distribution functions

$$F_n(x_1, x_2, ..., x_n) := \langle\Omega| : \rho(x_1)\rho(x_2)...\rho(x_n) : |\Omega\rangle, \tag{12.38}$$

where $x_j \in \mathbb{R}^m$, $1 \leq j \leq n$, and the normal ordering operation $: \cdot :$ is defined as

$$: \rho(x_1)\rho(x_2)...\rho(x_n) : = \prod_{j=1}^{n} \left(\rho(x_j) - \sum_{k=1}^{j} \delta(x_j - x_k) \right). \tag{12.39}$$

It is convenient to choose the Hamilton operator $\mathbb{H} : \Phi \longrightarrow \Phi$ in the [60, 146, 147] algebraic form

$$\mathbb{H} := \frac{1}{2}\int_{\mathbb{R}^m} K^+(x)\rho^{-1}(x)K(x)dx + V(\rho), \tag{12.40}$$

which is equivalent in Φ to the positive-definite operator expression

$$\mathbb{H} := \frac{1}{2}\int_{\mathbb{R}^m} (K^+(x) - A(x;\rho))\rho^{-1}(x)(K(x) - A(x;\rho))dx, \tag{12.41}$$

where $A(x;\rho) : \Phi \rightarrow \Phi$, $x \in \mathbb{R}^m$ is a specially chosen linear self-adjoint operator. The "potential" operator $V(\rho) : \Phi \longrightarrow \Phi$ is, in general, a polynomial (or analytic) functional of the density operator $\rho(x) : \Phi \longrightarrow \Phi$, and the operator is given as

$$K(x) := \nabla_x\rho(x)/2 + iJ(x), \tag{12.42}$$

where the self-adjoint "current" operator $J(x) : \Phi \longrightarrow \Phi$ can be defined (but non-uniquely) from the equality

$$\partial\rho/\partial t = \frac{1}{i}[\mathbb{H}, \rho(x)] = - < \nabla_x \cdot J(x) >, \tag{12.43}$$

valid for all $x \in \mathbb{R}^m$. Such an operator $J(x) : \Phi \longrightarrow \Phi$, $x \in \mathbb{R}^m$ exists owing to the commutation condition $[\mathbb{H}, \mathbb{N}] = 0$, which gives rise to the continuity relationship (12.43) - if one takes into account that the support $\mathrm{supp}\rho$ of the density operator $\rho(x) : \Phi \longrightarrow \Phi$, $x \in \mathbb{R}^m$ can be chosen arbitrarily since it does not depend on the potential operator $V(\rho) : \Phi \longrightarrow \Phi$. However, it does depend on the corresponding representation (12.33). Note also that representation (12.41) holds only under the condition that there exists a self-adjoint operator $A(x;\rho) : \Phi \longrightarrow \Phi$, $x \in \mathbb{R}^m$ such that

$$K(x)|\Omega\rangle = A(x;\rho)|\Omega\rangle \tag{12.44}$$

for all ground states $|\Omega\rangle \in \Phi$ corresponding to suitably chosen potential operators $V(\rho) : \Phi \longrightarrow \Phi$.

The self-adjointness of the operator $A(x;\rho) : \Phi \longrightarrow \Phi$, $x \in \mathbb{R}^m$, can be stated following schemes developed in [60, 147], under the additional condition that there exists a linear anti-unitary mapping $T : \Phi \longrightarrow \Phi$ satisfying the invariance conditions

$$T\rho(x)T^{-1} = \rho(x), \qquad T\,J(x)\,T^{-1} = -J(x), \qquad T|\Omega\rangle = |\Omega\rangle \tag{12.45}$$

for any $x \in \mathbb{R}^m$. Whence, owing to conditions (12.45), the following expressions

$$K^*(x)|\Omega\rangle = A(x;\rho)|\Omega\rangle = K(x)|\Omega\rangle \tag{12.46}$$

hold for any $x \in \mathbb{R}^m$, and imply the self-adjointness of the operator $A(x;\rho) : \Phi \longrightarrow \Phi$, $x \in \mathbb{R}^m$.

One can easily deduce from the above constructions and expression (12.43) that the generating Bogolubov functional (12.31) satisfies for all $x \in \mathbb{R}^m$ the functional-differential equation

$$[\nabla_x - i\nabla_x\varphi]\,\frac{1}{2i}\frac{\delta\mathcal{L}(\varphi)}{\delta\varphi(x)} = A\left(x;\frac{1}{i}\frac{\delta}{\delta\varphi}\right)\mathcal{L}(\varphi), \tag{12.47}$$

whose solutions satisfy the Fourier transform representation (12.32). In particular, a wide class of special so-called Poissonian white noise type solutions to the functional-differential equation (12.47) was obtained in [60, 147] by means of functional-operator methods in the following generalized form:

$$\mathcal{L}(\varphi) = \exp\left\{A\left(\frac{1}{i}\frac{\delta}{\delta\varphi}\right)\right\}\exp\left(\bar{\rho}\int_{\mathbb{R}^m}\{\exp[i\varphi(x)] - 1\}dx\right), \tag{12.48}$$

where $\bar{\rho} := \langle\Omega|\rho|\Omega\rangle \in \mathbb{R}_+$ is a Poisson distribution density parameter.

Now suppose that the basic Fock space $\Phi = \otimes_{j=1}^{s}\Phi^{(j)}$, where $\Phi^{(j)}$, $1 \leq j \leq s$, are Fock spaces corresponding to the different types of independent cyclic vectors $|\Omega_j\rangle \in \Phi^{(j)}$, $1 \leq j \leq s$. This, in particular, means that the suitably constructed creation and annihilation operators $a_j(x), a_k^+(y) : \Phi \longrightarrow \Phi$, $j,k = 1,\ldots,s$, satisfy the following commutation relations:

$$\begin{aligned} &[a_j(x), a_k(y)] = 0, \\ &[a_j(x), a_k^+(y)] = \delta_{jk}\delta(x-y) \end{aligned} \tag{12.49}$$

for any $x, y \in \mathbb{R}^m$.

Definition 12.10. A vector $|u\rangle \in \Phi$, $x \in \mathbb{R}^m$, is called **coherent** with respect to a mapping $u \in L^2(\mathbb{R}^m; \mathbb{R}^s) := M$, if it satisfies the eigenfunction condition

$$a_j(x)|u\rangle = u_j(x)|u\rangle \tag{12.50}$$

for each $j = 1, \ldots, s$ and all $x \in \mathbb{R}^m$.

It is easy to check that the coherent vectors $|u\rangle \in \Phi$ exist. In fact, the vector expression

$$|u\rangle := \exp\{(u, a^+)\}|\Omega\rangle, \tag{12.51}$$

where $(\cdot, \cdot)$ is the standard scalar product in the Hilbert space M, satisfies the defining condition (12.50), and moreover, the norm

$$\|u\|_\Phi := \langle u|u\rangle^{1/2} = \exp(\frac{1}{2}\|u\|^2) < \infty, \tag{12.52}$$

since $u \in M$ and its norm $\|u\| := (u, u)^{1/2}$ is bounded.

12.3 The Fock space embedding method, nonlinear dynamical systems and their complete linearization

Consider any function $u \in M := L^2(\mathbb{R}^m; \mathbb{R}^s)$ and observe that the Fock space embedding mapping

$$\xi : M \ni u \longrightarrow |u\rangle \in \Phi, \tag{12.53}$$

defined by means of the coherent vector expression (12.51) realizes a smooth isomorphism between the Hilbert spaces M and Φ. The inverse mapping $\xi^{-1} : \Phi \longrightarrow M$ is given by the following exact expression:

$$u(x) = \langle\Omega|a(x)|u\rangle, \tag{12.54}$$

valid for almost all $x \in \mathbb{R}^m$. Owing to condition (12.52), one finds from (12.54) that the corresponding function $u \in M$.

In the Hilbert space M, we now define a nonlinear dynamical system (which can, in general, be nonautonomous) in partial derivatives

$$du/dt = K[u], \tag{12.55}$$

where $t \in \mathbb{R}_+$ is the corresponding evolution parameter, $[u] := (t, x; u, u_x, u_{xx}, ..., u_{rx}), r \in \mathbb{Z}_+$, and $K : M \longrightarrow T(M)$ is Fréchet smooth. Assume also that the Cauchy problem

$$u|_{t=+0} = u_0 \tag{12.56}$$

is solvable for any $u_0 \in M$ in an interval $[0,T) \subset \mathbb{R}^1_+$ for some $T > 0$. From this the smooth evolution mapping is defined

$$T_t : M \ni u_0 \longrightarrow u(t|u_0) \in M, \tag{12.57}$$

for all $t \in [0,T)$.

It is now natural to consider the following commuting diagram

$$\begin{array}{ccc} M & \xrightarrow{\xi} & \Phi \\ T_t \downarrow & & \downarrow \mathbb{T}_t \\ M & \xrightarrow{\xi} & \Phi, \end{array} \tag{12.58}$$

where the mapping $\mathbb{T}_t : \Phi \longrightarrow \Phi$, $t \in [0,T)$, is defined from the conjugation relationship

$$\xi \circ T_t = \mathbb{T}_t \circ \xi. \tag{12.59}$$

Now take a coherent vector $|u_0\rangle \in \Phi$ corresponding to $u_0 \in M$, and construct the vector

$$|u\rangle := \mathbb{T}_t \cdot |u_0\rangle \tag{12.60}$$

for all $t \in [0,T)$. Since vector (12.60) is, by construction, coherent, that is

$$a_j(x)|u\rangle := u_j(x,t|u_0)|u\rangle \tag{12.61}$$

for each $1 \leq j \leq s$, $t \in [0,T)$ and almost all $x \in \mathbb{R}^m$, owing to the smoothness of the mapping $\xi : M \longrightarrow \Phi$ with respect to the corresponding norms in the Hilbert spaces M and Φ. We readily deduce that coherent vector (12.60) is differentiable with respect to the evolution parameter $t \in [0,T)$. Thus, one can easily find [209, 211, 319, 320] that

$$\frac{d}{dt}|u\rangle = \hat{K}[a^+, a]|u\rangle, \tag{12.62}$$

where

$$|u\rangle|_{t=+0} = |u_0\rangle \tag{12.63}$$

and a mapping $\hat{K}[a^+, a] : \Phi \longrightarrow \Phi$ is defined by the exact analytical expression

$$\hat{K}[a^+, a] := (a^+, K[a]). \tag{12.64}$$

As a result of the considerations above we obtain the following theorem.

Theorem 12.2. *Any smooth nonlinear dynamical system (12.55) in Hilbert space $M := L^2(\mathbb{R}^m; \mathbb{R}^s)$ is representable by means of the Fock space embedding isomorphism $\xi : M \longrightarrow \Phi$ in the completely linear form (12.62).*

We now make some comments concerning the solution to the linear equation (12.62) under the Cauchy condition (12.63). Since any vector $|u\rangle \in \Phi$ allows the series representation

$$|u\rangle = \bigoplus_{n=\sum_{j=1}^{s} n_j \in \mathbb{Z}_+} \frac{1}{(n_1!n_2!...n_s!)^{1/2}} \int_{(\mathbb{R}^m)^n} f^{(n)}_{n_1 n_2 \ldots n_s}(x_1^{(1)}, x_2^{(1)}, ..., x_{n_1}^{(1)}; \\ x_1^{(2)}, x_2^{(2)}, ..., x_{n_2}^{(2)}; ...; x_1^{(s)}, x_2^{(s)}, ..., x_{n_s}^{(s)}) \prod_{j=1}^{s} \left(\prod_{k=1}^{n_j} dx_k^{(j)} a_j^+(x_k^{(j)}) \right) |\Omega\rangle, \tag{12.65}$$

where for any $n = \sum_{j=1}^{s} n_j \in \mathbb{Z}_+$ functions

$$f^{(n)}_{n_1 n_2 \ldots n_s} \in \bigotimes_{j=1}^{s} L_s^2((\mathbb{R}^m)^{n_j}; \mathbb{C}) \simeq L_s^2(\mathbb{R}^{mn_1} \times \mathbb{R}^{mn_2} \times ...\mathbb{R}^{mn_s}; \mathbb{C}), \tag{12.66}$$

and the norm is

$$\|u\|_\Phi^2 = \sum_{n=\sum_{j=1}^{s} n_j \in \mathbb{Z}_+} \|f^{(n)}_{n_1 n_2 \ldots n_s}\|_2^2 = \exp(\|u\|^2). \tag{12.67}$$

Substituting (12.65) into equation (12.62), reduces (12.62) to an infinite recurrent set of linear evolution equations in partial derivatives on coefficient functions (12.66). This can often be solved [209, 319, 320] step-by-step analytically in exact form, thereby making it possible to obtain, owing to representation (12.54), the exact solution $u \in M$ to the Cauchy problem (12.56) for our nonlinear dynamical system in partial derivatives (12.55).

Remark 12.1. For applications of nonlinear dynamical systems such as (12.53) in mathematical physics problems, it is very important to construct their so-called conservation laws or smooth invariant functionals $\gamma : M \longrightarrow \mathbb{R}$ on M. The quantum mathematics technique described above suggests an effective algorithm for constructing these conservation laws.

Indeed, consider a vector $|\gamma\rangle \in \Phi$, satisfying the linear equation:

$$\frac{\partial}{\partial t}|\gamma\rangle + \hat{K}^*[a^+, a]|\gamma\rangle = 0. \tag{12.68}$$

Then, the following proposition [209, 319, 320] holds.

Proposition 12.4. *The functional*

$$\gamma := \langle u|\gamma\rangle \tag{12.69}$$

is a conservation law for the dynamical system (12.53), that is

$$d\gamma/dt|_K = 0 \tag{12.70}$$

along any orbit of the evolution mapping (12.57).

12.4 Summary remarks and preview

We have described the main mathematical preliminaries and properties of quantum mathematics suitable for linearizing a problem for a wide class of nonlinear dynamical systems in partial derivatives in Hilbert spaces. This problem was analyzed in considerable detail using the Gelfand–Vilenkin representation theory [59, 142] of infinite-dimensional groups and the Goldin–Menikoff–Sharp theory [146–148] of generating Bogolubov type functionals, and classifying these representations. The related problem of constructing Fock space representations and retrieving their creation-annihilation generating structure still needs a deeper investigation within the confines of the approach devised. Here we mention only that some aspects of this problem related to Poissonian white noise analysis were studied using generalizations of the Delsarte characters technique in a series of works [11, 29, 30, 200, 241].

It is also necessary to mention the related results obtained in [209–211, 321, 322, 367], which are devoted to the application of the Fock space embedding method to studying solutions to a variety of nonlinear dynamical systems and to constructing quantum computing algorithms. Applications to quantum computing are the subject of the next section.

12.5 A geometric approach to quantum holonomic computing algorithms

12.5.1 *Introduction*

In this section we show, using the approach of Samoilenko et al. [367], how the tools developed so far can be used as a mathematical foundation for quantum computing. Consider a two-dimensional vector subspace $F^{(2)} \subset \mathcal{H}$ of a complex Hilbert space $\mathcal{H}$ called a quantum bit (*qubit*) information space defined as

$$F^{(2)} := \operatorname{span}_{\mathbb{C}}\{|0>, |1> \in \mathcal{H}\} \tag{12.71}$$

with orthonormal base vectors $|0>, |1> \in \mathcal{H}$ with respect to the scalar product $< \cdot, \cdot >$ in $\mathcal{H}$ and also its n-th tensor product

$$\mathcal{H}^{(n)} := \bigotimes\nolimits_{k=1}^{n} F^{(2)} \tag{12.72}$$

which a Hilbert space of $\dim \mathcal{H}^{(n)} = 2^n$, called an *n-qubit* computation medium. Note that we shall use the standard “bra” and “ket” notation, which has already been initiated, throughout our analysis.

Now we shall consider one of simplest computational tasks in this medium: calculate the values, up to some required degree of accuracy, of a polynomially computable function $f : \mathbb{R} \to \mathbb{R}$, that is we must compute all the values $f(x)$, $x \in \mathbb{R}$, within a prescribed tolerance. To begin with, any $x \in \mathbb{R}$ has an essentially unique binary expansion, so we have a bijection

$$x \leftrightarrow \{x_i\}, \tag{12.73}$$

where

$$x = \sum\nolimits_{i\in\mathbb{Z}} x_i 2^i, \qquad x_i \in \{0,1\} \tag{12.74}$$

for all $i \in \mathbb{Z}$. The (truncated) sequence $(x_1, \ldots, x_n)_2$ can be embedded in the Hilbert space $\mathcal{H}^{(n)}$

$$\mathbb{R} \ni x \leftrightarrow (x_1, \ldots, x_n)_2 \leftrightarrow |x_1, ..., x_n >\in \mathcal{H}^{(n)}. \tag{12.75}$$

So, for any $x \in \mathbb{R}$ one can construct an n-qubit state via the induced mapping

$$\mathbb{R} \ni x \leftrightarrow |x_1, x_2, x_3, ..., x_n >:= |(x) >\in \mathcal{H}^{(n)}. \tag{12.76}$$

On the set $\{0,1\} \in \mathbb{Z}_2$ one has the usual algebraic operations: if $x, y \in \mathbb{Z}_2$, then

$$x \oplus y := x + y \ (\text{mod } 2) \tag{12.77}$$

so

$$\begin{aligned} &0 \oplus 0 = 0, \quad 1 \oplus 1 = 0, \quad 0 \oplus 1 = 1, \quad 1 + 0 = 1, \\ &0 \cdot 1 = 0, \quad 1 \cdot 1 = 1, \quad 1 \cdot 0 = 0. \end{aligned} \tag{12.78}$$

Definition 12.11. The following unitary transform of the Hilbert space $\mathcal{H}^{(n+k)}$ for a mapping $f : \mathbb{Z}_2^n \to \mathbb{Z}_2^k$, $n, k \in \mathbb{Z}_+$,

$$U_f : |(x), (a) >\mapsto |(x), (a) \oplus (f(x) >, \tag{12.79}$$

where the values $(f(x))$ and (a) are defined, respectively, as $(f_1(x), f_2(x), ..., f_k(x))_2$ and $(a_1, a_2, ..., a_k)_2$ for the mapping (12.76) leads to a quantum computation subject to any fixed test vector $|(a) > \in \mathcal{H}^{(k)}$, $k \in \mathbb{Z}_+$.

This computation is usually expressed [360] graphically as

$$\begin{array}{l} |(x) > \mapsto \\ |(a) > \mapsto \end{array} \boxed{U_f} \begin{array}{l} \mapsto \qquad |(x) > \\ \mapsto |(a) \oplus (f(x) > \end{array}. \tag{12.80}$$

From (12.79) and linearity, one readily obtains

$$U_f : \sum_{x\in\mathbb{Z}_+} \alpha_x|(x),(a) > \mapsto \sum_{x\in\mathbb{Z}_+} \alpha_x|(x),(a) \oplus (f(x) > \qquad (12.81)$$

for all $\alpha_x \in \mathbb{C},\ x \in \mathbb{Z}_+$. Thus, the unitary operator $U_f : \mathcal{H}^{(n+k)} \mapsto \mathcal{H}^{(n+k)}$ counts all desired values of pairs $|(x),(f(x)) >$, $x \in \mathbb{R}$.

Example 12.1. [360] Consider a function $f : \mathbb{Z}_2^n \mapsto \{0,1\} \subset \mathbb{Z}_2$, such that $f(x) = 1$ if the sign of an element $x \in X_1$ changes, and $f(x) = 0$ otherwise, where $\mathbb{Z}_2^n := X_0 \cup X_1$, $X_0 \cap X_1 = \varnothing$.

Take the test element $|(a) > \in \mathcal{H}^{(1)}$

$$|(a) > := \frac{1}{\sqrt{2}}(|0 > -|1 >) \qquad (12.82)$$

and compute the value

$$\begin{aligned} U_f(\sum_{x\in\mathbb{Z}_2^n} \alpha_x|(x) > \otimes\frac{1}{\sqrt{2}}(|0 > -|1 >)) = \frac{1}{\sqrt{2}}(&\sum_{x\in X_0} \alpha_x|(x), 0 > \\ &- \sum_{x\in X_0} \alpha_x|(x), |1 >) \end{aligned}$$

$$\begin{aligned} &= \frac{1}{\sqrt{2}}(\sum_{x\in X_0} \alpha_x|(x), 0 > - \sum_{x\in X_0} \alpha_x|(x), |1 >) + \frac{1}{\sqrt{2}}(\sum_{x\in X_1} \alpha_x|(x), 1 > \qquad (12.83) \\ &- \sum_{x\in X_1} \alpha_x|(x), |0 >) = (\sum_{x\in X_0} \alpha_x|(x) > - \sum_{x\in X_1} \alpha_x|(x) >) \otimes \frac{1}{\sqrt{2}}(|0 > -|1 >). \end{aligned}$$

Clearly, the U_f-operator (12.83) changes all the signs of $X_1 \ni x$-elements.

Algorithms of this type can effectively solve [360] the following important problems that usually require an inordinate amount of time using classical computers:

i) prime factorization of a large integer $x \in \mathbb{Z}_+$ (Shor [375]) and its application to decrypting messages encoded via the RSA system;

ii) search or sorting algorithm for finding an item in structured and unstructured data sets (Grover [160]; Hogg [175]);

iii) fast discrete Fourier transforms [375];

iv) finding periods of periodic functions ([375]; Kitaev et al. [194]).

The most important ingredient in quantum computing algorithms is the construction of the unitary transformations (12.79) and controlling its action on information data vectors from the proper quantum computation medium.

In what follows, we discuss some well-known examples that have recently been tackled using quantum computing algorithms.

Example 12.2. Public key Cryptography: (RSA)-cryptosystem. Take any message P which we want to send - in plaintext. Using a key to encipher it into ciphertext C, transmit it to the receiver who uses another key to decipher the C into P. A simple RSA scheme acts as follows: take any sufficiently large primes p and $q \in \mathbb{Z}_+$ and let $m := pq$. Knowing p and q, compute Euler function $\varphi(m) = (p-1)(q-1)$. Now choose an integer $e \in \mathbb{Z}_+$ coprime to $\varphi(m)$, i.e., $(\varphi(m), e) = 1$ (the greatest common divisor equals 1). Then the numbers m and e are published.

Consider a plaintext $P \in \mathbb{Z}_+$ (presented as a large integer) that is less than $m \to \mathbb{Z}_+$ and coprime to m, i.e., $(P, m) = 1$ (note that all integers $P < m$ are clearly coprime). The corresponding ciphertext is the unique integer $C := P^e (\text{mod } m)$, $0 < C < m$. It is important that prime factors p, q and $\varphi(m)$ are hidden from public. To decrypt the plaintext P from C, we need to find a receiver key - the integer $d \in \mathbb{Z}_+$, such that $ed \equiv 1 (\text{mod } \varphi(m))$, which can be found by using the Euclidean algorithm. Then, one has

$$C^d \equiv P^{ed} \equiv P^{(1+k\varphi(m))} \equiv P\ (P^{\varphi(m)})^k \equiv P(1)\ (\text{mod } m) :\Rightarrow P;$$

that is, the plain text P is decrypted.

Now we describe Shor's quantum counterpart of the algorithm described above. The main problem for classical computers to solve in a reasonable time is factoring an integer $m \in \mathbb{Z}_+$ into its primes. It is a classical result that the factorization problem reduces to finding the m-th order $r \in \mathbb{Z}_+$ of an element $a \in \mathbb{Z}_+$ in $\mathbb{Z}_m$, that is $a^r \equiv 1\ (\text{mod } m)$, $(a, m) = 1$. In general, $m \in \mathbb{Z}_+$ is taken to be odd as multiples of 2 are evidently not important, as well as $m \neq p^e$ for some $p \in \mathbb{Z}_+$ and power $e \in \mathbb{Z}_+$. By definition, $a^{e+sr} \equiv a^e\ (\text{mod } m)$ for any integers $e \in \mathbb{Z}_+$, $s \in \mathbb{Z}$, that is the function $f(k) := a^k\ (\text{mod } m)$ is r-periodic. This period $r \in \mathbb{Z}_+$ can be found easily making use of the Shor's QFT (Quantum Fourier Transform)-algorithm. To understand this transform consider an abstract abelian group $G = \{g_1, g_2, ..., g_n, ...\}$ such as $(\mathbb{Z}, +)$, $(\mathbb{Z}^*, \cdot)$, or $(\mathbb{Z}_n +, \cdot)$, denote by $\{\chi_i : 1 \leq i \leq \dim G\}$ its characters (the G-invariant functions on G in $\mathbb{C}^*$). The G-invariant functions $f : G \to \mathbb{C}$ form the linear space F_G, on which there exists the standard G-invariant scalar product $(f, h) := \sum_{i=1}^{\dim G} \bar{f}(g_i) h(g_i)$.

The corresponding norm of $f \in F_G$ is denoted as $||f||$, and the following lemma can be readily verified.

Lemma 12.1. *The characters $\chi_i \in F_G$ form an orthogonal basis of F_G, that is $\overline{\text{span}}_{\mathbb{C}}\{\chi_i : 1 \le i \le \dim G\} = F_G$ and $(\chi_i, \chi_j) = \delta_{ij}$, $1 \le i, j \le \dim G$. Thus any $f \in F_G$ can be uniquely expressed as $f = \sum_{i=1}^{\dim G} c_i \chi_i$, where $c_i \in \mathbb{C}$, $1 \le i \le \dim G$.*

Definition 12.12. A function $\hat{f} \in F_G$ is called the discrete Fourier transform of $f \in F_G$ if $c_i = \hat{f}(g_i)$, $1 \le i \le \dim\, G$. It follows that

$$\hat{f}(g_i) = \sum_{k=1}^{\dim G} \bar{\chi}_i(g_k) f(g_k)$$

for any $1 \le i \le \dim G$.

As a result, Parseval's identity $||f|| = ||\bar{f}||$ holds for any $f \in F_G$. Now we return to our quantum Hilbert space $\mathcal{H}$ of a sufficient large dimension such that all elements $g_i \in G$ can be represented by means of basis vectors $|g_i > \mathcal{H}$, $1 \le i \le \dim G$. A general element $g \in G$, $||g|| = 1$, is obviously represented in $\mathcal{H}$ as $|g >= \sum_{i=1}^{\dim G} c_i(g)|g_i >$, $< g_i|g_j >= \quad \delta_{ij}$, $1 \le i, j \le \dim G$, where $\sum_{k=1}^{\dim G} |c_i(g)| = 1$. One can now define for each i a function $\hat{f} \in F_G$, such that $\hat{f}(g_i) = c_i(g)$ and $||\hat{f}|| = 1$.

Definition 12.13. The Quantum Fourier Transform (QFT) is the operation $\sum_{i=1}^{\dim G} f(g_i)|g_i > \rightarrow \sum_{i=1}^{\dim G} \hat{f}(g_i)|g_i >$, where naturally $\hat{f}(g_i) = \sum_{k=1}^{\dim G} \bar{\chi}_i(g_k) f(g_k)$ for any $1 \le i \le \dim G$.

As a simple consequence of this definition, the QFT is a linear and unitary transformation. Whence, it follows readily that $|g_i > \rightarrow \sum_{k=1}^{\dim G} \chi_k(g_i)|g_k >$, $1 \le i \le \dim G$, so the corresponding unitary transformation matrix has the form $U_\chi := \{\bar{\chi}_i(g_j) : 1 \le i, j \le \dim G\}$. For example, let $G = \mathbb{Z}_n$, $\dim G = n$ and the corresponding characters be $\chi_y(x) = \exp(\frac{2\pi i}{n} xy)$, where $x, y \in \mathbb{Z}_n$. Then, the corresponding QFT on $\mathbb{Z}_n$ is the operation $|x > \overset{QFT}{\rightarrow} \frac{1}{\sqrt{n}} \sum_{y=0}^{n-1} \exp(-\frac{2\pi i}{n} xy)|y >$ for any $x, y \in \mathbb{Z}_n = \{0, 1, 2, ..., n-1\}$. Now embed $\mathbb{Z}_n$ in $\mathcal{H}$ as a basis $\{|k >\in \mathcal{H} : 0 \le k \le n-1\}$. For this assume that $n = n_1 n_2$ and $(n_1, n_2) = 1$. Then the Chinese Remainder Theorem implies that the following decomposition holds: $\mathbb{Z}_n \simeq \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$ and the isomorphic mapping $\Phi : \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \rightleftharpoons \mathbb{Z}_n$ acts as follows: $(k_1, k_2) = a_1 n_2 k_1 + a_2 n_1 k_2$, where a_1(resp. a_2) is the multiplicative inverse of n_2 in $\mathbb{Z}_{n_1}$ (of n_1 in $\mathbb{Z}_{n_2}$).

Now that we have made the QFT available for $\mathbb{Z}_{n_1}$ and $\mathbb{Z}_{n_2}$, one can easily construct the QFT for $\mathbb{Z}_{n!}$, taking into account that $U_{n_1} : |k_1 >\rightarrow \frac{1}{\sqrt{n_1}}\sum_{l_1=0}^{n_1-1}\exp(-\frac{2\pi i}{n_1}l_1k_1)|l_1 >$, $U_{n_2} : |k_2 >\rightarrow \frac{1}{\sqrt{n_2}}\sum_{l_2=0}^{n_2-1}\exp(-\frac{2\pi i}{n_2}l_2k_2)|l_2 >$, where $\mathbb{Z}_{n_1}\times\mathbb{Z}_{n_2}\ni(k_1,k_2)\leftrightarrows k = a_1n_2k_1 + a_2n_1k_2 \in \mathbb{Z}_n$. Thus, one obtains $|k_1,k_2 >\rightarrow |a_1k_1, a_2k_2 > = (U_{n_1}|k_1 >)\otimes(U_{n_2}|k_2 >) = \frac{1}{\sqrt{n_1n_2}}\sum_{l_1=0}^{n_1-1}\sum_{l_2=0}^{n_2-1}\exp[-\frac{2\pi i}{n_1n_2}\Phi(k_1,k_2)\Phi(l_1,l_2]|\Phi(l_1,l_2) >$. Consequently, we can also apply this decomposition recursively to both n_1 and $n_2 \in \mathbb{Z}_+$.

The above construction can be used to describe Shor's algorithm for finding the orders of integers in the following steps:

Step 1. To prepare the superposition $\frac{1}{\sqrt{m}}\sum_{k=0}^{m-1}|k,0 >,\ \ n,m = 2^l \in \mathbb{Z}_+$;

Step 2. To compute $k\rightarrow a^k (\text{mod } n)$ and due to the r-periodicity, $\frac{1}{\sqrt{m}}\sum_{k=0}^{m-1}|k,a^k >= \frac{1}{\sqrt{m}}\sum_{l=0}^{r-1}\sum_{q=0}^{s_l}|qr+l,a^l >$, where $s_l \max\{s : sr+l \leq m\}$, $\frac{m}{r}-1-\frac{l}{r}\leq s_l \leq \frac{m}{r}-\frac{l}{r}$;

Step 3. Compute the inverse QFT on $\mathbb{Z}_m$ to get $\frac{1}{\sqrt{m}}\sum_{l=0}^{r-1}\sum_{q=0}^{s_l}\frac{1}{\sqrt{m}}\sum_{p=0}^{m-1}\exp[\frac{2\pi pi}{m}(qr+l)]|p,a^l >= \frac{1}{m}\sum_{l=0}^{r-1}\sum_{p=0}^{m-1}e^{\frac{2\pi ip}{m}l}\sum_{q=0}^{s_l}e^{\frac{2\pi ip}{m}rq}|p,a^l >$;

Step 4. To obtain $p\in\mathbb{Z}_n$ from the above expression;

Step 5. Find the continued fraction expansion of $\frac{p}{m}$ in the form $\frac{p_i}{q_i}\rightarrow\frac{p}{m}$, $i\in\mathbb{Z}_+$, using the Euclidean algorithm, and at last output the smallest $q_{\hat{\imath}}\in\mathbb{Z}_+$ such that $a^{q_{\hat{\imath}}}\equiv 1\,(\text{mod } n)$ if such one exists.

For example, let $n = 15$ and $a = 7$ be the element whose period is to be found. Choose $m = 16 = 2^4 > 15$, and obviously $a = 7 \in \mathbb{Z}^*_{15}$. Take the group $G = \mathbb{Z}_{16}$, with $\dim G = 16$ and the Hilbert space $\mathcal{H} = \mathcal{H}^{(4)}$. Then, step-by-step, we compute that

Step 1: $\frac{1}{4}\sum_{k=0}^{14}|k,0 >\in\mathcal{H}$;

Step 2: $k\rightarrow 7^k(\text{mod } 15)$ gives $\frac{1}{4}(|0,1 > +|4,1 > +|8,1 > +|2,1 > +|1,7 > +|5,7 >$

$+|9,7 > +|13,7 > +|2,4 > +|6,4 > +|10,4 > +|14,4 > +|3,13 > +|7,13 >$

$+|11,13 > +|15,13 >)$;

Step 3: the inverse QFT yields in $\mathbb{Z}_{16} : \frac{1}{4}(|0,1 > +|4,1 > +|8,1 > +|12,1 > +|0,7 > +i|4,7 > -|8,7 > -i|12,7 > +|0,4 > -|4,4 > +|8,7 > -|12.4 > +|0,13 > -i|4,13 > -|8,13 > +i|12,13 >)$;

Step 4: Choose p-elements $p_1 = 0,\ p_2 = 4,\ p_3 = 8,\ p_4 = 12,$ which are present in the above expansion;

Step 5: The corresponding convergents for $\frac{4}{16} = \frac{1}{4}$ are $\{\frac{0}{1},\frac{1}{4}\}$, and for $\frac{12}{16} = \frac{3}{4}$ are $\{\frac{0}{1},\frac{1}{1},\frac{3}{4}\}$, giving rise to the correct period $r = 4$. Therefore, one has $7^r - 1 = 7^4 - 1 \equiv 0 \pmod{15}$ and $7^{\frac{r}{2}} - 1 = 7^{\frac{4}{2}} - 1 \equiv 3 \pmod{15}$, $7^{\frac{4}{2}} + 1 \equiv 5 \pmod{15}$, or $3 = \gcd(3, 15)$, $5 = \gcd(5, 15)$ are the true nontrivial factors of 15.

12.5.2 *Loop Grassmann manifolds*

The following observation due to [138] and [408] suggests a new approach for realizing quantum computing algorithms. Namely, take any self-adjoint projector operator $P : \mathcal{H} \to \mathcal{H}$, $P^2 = P$, and construct the operator

$$U := I - 2P. \tag{12.84}$$

Clearly, $U^+U = I$, so the mapping $U : \mathcal{H} \to \mathcal{H}$ is unitary. It is called [138] a *uniton* and plays a key role in the construction of quantum computing gates [360]. Since these projection operators, which usually depend on some parameters, belong to the loop Grassmann manifold $Gr\,(\mathcal{H})$ [138, 294, 395], the problem of studying them for encoding information into a quantum computation medium and its quantum computation assumes great importance (see [77, 138, 408]). With respect to this computational aspect, which is related to Grassmann manifolds, we shall next study some properties of Lax type Hamiltonian flows [326, 336] on symplectic loop Grassmannians in the context of the Marsden–Weinstein reduction scheme, dual momentum mappings and connection theory. We construct what is called a *holonomy quantum computing transformation* acting in a quantum computation medium by making use of special Casimir invariants of the canonical symplectic structure on the loop Grassmannian endowed with Uhlmann's [390] connection.

As is well known, the Grassmann manifold associated with a linear Hermitian space $\mathcal{H}$ over $\mathbb{C}$ is a compact space defined as (cf. [395])

$$Gr\,(\mathcal{H}) := \bigcup_m Gr_m\,(\mathcal{H}) \tag{12.85}$$

where $Gr_m\,(\mathcal{H})\,,\ 1 \leq m \leq \dim \mathcal{H}$ is the set of all Hermitian matrices satisfying the quadratic constraint

$$Gr_m\,(\mathcal{H}) := \{P \in Hom(\mathcal{H}) : P = P^2,\ \ P = P^+,\ \ SpP = m \in \mathbb{Z}_+\}. \tag{12.86}$$

The conjugation $+$ is taken with respect to the usual Hermitian scalar product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}$. The quadratic constraint implies that the eigenvalues of the matrix operator $P : \mathcal{H} \to \mathcal{H}$ are either 0 or 1. Each component $Gr_m\,(\mathcal{H})$

of the Grassmannian $Gr\,(\mathcal{H})$ can also be viewed as the coset space of the unitary group $U(\mathcal{H})$:

$$Gr_m\,(\mathcal{H}) = U(\mathcal{H})/\left(U(m)\times U(\mathcal{H}^{\perp})\right), \tag{12.87}$$

where $\mathcal{H}^{\perp}\subset\mathcal{H}$ is orthogonal to $U(m)\mathcal{H}$, i.e., $U(m)\mathcal{H}^{\perp}=0$, and the group $U(\mathcal{H})$ acts transitively on each component $Gr_m(\mathcal{H})$ via the action

$$a: P\rightarrow\ a^{+}Pa, \qquad a\in U(\mathcal{H}). \tag{12.88}$$

Since we are interested in integrable flows on the Grassmannian $Gr(\mathcal{H})$, it is necessary to define possible invariant symplectic structures both on the compact Grassmann manifold $M := Gr(\mathcal{H})$ and on the loop Grassmannian $C_{\mathbb{S}^1}(M)=\prod_{x\in\mathbb{S}^1}Gr(\mathcal{H})$, parametrized by the circle $\mathbb{S}^1$. The simplest such structure on $Gr(\mathcal{H})$ can be written as [348]

$$\Omega^{(2)} := Sp(P\,dP\wedge dP\,P), \tag{12.89}$$

which is obviously invariant with respect to the natural $U(\mathcal{H})-$ action (12.88). The following lemmas [336] suffice to show that the 2-form (12.89) is indeed symplectic and the action (12.88) is Hamiltonian.

Lemma 12.2. *The 2-form (12.89) on $Gr(\mathcal{H})$ is closed and nondegenerate.*

Lemma 12.3. *The unitary loop group $C_{\mathbb{S}^1}(U(\mathcal{H}))$ - action (12.88) on the loop Grassmann manifold $C_{\mathbb{S}^1}(M)$ is Hamiltonian with an equivariant momentum mapping $l: C_{\mathbb{S}^1}(M)\rightarrow C_{\mathbb{S}^1}(u^*(\mathcal{H}))$ defined as*

$$il(P)=P, \qquad l^2\,(P)+il\,(P)=0 \tag{12.90}$$

for all $P\in C_{\mathbb{S}^1}(M)$, where $C_{\mathbb{S}^1}(u^*(\mathcal{H}))$ is the adjoint space to the loop Lie algebra $C_{\mathbb{S}^1}(u(\mathcal{H}))$ of the unitary loop Lie group $C_{\mathbb{S}^1}(U(\mathcal{H}))$.

As a simple corollary of these lemmas, we can construct a vector field $K: C_{\mathbb{S}^1}(M)\rightarrow T(C_{\mathbb{S}^1}(M))$ generated by a Hamiltonian function H_X : $C_{\mathbb{S}^1}(M)\rightarrow\mathbb{R}$. Indeed, from the definition

$$i_K\Omega^{(2)} = -dH_X(P) \tag{12.91}$$

one readily obtains that for a fixed $X\in C_{\mathbb{S}^1}(u(\mathcal{H}))$

$$dP/dt := K[P] = i[P,X]. \tag{12.92}$$

The result (12.92) coincides with that obtained in [48, 49, 348], and has the desired Lax structure [4, 121, 243, 326, 350]. This form clearly still does not exploit the whole loop structure of the phase space $C_{\mathbb{S}^1}(M)$. To construct Lax flows on the loop Grassmannian $C_{\mathbb{S}^1}(M)$, we shall devise a powerful new technique based on the theory of dual momentum mappings suggested in [4].

12.5.3 *Symplectic structures on loop Grassmann manifolds and Casimir invariants*

A point $P \in C_{\mathbb{S}^1}(M)$ satisfying $P^2 = P$ can, owing to Lemma 12.3, be naturally embedded in the adjoint space $C_{\mathbb{S}^1}(u^*(\mathcal{H}))$, thereby defining the principal fiber bundle

$$\begin{array}{cccc} g^* & \xleftarrow{\;l\;} & C_{\mathbb{S}^1}(M) \quad \longleftarrow: & G \times C_{\mathbb{S}^1}(M) \\ & id & & \\ \downarrow \pi & \swarrow & & \\ & & & \\ C_{\mathbb{S}^1}(M) & & & \end{array} \tag{12.93}$$

where $G := C_{\mathbb{S}^1}(U(\mathcal{H}))$, g^* is the corresponding adjoint space to the Lie algebra $g := C_{\mathbb{S}^1}(u(\mathcal{H}))$ with respect to the ad-invariant symmetric and nondegenerate Killing form [3], and $\pi : g^* \to C_{\mathbb{S}^1}(M)$ is the corresponding natural projection onto $C_{\mathbb{S}^1}(M)$ generated by the constraint $l^2 + il = 0$ for $l \in g^*$ due to equality (12.90). Now we are in a position to view the symplectic structure (12.89) as a reduction of a natural Lie–Poisson structure on g^* (see [4, 121, 243, 326, 350]). Indeed, on the space g^* we have the standard Lie–Poisson structure

$$\{\gamma, \mu\}_{Lie} := (l, [\nabla\gamma(l), \nabla\mu\,(l)]), \tag{12.94}$$

defined for smooth functionals γ and $\mu \in \mathcal{D}(g^*)$, with ∇ the corresponding gradient map. Now, employing the constraint $il^2 + l = 0$ for $l \in g^*$, from (12.94) one can easily calculate, using standard Dirac reduction theory, the same symplectic structure as (12.89). Thus, we have proved the following result.

Theorem 12.3. *The Poisson bracket on the space $\mathcal{D}(C_{\mathbb{S}^1}(M))$ generated by symplectic structure (12.89) is the same as that obtained from the natural Lie–Poisson bracket (12.94) on $\mathcal{D}(g^*)$ via Dirac reduction with respect to the momentum mapping and the quadratic constraint (12.90).*

The above reduction approach can be naturally generalized to the case when the loop Lie algebra g is expanded so that it is centrally extended by means of the standard Maurer–Cartan cocycle [121]. The corresponding Lie–Poisson bracket on the expanded adjoint space $\hat{g}^*$ is given as

$$\{\gamma, \mu\}_0 := (l, [\nabla\gamma(l), \nabla\mu(l)]) + (\nabla\gamma(l), d\nabla\mu(l)\;/\;dx). \tag{12.95}$$

Therefore the Lie–Poisson bracket (12.95), upon imposing the constraint (12.90), induces a new symplectic structure $\hat{\Omega}_0^{(2)} \in \Lambda^2(C_{\mathbb{S}^1}(M))$, generalizing the symplectic structure (12.89) on the loop Grassmannian $C_{\mathbb{S}^1}(M)$, whose exact expression shall not be needed. Here, we shall only make some interesting and useful extension of the construction above.

It is easy to prove [243] that for any $\lambda \in \mathbb{C}\backslash\{0\}$, the following brackets

$$\{\gamma, \mu\}_\lambda := (l + \lambda J, [\nabla\gamma(l), \nabla\mu(l)]) + (\nabla\gamma(l), d\nabla\mu(l)\ /\ dx), \tag{12.96}$$
$$\{\gamma, \mu\}_\lambda := (l, [\nabla\gamma(l), \nabla\mu(l)]) + \lambda^{-1}(\nabla\gamma(l), d\nabla\mu(l)\ /\ dx)$$

define on $\hat{g}^*$ Poisson brackets pencils, which can also be reduced on the Grassmannian $C_{\mathbb{S}^1}(M)$ if a matrix $J \in g^*$ is constant. Accordingly one obtains pencils of symplectic structures $\hat{\Omega}^{(2)}_\lambda \in \Lambda^2(C_{\mathbb{S}^1}(M))$, $\lambda \in \mathbb{C}$, parametrized by constant matrices $J \in \hat{g}^{*\cdot}$ The corresponding Lax flows on the Grassmannian $C_{\mathbb{S}^1}(M)$ can be constructed as Hamiltonian systems $C_{\mathbb{S}^1}(M)$ generated by Casimir functionals of the Poisson brackets pencil (12.96) reduced on $C_{\mathbb{S}^1}(M)$. Clearly, the Casimir invariants of (12.96) can be determined [121, 243] as solutions to the following Novikov–Marchenko equations:

$$d\nabla H(\lambda)(l)\ /\ dx = [l + \lambda J, \nabla H(\lambda)(l)], \tag{12.97}$$
$$d\nabla H(\lambda)(l)\ /\ dx = \lambda[l, \nabla H(\lambda)(l)],$$

where $H(\lambda) \in I_\lambda(\hat{g}^*)$, $\lambda \in \mathbb{C}$, is a Casimir functional of the bracket (12.96). Assuming further that the Poisson bracket $\{\cdot,\cdot\}_1 := d/d\lambda\,\{\cdot,\cdot\}_\lambda\,|_{\lambda=0}$ is endowed with a Casimir invariant $H_0 \in I_1(\hat{g}^*)$, one can try to construct the asymptotic expansion of the functional $H(\lambda) \in I_\lambda(\hat{g}^*)$ with the leading term $H_0 \in I_1(\hat{g}^*)$ as follows:

$$H(\lambda) \ \sim\ H_0 + \sum_{j\geqslant 1} H_j \lambda^{-j}. \tag{12.98}$$

The coefficients $H_j \in \mathcal{D}(\hat{g}^*)$, $j \in \mathbb{Z}_{+,}$ apparently comprise an involutive hierarchy of functionals with respect to both the bracket $\{\cdot,\cdot\}_0$ and the bracket $\{\cdot,\cdot\}_1$. The same is true for the corresponding symplectic structures $\hat{\Omega}^{(2)}_0$ and $\hat{\Omega}^{(2)}_1$ reduced on the Grassmannian $C_{\mathbb{S}^1}(M)$.

We note that finding the asymptotic solution (12.98) is a quite complex and interesting analytical problem that remains open. One possible approach to this task consists in some additional invariants subject to reducing the phase space $\hat{g}^*$ on an appropriate subspace $\hat{g}^*_{red} \subset \hat{g}^*$ defined as a quotient space with respect to a foliation induced by a distribution [243] on $\hat{g}^{*\cdot}$. For instance, with respect to the first Poisson structure in (12.96) let us denote by $\hat{g}^*_J \in \hat{g}^*$ a maximal fixed integral submanifold of the distribution

$$D_{1:} = \{\ K \in T(\hat{g}^*) : K(l) = [J, \nabla\gamma(l)], \quad l \in \hat{g}^*, \quad \gamma \in \mathcal{D}(\hat{g}^*)\}, \tag{12.99}$$

which is clearly integrable, that is $[D_1\ ,\ D_1]\ \subset D_1$. Now define the distribution D_0 on $\hat{g}^*$ as

$$D_0 := \{K \in T(\hat{g}^*) : K(l) = [l - d/dx, \nabla H_0], \quad l \in \hat{g}, \quad H_0 \in\ I_1(\hat{g}^*), \tag{12.100}$$

which is also integrable ($[D_0, D_0] \subset D_0$). The set of maximal integral submanifolds of distribution (12.100) results in the foliation $\hat{g}^*_J \backslash D_0$ with leaves that are intersections of $\hat{g}^*_J$ with leaves of the distribution (12.100). Assuming that $\hat{g}^*_J \backslash D_0$ is sufficiently regular, we obtain the quotient manifold $\hat{g}^*_{red}{:=}\hat{g}^*_J / \left(\hat{g}^*_J \backslash D_0\right)$. It follows from (12.99) that $\hat{g}^*_J = \hat{g}^{\perp}_J$, where $\hat{g}_J \subset \hat{g}$ is the isotropy Lie subalgebra with respect to the element $J \in \hat{g}^*$, that is $\mathrm{ad}^*_{\hat{g}_J} J = 0$, and $\hat{g}^{\perp}_J \subset \hat{g}^*$ is its orthogonal subspace with respect to the usual Killing scalar product on $\hat{g}$. For the quotient $\hat{g}^*_{red}$ to be defined invariantly it is necessarily that $D_0\left(\hat{g}^*_J\right) \subset \hat{g}^*_J$, which is assumed to be fulfilled for the chosen element $J \in \hat{g}^*$.

We now proceed to study in greater detail applications of the above reduction scheme to quantum computing problems.

12.5.4 *An intrinsic loop Grassmannian structure and dual momentum maps*

As is well known [121, 326, 352], a large class of integrable dynamical systems on an infinite-dimensional manifolds can be obtained by constructing momentum maps into the dual spaces of certain Lie algebras via the Lie-algebraic approach. To proceed in the case of our loop Grassmann manifold $C_{\mathbb{S}^1}(M)$, let us represent $C_{\mathbb{S}^1}(M)$ as a manifold reduced from the matrix manifold $\tilde{C}_{\mathbb{S}^1}(M) := C_{\mathbb{S}^1}(M_{m,N}) \times C_{\mathbb{S}^1}(M_{m,N})$ and endowed with a canonical symplectic structure generalizing the one used in [4], [316]

$$\tilde{\Omega}^{(2)} := \int_0^{2\pi} dx\ Sp(dF \wedge dQ^\top), \tag{12.101}$$

where superscript $\top$ denotes the usual matrix transpose, and $(F, Q) \in C_{\mathbb{S}^1}(M_{m,N}) \times C_{\mathbb{S}^1}(M_{m,N})$ is the space of all loop matrices from $Hom(\mathbb{C}^N; \mathbb{C}^m)$ with maximal rank equal to $m \in \{1, \ldots, N\}$. An arbitrary point in $C_{\mathbb{S}^1}(M)$ can now obviously be represented as the composition $(F, Q) \to P := Q^\top F$ subject to

$$F^\top Q = (Q^\top F)^*, \qquad F^\top Q F^\top Q = F^\top Q, \tag{12.102}$$

which suffice for the embedding $P \in C_{\mathbb{S}^1}(M)$, where $*$ denotes complex conjugation. When $Q = F^* \in C_{\mathbb{S}^1}(M_{m,N})$, one finds that $P = F^+ F \in$

$C_{\mathbb{S}^1}(M)$ if and only if $F^+FF^+F = F^+F$ for any $F \in C_{\mathbb{S}^1}(M)$ with the scalar constraint $Sp(F^+F) = m$ being constant on $C_{\mathbb{S}^1}(M)$.

There is a natural symplectic action of the loop Lie group $G = C_{\mathbb{S}^1}(U(\mathcal{H}))$ on $\tilde{C}_{\mathbb{S}^1}(M)$ defined as

$$a : (F, Q) \ \rightarrow \ (F_a, Q_a), \tag{12.103}$$

where

$$F_a := \ Fa^+ \ , \qquad Q_a := aQ^\top \tag{12.104}$$

for any $a \in G$, $(F, Q) \in \tilde{C}_{\mathbb{S}^1}(M)$. The next result characterizes the action (12.103) in terms of the momentum map.

Theorem 12.4. *The action (12.103) on $\tilde{C}_{\mathbb{S}^1}(M)$ endowed with symplectic structure (12.101) is Hamiltonian with respect to an equivariant momentum mapping $l : \tilde{C}_{\mathbb{S}^1}(M) \longrightarrow \hat{g}^*$ defined as*

$$l\,(F, Q) = Q^\top F. \tag{12.105}$$

Proof. A proof follows easily using methods from Chapter 2. $\square$

Since on the adjoint space $\hat{g}^*$ there exists the natural Lie-Poisson structure (12.94), its reduction on the manifold $\tilde{C}_{\mathbb{S}^1}(M)$ gives rise to the symplectic structure (12.101). Taking into account constraints (12.102) and the result (12.105), we can now find a symplectic structures pencil on the loop Grassmannian $C_{\mathbb{S}^1}(M)$ reducing Poisson brackets pencil (12.96) with respect to the constraints

$$l^+ = -l, \quad l^2 + il = 0, \quad i\,Sp\,l = m \in \mathbb{Z}_+ \,.. \tag{12.106}$$

The map (12.105) allows us to construct the following analog of the fiber bundle (12.93):

$$\begin{array}{ccccc} G \times \tilde{C}_{\mathbb{S}^1}(M): & \longrightarrow & \tilde{C}_{\mathbb{S}^1}(M) & \xrightarrow{\tilde{\pi}} & C_{\mathbb{S}^1}(M) \\ & & \downarrow l & \overset{\pi}{\nearrow} & \\ & & & & \\ & & \hat{g}^* & & \end{array} \tag{12.107}$$

where the projections π and $\tilde{\pi}$ satisfy the condition $\pi \circ l = \tilde{\pi}$ with respect to the constraints (12.106). Since the momentum map (12.105) is not injective, there are certain difficulties in finding a symplectic structures pencil $\tilde{\Omega}^{(2)}_\lambda \in \Lambda^2\left(\tilde{C}_{\mathbb{S}^1}(M)\right)$, $\lambda \in \mathbb{C}$, corresponding to the Poisson brackets pencil (12.96) on g^*, and generalizing the reduced bracket (12.101) with respect to the constraint $F^\top Q = \left(Q^\top F\right)^*$. The construction works trivially when $Q = F^*$, and this leads directly to the natural symplectic structure

$$\tilde{\Omega}^{(2)} := \int_0^{2\pi} dx \; Sp(dF \wedge dF^+), \tag{12.108}$$

on the loop space $\tilde{C}_{\mathbb{S}^1}(M^*) := \left\{(F, Q = F^*) \subset \tilde{C}_{\mathbb{S}^1}(M)\right\}$. This symplectic structure (12.108) is obtained now from the Lie–Poisson bracket (12.94) on g^* constrained to the corresponding symplectic leaves [3] where it is not degenerate with respect to the diagram (12.107) if uses the general formulas

$$i\delta\gamma/\delta F = (\delta\gamma/\delta l)^\top Q^\top, \qquad i\delta\gamma/\delta Q = (\delta\gamma/\delta l)\, F^\top, \tag{12.109}$$

with $il = il\,(F,Q) := Q^\top F \in g^*$ and $\gamma \in \mathcal{D}\,(g^*)$ an arbitrary smooth functional. Upon Dirac reducing the symplectic structure (12.108) on the loop Grassmannian $C_{\mathbb{S}^1}(M)$ with respect to the second constraint $F^+F = F^+FF^+F$, where $(F, F^+) \in \tilde{C}_{\mathbb{S}^1}(M)$, one finds explicitly a symplectic structure that can be further reduced to the symplectic structure (12.89).

Note that original symplectic structure (12.101) is also invariant with respect to the following $G\,(m)$ - loop group action:

$$F :\longrightarrow F_a := aF, \qquad Q^\top :\longrightarrow Q_a^\top := Q^\top a^+, \tag{12.110}$$

where $a \in G\,(m) := C_{\mathbb{S}^1}\,(U\,(m))$, and $m :=$ rank (F,Q). The action (12.110) induces the momentum map on $\tilde{C}_{\mathbb{S}^1}(M) \ni (F,Q)$ defined as

$$q : (F,Q) \longrightarrow FQ^\top - m^{-1} Sp(FQ^\top) I_m \in g^*\,(m)\,. \tag{12.111}$$

There is also the following fiber bundle naturally associated to the mapping (12.111):

$$\begin{array}{ccccc} G\,(m) \times \tilde{C}_{\mathbb{S}^1}(M): & \longrightarrow & \tilde{C}_{\mathbb{S}^1}(M) & \xrightarrow{\tilde{\pi}} & C_{\mathbb{S}^1}(M) \\ & & \downarrow q & \nearrow^{\pi} & \\ & & \hat{g}^*\,(m) & & \end{array} \tag{12.112}$$

Since rank $(F,Q) = m$, it is clear that rank $\left(FQ^T\right) = m$, $\det FQ^\top \neq 0$ and $Sp\; q = 0$. Assuming further that $q \equiv 0$, one can easily show that the momentum map $l\,(F,Q) \in g^*$ satisfies the constraint $il^2 + l = 0$ identically. Thus, we can reduce the loop manifold $\tilde{C}_{\mathbb{S}^1}(M)$ on the Grassmann manifold $C_{\mathbb{S}^1}(M)$ with respect to the constraints $FQ^\top = I_m, \quad FQ^\top = Q^*F^+$. The corresponding reduced symplectic structure on $C_{\mathbb{S}^1}(M)$ is due to (12.89) if the sufficient condition $Q = F^*$ for the constraint, $-l^+ = l$, is imposed. Moreover, owing to the compatibility of diagrams (12.107) and (12.112) with respect to the mutual loop group actions, the next result follows directly.

Theorem 12.5. *The Lie–Poisson bracket (12.94) reduced on the Grassmann manifold $C_{\mathbb{S}^1}(M)$ with respect to the constraint $il^2 + l = 0$ and the momentum map $il(F, F^+) = F^+F = P \in C_{\mathbb{S}^1}(M)$ coincides with the Poisson bracket related to symplectic structure (12.108) reduced invariantly on the manifold $C_{\mathbb{S}^1}(M)$ with respect to the first class constraint $q(F, F^+) = FF^+ \equiv I_m$, via the usual Marsden–Weinstein procedure* [3].

As an immediate corollary, we can construct a symplectic structure on $\tilde{C}_{\mathbb{S}^1}(M)$ of the type we seek, generated by Poisson bracket (12.95) on the adjoint space $\hat{g}^*$ to the centrally extended loop Lie algebra $\hat{g} = g \oplus \mathbb{C}$ by means of the standard Maurer–Cartan 2-cocycle on g. With a hierarchy of Casimir functionals via the scheme devised in Chapter 2, we can construct integrable Lax flows on the loop Grassmann manifold $C_{\mathbb{S}^1}(M)$. Returning now to diagram (12.112), we now study a natural connection arising on the corresponding fiber bundle via the method suggested in [89, 106]. In particular, the constraint $FF^+ = I_m$ imposed on the loop manifold $\tilde{C}_{\mathbb{S}^1}(M^*)$ defines the injection projection $\tilde{\pi} : \tilde{C}_{\mathbb{S}^1}(M^*) \to C_{\mathbb{S}^1}(M)$ into the loop Grassmann manifold $C_{\mathbb{S}^1}(M)$ endowed with the symplectic structure $\hat{\Omega}^{(2)} \in \Lambda^2\left(C_{\mathbb{S}^1}(M)\right)$, obtained by means of the Dirac reduction procedure from the usual Lie-Poisson bracket (12.95) on $\hat{g}^*$.

To construct the desired connection on $\tilde{C}_{\mathbb{S}^1}(M^*)$, one needs to define in a natural way the tangent space to the fibers of the bundle (12.112) at a point $(F, F^+) \in \tilde{C}_{\mathbb{S}^1}(M^*) \times G(m)$ over the element $P = F^+F \in C_{\mathbb{S}^1}(M)$. From simple calculations we find that a vector $X_v \in T\left(\tilde{C}_{\mathbb{S}^1}(M^*)\right)$ is vertical; that is, $\tilde{\pi}_*(X_v) = 0$ at any point $(F, F^+) \in \tilde{C}_{\mathbb{S}^1}(M^*)$ if $\tilde{\pi}(F, F^+) := F^+F = P \in C_{\mathbb{S}^1}(M)$, or

$$X_v^+F + F^+X_v = 0. \tag{12.113}$$

We shall call a vector $X_h \in T\left(\tilde{C}_{\mathbb{S}^1}(M^*)\right)$ *horizontal* if it is orthogonal (with respect to the standard Sp-type Killing scalar product) to the space of vertical vectors defined above. Thus, we have

$$\int_0^{2\pi} dx\ Sp(X_h^+ \cdot AF) = 0 \tag{12.114}$$

for any $A \in C_{\mathbb{S}^1}(u(m))$ at $(F, F^+) \in \tilde{C}_{\mathbb{S}^1}(M^*)$. Since $A^+ = -A$ for any $A \in C_{\mathbb{S}^1}(u(m))$, from (12.114) we easily find that

$$FX_h^+ - X_h F^+ = 0, \tag{12.115}$$

and this leads immediately to the following result.

Lemma 12.4. *The tangent space $T\left(\tilde{C}_{\mathbb{S}^1}(M^*)\right)$ splits invariantly with respect to the $G(m)$-action on the manifold $\tilde{C}_{\mathbb{S}^1}(M^*)$ into the direct sum $T_h\left(\tilde{C}_{\mathbb{S}^1}(M^*)\right) \oplus T_v\left(\tilde{C}_{\mathbb{S}^1}(M^*)\right) = T\left(\tilde{C}_{\mathbb{S}^1}(M^*)\right)$, where*

$$\begin{aligned} T_h\left(\tilde{C}_{\mathbb{S}^1}(M^*)\right) &= \left\{X \in \tilde{C}_{\mathbb{S}^1}(M^*):\ XF^+ - FX^+ = 0\right\}, \\ T_v\left(\tilde{C}_{\mathbb{S}^1}(M^*)\right) &= \left\{X \in \tilde{C}_{\mathbb{S}^1}(M^*):\ X^+F + F^+X = 0\right\} \end{aligned} \tag{12.116}$$

at any point $(F, F^+) \in \tilde{C}_{\mathbb{S}^1}(M^*)$.

Since the horizontal subspace $T_h\left(\tilde{C}_{\mathbb{S}^1}(M^*)\right)$ has to be $G(m)$-invariant, that is $F_aF_a^+ = 1$ and $X_aF_a^+ - F_aX_a^+ = 0$, where $F_a := FA_a^+,\ \ F_a^+ := A_aF^+$ and $X_a := XA_a^+,\ \ X_a^+ := A_aX^+$ for some Lie algebras homomorphism $\rho:\ \ g(m) \to g$, $\rho(a) := A_a \in g$, $a \in g(m)$, one can define a connection 1-form $\omega \in \Lambda^1\left(\tilde{C}_{\mathbb{S}^1}(M^*)\right) \otimes g(m)$, satisfying the characteristic conditions: $\omega(X_h = FA_a^+) = 0$, $\omega(X_v = aF) = a \in g(m)$ for any $X_h \in T_h(\tilde{C}_{\mathbb{S}^1}(M^*))$, $X_v \in T_v\left(\tilde{C}_{\mathbb{S}^1}(M^*)\right)$. The results of [106, 390] imply that we can now construct a general Uhlmann type expression determining the above connection form $\omega \in \Lambda^1\left(\tilde{C}_{\mathbb{S}^1}(M^*)\right) \otimes g(m)$ as follows:

$$dF\, F^+ - F\, dF^+ = \frac{1}{2}\left(FF^+\omega + \omega FF^+\right). \tag{12.117}$$

And similarly as in [89, 106], the connection 1-form ω found from (12.117) satisfies the desired gauge transformation property

$$r_a^*\omega = a^+da + ad_*\left(a^+\right)\omega \tag{12.118}$$

for any $a \in G(m)$. If the normalizing condition $FF^+ = I_m$ is satisfied on $\tilde{C}_{\mathbb{S}^1}(M^*)$, equation (12.117) together with the 1-form identity

$$dF\, F^+ + F\, dF^+ = 0 \tag{12.119}$$

determine the desired connection 1-form ω explicitly. Here we must notice that the connection above was obtained from the diagram (12.112) in the case where the loop Lie group $G(m)$ acts upon the loop manifold $\tilde{C}_{\mathbb{S}^1}(M^*)$, where its central extension structure has so far been ignored. If the above central extension of the loop group $G(m)$ is taken into account, we must use the resulting symplectic structure $\hat{\Omega}^{(2)} \in \Lambda^2\left(\tilde{C}_{\mathbb{S}^1}(M^*)\right)$ for explicit calculation of the corresponding horizontal and vertical vectors from the tangent space $T\left(\tilde{C}_{\mathbb{S}^1}(M^*)\right)$ as shown above. The constraint $FF^+ = I_m$, imposed on the manifold $\tilde{C}_{\mathbb{S}^1}(M^*)$ is invariant with respect to the usual Dirac reduction of the symplectic structure (12.108) due to the identity

$$\{Q(A), Q(B)\} = Q([A, B]) = 0 \tag{12.120}$$

for any $A, B \in g(m)$ where $Q(\cdot) = (q, \cdot)$. However, it fails to be invariant for the reduction of the corresponding centrally extended symplectic structure $\hat{\Omega}^{(2)} \in \Lambda^2\left(\tilde{C}_{\mathbb{S}^1}(M^*)\right)$ obtained from the Lie–Poisson bracket (12.95) projected onto the loop manifold $\tilde{C}_{\mathbb{S}^1}(M^*)$. The details of the reduction procedure in this case shall not be presented here.

Observe that Lax flows on Grassmann manifolds were treated in [48, 49, 316, 348] and also in [171, 289, 369] using mainly classical algebraic techniques. One result of interest that can be extracted from considerations in [369] is a proof that the following nonlinear dynamical system on the loop Grassmann manifold $C_{\mathbb{S}^1}(M)$

$$dP/dt = i[P_{xx}, P] \tag{12.121}$$

is Lax integrable. In the sequel we will derive the flows such as (12.121) in the context of the dual momentum map approach devised above and make use of them for describing what are called holonomic quantum computation media.

12.5.5 *Holonomy group structure of the quantum computing medium*

Consider the fiber bundle $G(m) \times \tilde{C}_{S^1}(M^*) \xrightarrow{\tilde{\pi}} C_{\mathbb{S}^1}(M^*)$, where

$$\tilde{\pi}(F, \tilde{F}) := F^+F \in C_{\mathbb{S}^1}(M^*) \tag{12.122}$$

for any $(F, \tilde{F}) \in \tilde{C}_{S^1}(M^*)$. Since the manifold $\tilde{C}_{S^1}(M^*)$ is symplectic with the symplectic structure (12.108), one can reduce it upon the loop Stiefel manifold $St_{S^1}(M^*)$ [395] subject to the momentum map $q : \tilde{C}_{S^1}(M^*) \to g^*(m)$ with $q(F, F^*) = 0$, $Sp(FF^+) = m$. Hence, we compute that

$$St_{S^1}(M^*) = \{(F, F^+) \in \tilde{C}_{S^1}(M^*) : FF^+ = I_m\}. \tag{12.123}$$

Since the loop group $G(m)$ acts also on the Stiefel manifold $St_{S^1}(M^*)$ leaving it invariant, we can restrict the projection $\tilde{\pi} : \tilde{C}_{S^1}(M^*) \to C_{\mathbb{S}^1}(M^*)$ to $St_{S^1}(M^*)$, obtaining thereby the universal loop Grassmannian fiber bundle

$$G(m) \times St_{S^1}(M^*) \xrightarrow{\tilde{\pi}} C_{\mathbb{S}^1}(M) \tag{12.124}$$

with the base space being precisely the loop Grassmannian $C_{\mathbb{S}^1}(M)$.

Assume now that an element $P \in C_{\mathbb{S}^1}(M)$ acts on the unitary vector space $\mathcal{H}^{(N)}$ of dimension $\dim \mathcal{H}^{(N)} = N$ and consider the image subspace $\operatorname{im} P \subset \mathcal{H}^{(N)}$, that is for any $f \in \mathcal{H} := \operatorname{im} P$ the condition $Pf = f$ holds. In this way we obtain the vector bundle associated with (12.124),

$$G(m) \times \mathcal{H}(St_{S^1}(M^*) \xrightarrow{\tilde{\pi}} C_{\mathbb{S}^1}(M), \tag{12.125}$$

whereby the total fiber space is

$$\mathcal{H}(St_{S^1}(M^*) := \{(F, F^+) \times \mathcal{H}) : (F, F^+) \in St_{S^1}(M^*), \mathcal{H} = \operatorname{im} P\}, \tag{12.126}$$

and the group action $(A \circ (F, F^+) \times \mathcal{H}) := ((AF, F^+A^+) \times A^{-1} \circ \mathcal{H})$ for any $A \in G(m)$ is specified as follows: Let vectors $f_j \in \mathcal{H}$ be a basis for the space $\mathcal{H}$ =$\operatorname{im} P$, so any vector $f \in \mathcal{H}$ can be uniquely expanded as $f = \sum_{j=1}^{m} f_j c_j(f)$ with complex coefficients $c_j(f) \in \mathbb{C}$, $1 \le j \le m$. Then the action is $A \circ f = \sum_{j,k=1}^{m} f_j c_k(f) A_{jk}$ for any m-dimensional matrix representation of $A \in G(m)$. On the other hand, consider the universal loop Grassmannian bundle (12.124) and calculate the following quantity:

$$PF^+ = (F^+F)F^+ = F^+(FF^+) = F^+ \tag{12.127}$$

since for any $(F, F^+) \in St_{S^1}(M^*)$ we have $FF^+ = I_m$. Thus, we can identify the matrix $F^+ \in C_{S^1}(M_{N\times m})$ as that consisting of base vectors $f_j \in \operatorname{im} P$, $1 \le j \le m$, constructed above. Then one can define the matrix

$$F^+ = (f_1, f_2, ..., f_m) := |F> \tag{12.128}$$

for any $(F, F^+) \in St_{S^1}(M^*)$ and write

$$P = |F >< F|, \quad \mathcal{H} = \operatorname{span}_{\mathbb{C}} |F> . \tag{12.129}$$

Now we study some special flows on $C_{S^1}(M)$ that are related to the Casimir functionals of the second Poissonian pencil structure (12.96), namely

$$\{\gamma, \mu\}_\lambda := (l, [\nabla\gamma(l), \nabla\mu(l)]) + \lambda^{-1}(\nabla\gamma(l), d\nabla\mu(l)/dx) \tag{12.130}$$

for any $\lambda \in \mathbb{C}\backslash\{0\}$ and $\gamma, \mu \in \mathcal{D}(\hat{g}^*)$. The corresponding equation for Casimir functionals is

$$d\nabla H(\lambda)(l) \;/\; dx = \lambda[l, \nabla H(\lambda)(l)] \tag{12.131}$$

for all $\lambda \in \mathbb{C}$ and $l \in \hat{g}^*$. Making use of the expansion (12.98) and the condition $P^2 = P$, one computes directly from (12.131) that at $il = P$

$$d\nabla H_j(P)/dx = i[P, \nabla H_{j+1}], \qquad i \bigtriangledown H_{j+1}(P) = [d \bigtriangledown H_j(P)/dx, P] \tag{12.132}$$

for all $j \in \mathbb{Z}_+$, where $\nabla H_0(P) = P$. Subject to the Poissonian structure (12.130) reduced upon the loop Grassmannian $C_{\mathbb{S}^1}(M)$, one finds the following infinite hierarchy of Hamiltonian flows on $C_{\mathbb{S}^1}(M)$:

$$dP/dt_j = i[P, \nabla H_j(P)], \tag{12.133}$$

where $t_j \in \mathbb{R}$, $j \in \mathbb{Z}_+$, are the corresponding evolution parameters. For $j = 1$ one obtains on $C_{\mathbb{S}^1}(M)$ from (12.133) the shifting nonlinear Hamiltonian flow $dP/dt_j = dP/dx$, and for $j = 2$ the above hierarchy yields the flow coinciding with that of (12.121).

It is easy to see from (12.133) that the quantity $SpP = \dim \operatorname{im} P = m$ is invariant with respect to all flows d/dt_j, $j \in \mathbb{Z}_+$. This obviously means that for any $t_j \in \mathbb{R}$, $j \in \mathbb{Z}_+$,

$$P(t_j) = W^+(t_j)P_0W(t_j), \tag{12.134}$$

where $P_0 \in C_{\mathbb{S}^1}(M)$ is constant and $W(t_j) \in G = C_{\mathbb{S}^1}(U(\mathcal{H}^{(N)}))$ satisfies the following condition derived from (12.133):

$$\nabla H_j(P) = W^+(t_j)dW(t_j)/dt_j. \tag{12.135}$$

The expressions (12.135) are of great importance [138], [408], [360] for holonomic quantum computational algorithms. Namely, from (12.119) and (12.120) one sees that the connection 1-form on the loop Stiefel fiber bundle (12.124) at the fixed point $(F_0, F_0^+) \in St_{S^1}(M^*)$ takes the form

$$\omega(F_0, F_0^*) = -F_0dF_0^+ \in g(m). \tag{12.136}$$

Let us now recall that $G = C_{S^1}(U(\mathcal{H}))$ acts naturally on the loop Stiefel manifold $St_{S^1}(M^*)$ as $W \circ (F_0, F_0^+) = (F_0W, F_0^*W^*)$, where $W \in G$ and $(F_0, F_0^+) \in St_{S^1}(M^*)$. Under this action an element $P_0 := F_0^+F_0 \in C_{\mathbb{S}^1}(M)$ transforms into the element $W \circ P_0 = W^+F_0^+F_0W = W^+P_0W \in C_{\mathbb{S}^1}(M)$, $W \in G$, coinciding with the transformations (12.88) and (12.134). On the other hand, the connection 1-form (12.136) transforms as

$$\omega(W(F_0, F_0^+)) = \omega(F_0, F_0^+) + < F_0|dWW^+|F_0 > \tag{12.137}$$

for any $(F_0, F_0^+) \in St_{S^1}(M^*)$ and $W \in G$. Then, from expressions (12.137) and (12.135) one obtains that for $P_0 = F_0^+F_0$ at the point $P = W^+P_0W \in C_{\mathbb{S}^1}(M)$

$$\omega(W(F_0, F_0^+)) = \omega(F_0, F_0^+) + \sum_{j \in \mathbb{Z}_+} < F_0|W\nabla H_j(P)W^+|F_0 > dt_j. \tag{12.138}$$

Hence, making use of (12.137) one can easily calculate the corresponding curvature 2-form

$$\Omega(W(F_0,F_0^+)) := d\omega + \omega\wedge\omega$$
$$=< F|W^+dW\wedge(P-I_N)W^+dW|F>, \tag{12.139}$$

where $F := F_0W$, $W\in G$. Since it follows directly from (12.135) that on the smoothly embedded parameter submanifold $C_{\mathbb{S}^1}(M_{(t)})\subset C_{\mathbb{S}^1}(M)$ the matrix differential 1-form has the form

$$W^+(t)dW(t) = \sum_{j\in\mathbb{Z}_+} W^+(t)\frac{\partial W(t)}{\partial t_j}dt_j = \sum_{j\in\mathbb{Z}_+}\nabla H_j(P)dt_j, \tag{12.140}$$

the expression (12.139) yields the matrix 2-form

$$\Omega(F,F^+) = \sum_{j,k\in\mathbb{Z}_+} < F|\nabla H_j(P)(P-I_N)\nabla H_k(P)|F> dt_j\wedge dt_k. \tag{12.141}$$

Take now any loop σ in the parameter submanifold $C_{\mathbb{S}^1}(M_{(t)})$ above. Then one can calculate the corresponding holonomy group mapping [89] in the following chronologically ordered symbolic form

$$\Gamma_\omega(\sigma) := \mathcal{P}\exp\{\oint_\sigma \omega\}\in Hol(\omega;G(m)) \tag{12.142}$$

as a result of parallel transporting any point $(F_0,F_0^+)\in St_{S^1}(M^*)$, such that $P_0 = F_0^+F_0\in C_{\mathbb{S}^1}(M_{(t)})$, along this loop. The simplest case is when a loop $\sigma\subset C_{\mathbb{S}^1}(M_{(t)})$ depends locally only on two parameters such as x and $t\in\mathbb{R}$. Then the holonomy group mapping (12.142) reduces to the symbolic expression

$$\Gamma_\omega(\sigma) := \mathcal{P}\exp\{\oint_\sigma\omega\} = \mathcal{P}\exp\{\int_{D(\sigma)}\Omega(F,F^+)\} \tag{12.143}$$
$$= \mathcal{P}\exp\{\int_{D(\sigma)}\langle F|(\nabla H_0(P)(P-I_N)\nabla H_1(P)-$$
$$\nabla H_1(P)(P-I_N)\nabla H_0(P))|F\rangle\, dx\wedge dt\},$$

where $D(\sigma)$ is any smooth two-dimensional disk in the parameter submanifold $C_{\mathbb{S}^1}(M_{(t)})$ such that the boundary $\partial D(\sigma)=\sigma$. Thus, in the holonomic quantum computation medium $G(m)\times St_{S^1}(M^*)\xrightarrow{\tilde{\pi}} C_{\mathbb{S}^1}(M_{(t)})$ we have constructed two important objects: the first one is the encoding of information into a vacuum space $\operatorname{im}P_0\subset QTRH^{(N)}$ characterized by a pair of matrices $(F_0,F_0^*)\in St_{S^1}(M^*)$, and the second one, properly processing or computing the information as a mapping

$$\Gamma_\omega(\sigma): f\to\Gamma_\omega(\sigma)\circ f \tag{12.144}$$

for any given information vector $f \in \mathcal{H}$ $\operatorname{im} P_0$ and the corresponding closed loop $\sigma \subset C_{\mathbb{S}^1}(M_{(t)})$. These two operations in the quantum holonomic medium $G(m) \times \mathcal{H}(St_{S^1}(M^*)) \xrightarrow{\tilde{\pi}} C_{\mathbb{S}^1}(M_{(t)})$ can be effectively realized by making use of the standard uniton gates [360] and adapting them to specific problem oriented calculations.

12.5.6 *Holonomic quantum computations: Examples*

Example 12.3. Two-mode quantum-optical model

As shown in [294] and [138], it is possible to realize optical holonomic quantum calculations using the structure group $G(m) = C_{\mathbb{S}^1}(U(m))$, $m \in \mathbb{Z}_+$. For this case, with $m = 4$, let us consider a selfadjoint operator $\hat{H}_0 : \mathcal{H}^{(2)} \to \mathcal{H}^{(2)}$, $\dim_{\mathbb{C}} \mathcal{H}^{(2)} = 4$, in the standard quantum optical form

$$\hat{H}_0 = \hat{N}_1(\hat{N}_1 - 1) \oplus \hat{N}_2(\hat{N}_2 - 1) \tag{12.145}$$

with factors

$$\hat{N}_1 := a_1^+ a_1, \qquad \hat{N}_2 := a_2^+ a_2, \tag{12.146}$$

defined in terms of the usual [311] creation and annihilation Bose-operators $a_j, a_k^+ : \mathcal{H}^{(2)} \to \mathcal{H}^{(2)}$, $j, k = 1, 2$, which satisfy the commutator relationships

$$[a_j, a_k^+] = \delta_{jk}, \quad [a_j, a_k] = 0 = [a_j^+, a_k^+]. \tag{12.147}$$

The space $\mathcal{H} := \operatorname{span}_{\mathbb{C}} |F_0>$ can be naturally generated by the kernel $\ker \hat{H}_0$ as

$$\hat{H}_0 |F_0> := 0, \tag{12.148}$$

where it is clear that $\mathcal{H} \simeq \mathbb{C}^4$ and $|F_0>$ is given as

$$|F>= (|0,0>, |0,1>, |1,0>, |1,1>), \tag{12.149}$$

since, by definition, we set

$$\hat{N}_{1,2}|0>= 0, \qquad \hat{N}_{1,2}|1>= |1> . \tag{12.150}$$

Now let two unitary operators $A(x)$ and $B(y)$ with parameters $x, y \in \mathbb{C}$ be defined as

$$A(x) := \exp(\bar{y} a_1 a_2 - y a_1^+ a_2^+), \quad B(y) := \exp(\bar{x} a_1 a_2^+ - y a_1^+ a_2) \tag{12.151}$$

and construct the quantum computation medium as

$$F := F_0 W, \qquad W := A(x)B(y). \tag{12.152}$$

Then we compute that the corresponding curvature 2-form (12.139) is

$$\begin{aligned}
\Omega(F,F^+) = \{(1+\frac{\sin(2|x|)}{2|x|}\frac{\overline{y}\sinh(2|y|)}{2|y|}\alpha_2 + \frac{\overline{x}^2}{|x|^2}(-1+ \\
\frac{\sin(2|x|)}{2|x|})\frac{\overline{y}\sinh(2|y|)}{2|y|}\alpha_1\}dx\wedge dy \\
+\{\frac{x}{|x|^2}(-1+\cos(2|x|))\cosh(2|y|)\alpha_2 - \frac{\sin(2|x|)}{|x|}(1+\cosh^2(2|y|))\alpha_3 \\
+\frac{\overline{x}}{|x|^2}(-1+\cos(2|x|)\cosh(2|y|)\alpha_1\}dx\wedge d\overline{x} \\
+\{(1+\frac{\sin(2|x|)}{2|x|})\frac{y\sinh(2|y|)}{2|y|}\alpha_2 + \frac{\overline{x}^2}{|x|^2}(-1+ \\
\frac{\sin(2|x|)}{2|x|})\frac{y\sinh(2|y|)}{2|y|}\alpha_1\}dx\wedge d\overline{y} \qquad (12.153) \\
+\{(1+\frac{\sin(2|x|)}{2|x|})\frac{\overline{y}\sinh(2|y|)}{2|y|}\alpha_1 \\
+\frac{\overline{x}^2}{|x|^2}(-1+\frac{\sin(2|x|)}{2|x|})\frac{\overline{y}\sinh(2|y|)}{2|y|}\alpha_2\}dy\wedge d\overline{x} \\
-\{(1+\frac{\sin(2|x|)}{2|x|})\frac{y\sinh(2|y|)}{2|y|}\alpha_1 \\
+\frac{x^2}{|x|^2}(-1+\frac{\sin(2|x|)}{2|x|})\frac{y\sinh(2|y|)}{2|y|}\alpha_2\}d\overline{x}\wedge d\overline{y} \\
-\frac{\sinh(2|y|)}{|y|}(2\alpha_6 - I_4)dy\wedge d\overline{y},
\end{aligned}$$

where

$$\alpha_1 = \begin{pmatrix} 0&0&0&0\\ 0&0&1&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix},\ \alpha_2 = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&1&0&0\\ 0&0&0&0 \end{pmatrix},$$

$$\alpha_3 = \begin{pmatrix} 0&0&0&0\\ 0&1/2&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix},\ \alpha_4 = \begin{pmatrix} 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix},$$

$$\alpha_5 = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 1&0&0&0 \end{pmatrix},\ \alpha_6 = \begin{pmatrix} 1/2&0&0&0\\ 0&0&0&0\\ 0&0&1&0\\ 0&0&0&3/2 \end{pmatrix}.$$

Making use of (12.153) and the classical Ambrose–Singer theorem [12], it is straightforward to show that the holonomy Lie algebra $hol(\omega)$ of the corresponding connection (12.137) is

$$hol(\omega) = \mathrm{su}(2) \times \mathrm{u}(1) \subset \mathrm{u}(4), \tag{12.154}$$

so the holonomy group $Hol(\omega) \subset U(4)$, and it follows that our connection on the universal Grassmann fiber bundle is not reducible. Consequently, there are some quantum calculations that cannot be realized in the holonomy group $Hol(\omega)$, as is suggested in the expression (12.144).

With respect to the exact nature of the connection 1-form (12.137), one readily obtains [138] the expression

$$\omega(F, F^{+}) = -F_0 dF_0^{+} + \Gamma_{(x)} dx + \Gamma_{(y)} dy - \Gamma_{(x)}^{+} d\bar{x} - \Gamma_{(y)}^{+} d\bar{y}, \tag{12.155}$$

where

$$\begin{aligned} \Gamma_{(x)} &= -\frac{1}{2}(1 + \frac{\sin(2|x|)}{2|x|}) \cosh(2|y|)\alpha_2 + \frac{\overline{x}}{2|x|^2}(1 - \cos(2|x|)\alpha_3 \\ &\quad - \frac{\overline{x}^2}{2|x|^2}(-1 + \frac{\sin(2|x|)}{2|x|}) \cosh(2|y|)\alpha_1, \\ \Gamma_{(y)} &= -\frac{1}{2}(1 + \frac{\sin(2|y|)}{2|y|})\alpha_5 + \frac{\overline{y}}{2|y|^2}(1 - \cosh(2|y|)\alpha_6 \\ &\quad + \frac{\overline{y}^2}{2|y|^2}(-1 + \frac{\sinh(2|y|)}{2|y|})\alpha_4 \end{aligned} \tag{12.156}$$

are generalized Christoffel type matrices taking values in the structural Lie algebra $\mathcal{G}(4)$. Whence, from the expressions (12.155), (12.156) and (12.143), we can compute all of the homotopy classes of the holonomy group $Hol(\omega)$ elements and encode them by means of some specially constructed uniton operators for quantum calculations. But this is another problem still under investigation.

Example 12.4. Lax flow model

Consider flows (12.133) generated by Hamiltonian functions $H_j : C_{\mathbb{S}^1}(M) \to \mathbb{R}$, $j \in \mathbb{Z}_+$, on the Grassmann loop manifold $C_{\mathbb{S}^1}(M)$ satisfying relationships (12.132). Since all flows (12.133) are by definition mutually commutative, it would be natural to speculate that the related curvature vanishes on $C_{\mathbb{S}^1}(M)$. This would mean that the flows (12.133) realize parallel transport along all the directions parametrized by the independent evolution variables $t_j \in \mathbb{R}$, $j \in \mathbb{Z}_+$.

To show that it is not the case, we first note that

$$\bigtriangledown H_{j+1}(P) = [d \bigtriangledown H_j(P)/dx, P] \tag{12.157}$$

holds for all $j \in \mathbb{Z}_+$. Therefore, substitution of (12.157) into expression (12.141) yields

$$\Omega(F,F^+) = \sum_{j,k\in\mathbb{Z}_+} < F|[d\bigtriangledown H_{j-1}(P)/dx, P](P$$

$$-I_4)[d\bigtriangledown H_{k-1}(P)/dx, P]|F > dt_j \wedge dt_k = \sum_{j,k\in\mathbb{Z}_+} < F|(P\bigtriangledown H_{j-1}(P)/dx$$

$$-\bigtriangledown H_{j-1}(P)/dxP)(P-I_4)(P\bigtriangledown H_{k-1}(P)/dx \bigtriangledown H_{k-1}(P)/dxP)|F > dt_j \wedge dt_k \tag{12.158}$$

$$= \sum_{j,k\in\mathbb{Z}_+} < F|d\bigtriangledown H_{j-1}(P)/dx(P-I_4)d\bigtriangledown H_{k-1}(P)/dx|F > dt_j \wedge dt_k,$$

where we have made used the relationships $P|F >= |F >$ and $P(I_4 - P) = 0$.

Assume now that $j,k = 1,2$, $t_1 := x$, $t_2 := t \in \mathbb{R}$; then the resulting curvature 2-form (12.158) satisfies

$$\Omega(F,F^+) = \{< F|P_x(P-I_4)P_{xx}|F > - < F|P_{xx}(P-I_4)P_x|F >\}dx \wedge dt, \tag{12.159}$$

which is not identically zero on the parametrized Grassmann loop manifold $C_{\mathbb{S}^1}(M_{(x,t)})$. Consequently, the holonomy group $Hol(\omega)$ of the connection (12.137) is not trivial, and this is important for the construction of quantum computing algorithms based on quantum optical considerations described above.

The expression (12.159) can be used to calculate, via the Ambrose–Singer theorem [12], the corresponding holonomy Lie algebra $hol(\omega)$ as well as studying the properties that ensure the important irreducibility equality $hol(\omega) = u(m)$, where $m = \dim(\operatorname{im} P)$, $P \in C_{\mathbb{S}^1}(M_{(x,t)})$. These and related questions require a more complicated geometrical analysis which can in large part be found in the recent literature.

This page intentionally left blank

Chapter 13

Lagrangian and Hamiltonian Analysis of Relativistic Electrodynamic and String Models

13.1 Introductory setting

Classical electrodynamics is nowadays considered [124, 225, 302] as the most fundamental physical theory, largely owing to the depth of its theoretical foundations and wealth of experimental verifications. Electrodynamics is essentially characterized by its Lorentz invariance from a theoretical perspective, and this very important property has had a revolutionary influence [57, 100, 124, 225, 302] on the whole development of physics. In spite of the breadth and depth of theoretical understanding of electromagnetism, there remain several fundamental open problems and gaps in comprehension related to the true physical nature of Maxwell's theory when it comes to describing electromagnetic waves as quantum photons in a vacuum: These start with the difficulties in constructing a successful Lagrangian approach to classical electrodynamics that is free of the Dirac–Fock–Podolsky inconsistency [100, 105, 321], and end with the problem of devising its true quantization theory without such artificial constructions as a Fock space with "indefinite" metrics, the Lorentz condition on "average", and regularized "infinities"[57] of S-matrices. Moreover, there are the related problems of obtaining a complete description of the structure of a vacuum medium carrying the electromagnetic waves, and deriving a theoretically and physically valid Lorentz force expression for a moving charged point particle interacting with and external electromagnetic field. To describe the essence of these problems, let us begin with the classical Lorentz force expression

$$F := qE + qu \times B, \tag{13.1}$$

where $q \in \mathbb{R}$ is a particle electric charge, $u \in \mathbb{E}^3$ is its velocity vector, expressed here in the light speed c units,

$$E := -\partial A/\partial t - \nabla \varphi \tag{13.2}$$

is the corresponding external electric field and

$$B := \nabla \times A \tag{13.3}$$

is the corresponding external magnetic field, acting on the charged particle, expressed in terms of suitable vector $A : M^4 \to \mathbb{E}^3$ and scalar $\varphi : M^4 \to \mathbb{R}$ potentials. Here "∇" is the standard gradient operator with respect to the spatial variable $r \in \mathbb{E}^3$, "$\times$" is the usual vector product in three-dimensional Euclidean vector space $\mathbb{E}^3$, which is naturally endowed with the classical scalar product $< \cdot, \cdot >$. These potentials are defined on the Minkowski space $M^4 := \mathbb{R} \times \mathbb{E}^3$, which models a chosen laboratory reference system $\mathcal{K}$. Now, it is a well-known fact [123, 225, 302, 387] that the force expression (13.1) does not take into account the dual influence of the charged particle on the electromagnetic field and should be considered valid only if the particle charge $q \to 0$. This also means that expression (13.1) cannot be used for studying the interaction between two different moving charged point particles, as was pedagogically demonstrated in [225].

Other questionable inferences, which strongly motivated the analysis here, are related both with an alternative interpretation of the well-known *Lorentz condition*, imposed on the four-vector of electromagnetic potentials $(\varphi, A) : M^4 \to \mathbb{R} \times \mathbb{E}^3$ and the classical Lagrangian formulation [225] of charged particle dynamics under an external electromagnetic field. The Lagrangian approach is strongly dependent on an important Einsteinian notion of the rest reference system $\mathcal{K}_r$ and the related least action principle, so before explaining it in more detail, we first analyze the classical Maxwell electromagnetic theory from a strictly dynamical point of view.

Let us consider the additional Lorentz condition

$$\partial \varphi/\partial t + < \nabla, A >= 0, \tag{13.4}$$

imposed *a priori* on the four-vector of potentials $(\varphi, A) : M^4 \to \mathbb{R} \times \mathbb{E}^3$, which satisfy the Lorentz invariant wave field equations

$$\partial^2 \varphi/\partial t^2 - \nabla^2 \varphi = \rho, \quad \partial^2 A/\partial t^2 - \nabla^2 A = J, \tag{13.5}$$

where $\rho : M^4 \to \mathbb{R}$ and $J : M^4 \to \mathbb{E}^3$ are, respectively, the charge and current densities of the ambient matter, which satisfy the charge continuity equation

$$\partial \rho/\partial t + < \nabla, J >= 0. \tag{13.6}$$

Then the classical electromagnetic Maxwell field equations [123, 225, 302, 387]

$$\nabla \times E + \partial B/\partial t = 0, \qquad <\nabla, E>= \rho,$$
$$\nabla \times B - \partial E/\partial t = J, \qquad <\nabla, B>= 0, \tag{13.7}$$

hold for all $(t, r) \in M^4$ with respect to the chosen reference system $\mathcal{K}$.

Notice here that Maxwell's equations (13.7) do not directly reduce, via definitions (13.2) and (13.3), to the wave field equations (13.5) without the Lorentz condition (13.4). This fact is very important, and suggests that when it comes to a choice of governing equations, it may be reasonable to replace Maxwell's equations (13.7) and (13.6) with the Lorentz condition (13.4), (13.5) and the continuity equation (13.6). From the assumptions formulated above, one is led to the following result.

Proposition 13.1. *The Lorentz invariant wave equations (13.5) for the potentials* $(\varphi, A) : M^4 \to \mathbb{R} \times \mathbb{E}^3$*, together with the Lorentz condition (13.4) and the charge continuity relationship (13.5), are equivalent to the Maxwell field equations (13.7).*

Proof. Substituting (13.4), into (13.5), one easily obtains

$$\partial^2\varphi/\partial t^2 = - <\nabla, \partial A/\partial t>=<\nabla, \nabla\varphi> + \rho, \tag{13.8}$$

which implies the gradient expression

$$<\nabla, -\partial A/\partial t - \nabla\varphi>= \rho. \tag{13.9}$$

Taking into account the electric field definition (13.2), expression (13.9) reduces to

$$<\nabla, E>= \rho, \tag{13.10}$$

which is the second of the first pair of Maxwell's equations (13.7). Now upon applying $\nabla\times$ to definition (13.2), we find, owing to definition (13.3), that

$$\nabla \times E + \partial B/\partial t = 0, \tag{13.11}$$

which is the first of the first pair of the Maxwell equations (13.7). Applying $\nabla\times$ to the definition (13.3), one obtains

$$\begin{aligned}\nabla \times B &= \nabla \times (\nabla \times A) = \nabla <\nabla, A> - \nabla^2 A \\ &= -\nabla(\partial\varphi/\partial t) - \partial^2 A/\partial t^2 + (\partial^2 A/\partial t^2 - \nabla^2 A) \\ &= \frac{\partial}{\partial t}(-\nabla\varphi - \partial A/\partial t) + J = \partial E/\partial t + J,\end{aligned} \tag{13.12}$$

which leads directly to

$$\nabla \times B = \partial E/\partial t + J,$$

which is the first of the second pair of the Maxwell equations (13.7). The final *no magnetic charge* equation

$$< \nabla, B >=< \nabla, \nabla \times A >= 0,$$

in (13.7) follows directly from the elementary identity $< \nabla, \nabla \times >= 0$, thereby completing the proof. □

This proposition allows us to consider the potential functions $(\varphi, A) : M^4 \rightarrow \mathbb{R} \times \mathbb{E}^3$ as fundamental ingredients of the ambient *vacuum field medium*, by means of which we can try to describe the related physical behavior of charged point particles embedded in space-time M^4. The following observation provides strong support for this approach.

Observation. The Lorentz condition (13.4) actually means that the scalar potential field $\varphi : M^4 \rightarrow \mathbb{R}$ has a continuity relationship, whose origin lies in a new field conservation law and characterizes the deep intrinsic structure of the vacuum field medium.

To make this observation more transparent and precise, let us recall the definition [123, 225, 302, 387] of the electric current $J : M^4 \rightarrow \mathbb{E}^3$ in the dynamical form

$$J := \rho v, \tag{13.13}$$

where the vector $v : M^4 \rightarrow \mathbb{E}^3$ is the corresponding charge velocity. Thus, the following continuity relationship

$$\partial \rho/\partial t + < \nabla, \rho v >= 0 \tag{13.14}$$

holds, which can easily be rewritten [88] as the integral conservation law

$$\frac{d}{dt} \int_{\Omega_t} \rho d^3 r = 0 \tag{13.15}$$

for the charge inside any bounded domain $\Omega_t \subset \mathbb{E}^3$, moving in the space-time M^4 with respect to the natural evolution equation

$$dr/dt := v. \tag{13.16}$$

Following the above reasoning, we are led to the following result.

Proposition 13.2. *The Lorentz condition (13.4) is equivalent to the integral conservation law*

$$\frac{d}{dt} \int_{\Omega_t} \varphi d^3 r = 0, \tag{13.17}$$

where $\Omega_t \subset \mathbb{E}^3$ *is any bounded domain moving with respect to the evolution equation*

$$dr/dt := v, \tag{13.18}$$

which represents the velocity vector of local potential field changes propagating in Minkowski space-time M^4.

Proof. Consider first the corresponding solutions to potential field equations (13.5), taking into account condition (13.13). Owing to the results from [123, 225], one finds that

$$A = \varphi v, \tag{13.19}$$

which gives rise to the following form of the Lorentz condition (13.4):

$$\partial\varphi/\partial t + < \nabla, \varphi v >= 0. \tag{13.20}$$

This can obviously be rewritten [88] as the integral conservation law (13.17), so the proof is complete. □

The above proposition suggests a physically motivated interpretation of electrodynamic phenomena in terms of what should naturally be called *the vacuum potential field*, which determines the observable interactions between charged point particles. More precisely, we can endow the ambient vacuum medium with a scalar potential field function $W := q\varphi : M^4 \to \mathbb{R}$, satisfying the governing *vacuum field equations*

$$\partial^2 W/\partial t^2 - \nabla^2 W = 0, \quad \partial W/\partial t + < \nabla, Wv >= 0, \tag{13.21}$$

taking into account that there are no external sources besides material particles, which possess only a virtual capability for disturbing the vacuum field medium. Moreover, this vacuum potential field function $W : M^4 \to \mathbb{R}$ allows the natural potential energy interpretation, whose origin should be assigned not only to the charged interacting medium, but also to any other medium possessing interaction capabilities, including for instance, material particles interacting through the gravity.

The latter leads naturally to the next important step, consisting in deriving the equation governing the corresponding potential field $\bar{W} : M^4 \to \mathbb{R}$ assigned to the vacuum field medium in a neighborhood of any spatial point moving with velocity $u \in \mathbb{E}^3$ and located at $R(t) \in \mathbb{E}^3$ at time $t \in \mathbb{R}$. As can be readily shown [321, 322], the corresponding evolution equation governing the related potential field function $\bar{W} : M^4 \to \mathbb{R}$, has the form

$$\frac{d}{dt}(-\bar{W}u) = -\nabla\bar{W}, \tag{13.22}$$

where $\bar{W} := W(r,t)|_{r\to R(t)}$, $u := dR(t)/dt$ at point particle location $(R(t),t) \in M^4$.

Similarly, if there are two interacting point particles located at points $R(t)$ and $R_f(t) \in \mathbb{E}^3$ at time $t \in \mathbb{R}$ and moving, respectively, with velocities $u := dR(t)/dt$ and $u_f := dR_f(t)/dt$, the corresponding potential field function $\bar{W} : M^4 \to \mathbb{R}$ for the particle located at point $R(t) \in \mathbb{E}^3$ should satisfy

$$\frac{d}{dt}[-\bar{W}(u-u_f)] = -\nabla\bar{W}. \tag{13.23}$$

The dynamical potential field equations (13.22) and (13.23) appear to have important properties and can be used as a means for representing classical electrodynamics. Consequently, we shall proceed to investigate their physical properties in more detail and compare them with classical results for Lorentz type forces arising in the electrodynamics of moving charged point particles in an external electromagnetic field.

We take our inspiration from [98, 116, 143, 168, 215, 399] and especially the interesting studies [79, 80] of the classical problem of reconciling gravitational and electrodynamic charges within the Mach–Einstein ether paradigm. First, we revisit the classical Mach–Einstein relativistic electrodynamics of a moving charged point particle, and second, we study the resulting electrodynamic theories associated with our vacuum potential field dynamical equations (13.22) and (13.23), making use of the fundamental Lagrangian and Hamiltonian formalisms which were specially devised for this in [64, 323]. The results obtained are used to apply the canonical Dirac quantization procedure to the corresponding energy conservation laws associated to the electrodynamic models considered.

13.2 Classical relativistic electrodynamics revisited

The classical relativistic electrodynamics of a freely moving charged point particle in Minkowski space-time $M^4 := \mathbb{R}\times\mathbb{E}^3$ is based on the Lagrangian approach [123, 225, 302, 387] with Lagrangian function

$$\mathcal{L} := -m_0(1-u^2)^{1/2}, \tag{13.24}$$

where $m_0 \in \mathbb{R}_+$ is the so-called particle rest mass and $u \in \mathbb{E}^3$ is its spatial velocity in the Euclidean space $\mathbb{E}^3$, expressed here and in the sequel in light speed units (with light speed c). The least action principle in the form

$$\delta S = 0, \quad S := -\int_{t_1}^{t_2} m_0(1-u^2)^{1/2}dt \tag{13.25}$$

for any fixed temporal interval $[t_1, t_2] \subset \mathbb{R}$ gives rise to the well-known relativistic relationships for the mass

$$m = m_0(1 - u^2)^{-1/2}, \tag{13.26}$$

momentum

$$p := mu = m_0 u(1 - u^2)^{-1/2} \tag{13.27}$$

and the energy of the particle

$$\mathcal{E}_0 = m = m_0(1 - u^2)^{-1/2}. \tag{13.28}$$

It follows from [225, 302], that the origin of the Lagrangian (13.24) can be extracted from the action

$$S := -\int_{t_1}^{t_2} m_0(1 - u^2)^{1/2} dt = -\int_{\tau_1}^{\tau_2} m_0 d\tau, \tag{13.29}$$

on a suitable temporal interval $[\tau_{1,}\tau_2] \subset \mathbb{R}$, where

$$d\tau := dt(1 - u^2)^{1/2} \tag{13.30}$$

and $\tau \in \mathbb{R}$ is the so-called proper temporal parameter assigned to a freely moving particle with respect to the rest reference system $\mathcal{K}_r$. The action (13.29) is rather questionable from the dynamical point of view, since it is physically defined with respect to the rest reference system $\mathcal{K}_r$, giving rise to the constant action $S = -m_0(\tau_2 - \tau_1)$, as the limits of integration $\tau_1 < \tau_2 \in \mathbb{R}$ were taken to be fixed from the very beginning. Moreover, considering this particle to have charge $q \in \mathbb{R}$ and be moving in Minkowski space-time M^4 under the action of an electromagnetic field $(\varphi, A) \in \mathbb{R} \times \mathbb{E}^3$, the corresponding classical (relativistic) action functional is chosen (see [64, 323, 123, 225, 302, 387]) as follows:

$$S := \int_{\tau_1}^{\tau_2} [-m_0 d\tau + q < A, \dot{r} > d\tau - q\varphi(1 - u^2)^{-1/2} d\tau], \tag{13.31}$$

with respect to the *rest reference system*, parametrized by the Euclidean space-time variables $(\tau, r) \in \mathbb{E}^4$, where we have denoted $\dot{r} := dr/d\tau$ in contrast to the definition $u := dr/dt$. The action (13.31) can be rewritten with respect to the laboratory reference system $\mathcal{K}$ moving with velocity vector $u \in \mathbb{E}^3$ as

$$S = \int_{t_1}^{t_2} \mathcal{L} dt, \quad \mathcal{L} := -m_0(1 - u^2)^{1/2} + q < A, u > -q\varphi, \tag{13.32}$$

on a suitable temporal interval $[t_1, t_2] \subset \mathbb{R}$, which gives rise to the following [123, 225, 302, 387] dynamical expressions

$$P = p + qA, \quad p = mu, \quad m = m_0(1 - u^2)^{-1/2}, \tag{13.33}$$

for the particle momentum and

$$\mathcal{E}_0 = [m_0^2 + (P - qA)^2]^{1/2} + q\varphi \tag{13.34}$$

for the particle energy, where $P \in \mathbb{E}^3$ is the common momentum of the particle and the ambient electromagnetic field at a space-time point $(t, r) \in M^4$.

The expression (13.34) for the particle energy $\mathcal{E}_0$ also is open to question, since the potential energy $q\varphi$, entering additively, has no effect on the particle mass $m = m_0(1 - u^2)^{-1/2}$. This was noticed by L. Brillouin [75], who remarked that the fact that the potential energy has no effect on the particle mass tells us that "... any possibility of existence of a particle mass related with an external potential energy, is completely excluded". Moreover, it is necessary to stress here that the least action principle (13.32), formulated with respect to the laboratory reference system $\mathcal{K}$ time parameter $t \in \mathbb{R}$, appears logically inadequate, for there is a strong physical inconsistency with other time parameters of the Lorentz equivalent reference systems. This was first mentioned by R. Feynman in [124], in his efforts to rewrite the Lorentz force expression with respect to the rest reference system $\mathcal{K}_r$. This and other special relativity theory and electrodynamics problems stimulated many prominent physicists of the past [74, 75, 124, 302, 395] and present [34, 79, 98, 116, 156, 215, 234, 236, 235, 237, 256, 255, 280, 351, 399] to try to develop alternative relativity theories based on completely different space-time and matter structure principles.

There is also another controversial inference from the action expression (13.32). As one can easily show [123, 225, 302, 387], the corresponding dynamical equation for the Lorentz force is given as

$$dp/dt = F := qE + qu \times B. \tag{13.35}$$

We have defined here, as before,

$$E := -\partial A/\partial t - \nabla\varphi \tag{13.36}$$

for the corresponding electric field and

$$B := \nabla \times A \tag{13.37}$$

for the related magnetic field, acting on the charged point particle q. The expression (13.35) implies, in particular, that the Lorentz force F depends linearly on the particle velocity vector $u \in \mathbb{E}^3$, and so there is a strong dependence on the reference system with respect to which the charged particle q moves. Attempts to reconcile this and some related controversies [75, 124, 351, 196] forced Einstein to devise his special relativity theory and proceed further to creating his general relativity theory trying to explain the gravity by means of geometrization of space-time and matter in the Universe. Here we must mention that the classical Lagrangian function $\mathcal{L}$ in (13.32) is written in terms of a combination of terms expressed by means of both the Euclidean rest reference system variables $(\tau, r) \in \mathbb{E}^4$ and arbitrarily chosen Minkowski reference system variables $(t, r) \in M^4$.

These problems were recently analyzed using a completely different "no-geometry" approach [321, 322, 351], where new dynamical equations were derived, which were free of the controversial elements mentioned above. Moreover, this approach avoided the introduction of the Lorentz transformations of the space-time reference systems with respect to which the action functional (13.32) is invariant. From this point of view, there are interesting conclusions in [186], in which Galilean invariant Lagrangians possessing intrinsic Poincaré–Lorentz symmetry are re-analyzed. Next, we revisit the results obtained in [321, 322] from the classical Lagrangian and Hamiltonian formalisms [64] in order to shed new light on the physical underpinnings of the vacuum field theory approach to the study of combined electromagnetic and gravitational effects.

13.3 Vacuum field theory electrodynamics: Lagrangian analysis

13.3.1 *Motion of a point particle in a vacuum - an alternative electrodynamic model*

In the vacuum field theory approach to combining electromagnetism and the gravity, devised in [321, 322], the main vacuum potential field function $\bar{W} : M^4 \rightarrow \mathbb{R}$, related to a charged point particle q satisfies the dynamical equation (13.21), namely

$$\frac{d}{dt}(-\bar{W}u) = -\nabla \bar{W} \tag{13.38}$$

in the case when the external charged particles are at rest, where, as above, $u := dr/dt$ is the particle velocity with respect to some reference system.

To analyze the dynamical equation (13.38) from the Lagrangian point of view, we write the corresponding action functional as

$$S := -\int_{t_1}^{t_2} \bar{W} dt = -\int_{\tau_1}^{\tau_2} \bar{W}(1+\dot{r}^2)^{1/2}\ d\tau, \tag{13.39}$$

expressed with respect to the rest reference system $\mathcal{K}_r$. Fixing the proper temporal parameters $\tau_1 < \tau_2 \in \mathbb{R}$, one finds from the least action principle ($\delta S = 0$) that

$$\begin{aligned} p &:= \partial\mathcal{L}/\partial\dot{r} = -\bar{W}\dot{r}(1+\dot{r}^2)^{-1/2} = -\bar{W}u, \\ \dot{p} &:= dp/d\tau = \partial\mathcal{L}/\partial r = -\nabla\bar{W}(1+\dot{r}^2)^{1/2}, \end{aligned} \tag{13.40}$$

where, owing to (13.39), the corresponding Lagrangian function is

$$\mathcal{L} := -\bar{W}(1+\dot{r}^2)^{1/2}. \tag{13.41}$$

Recalling now the definition of the particle mass

$$m := -\bar{W} \tag{13.42}$$

and the relationships

$$d\tau = dt(1-u^2)^{1/2},\ \dot{r}d\tau = udt, \tag{13.43}$$

from (13.40) we easily obtain the dynamical equation (13.38). Moreover, one now readily finds that the dynamical mass, defined by means of expression (13.42), is given as

$$m = m_0(1-u^2)^{-1/2},$$

which coincides with the equation (13.26) of the preceding section. Now it is easy to verify the following proposition using the above results.

Proposition 13.3. *The alternative freely moving point particle electrodynamic model (13.38) allows the least action formulation (13.39) with respect to the "rest" reference system variables, where the Lagrangian function is given by expression (13.41). Its electrodynamics is completely equivalent to that of a classical relativistic freely moving point particle described above.*

13.3.2 *Motion of two interacting charge systems in a vacuum–an alternative electrodynamic model*

We now consider the case when the charged point particle q moves in space-time with velocity vector $u \in \mathbb{E}^3$ and interacts with another external charged point particle, moving with velocity vector $u_f \in \mathbb{E}^3$ in a common reference system $\mathcal{K}$. As was shown in [321, 322], the corresponding dynamical equation for the vacuum potential field function $\bar{W} : M^4 \rightarrow \mathbb{R}$ is

$$\frac{d}{dt}[-\bar{W}(u-u_f)] = -\nabla \bar{W}. \tag{13.44}$$

As the external charged particle moves in the space-time, it generates the related magnetic field $B := \nabla \times A$, whose magnetic vector potential $A : M^4 \rightarrow \mathbb{E}^3$ is defined, owing to the results of [321, 322, 351], as

$$qA := \bar{W} u_f. \tag{13.45}$$

Whence, it follows from (13.40) that the particle momentum $p = -\bar{W}u$, equation (13.44) is equivalent to

$$\frac{d}{dt}(p + qA) = -\nabla \bar{W}. \tag{13.46}$$

To represent the dynamical equation (13.46) in the classical Lagrangian formalism, we start from the following action functional, which naturally generalizes the functional (13.39):

$$S := -\int_{\tau_1}^{\tau_2} \bar{W}(1 + |\dot{r} - \dot{\xi}|^2)^{1/2} \, d\tau, \tag{13.47}$$

where we have denoted by $\dot{\xi} = u_f dt/d\tau$, $d\tau = dt(1-(u-u_f)^2)^{1/2}$, which takes into account the relative velocity of the charged point particle q with respect to the reference system $\mathcal{K}'$, moving with velocity vector $u_f \in \mathbb{E}^3$ in the reference system $\mathcal{K}$. It is clear in this case the charged point particle q moves with velocity vector $u - u_f \in \mathbb{E}^3$ with respect to the reference system $\mathcal{K}'$ in which the external charged particle is at rest.

Now we compute the least action variational condition $\delta S = 0$, taking into account that, owing to (13.47), the corresponding Lagrangian function is given as

$$\mathcal{L} := -\bar{W}(1 + (\dot{r} - \dot{\xi})^2)^{1/2}. \tag{13.48}$$

Hence, the common momentum of the particles is

$$\begin{aligned} P := \partial \mathcal{L}/\partial \dot{r} &= -\bar{W}(\dot{r} - \dot{\xi})(1 + (\dot{r} - \dot{\xi})^2)^{-1/2} \\ &= -\bar{W}\dot{r}(1 + (\dot{r} - \dot{\xi})^2)^{-1/2} + \bar{W}\dot{\xi}(1 + (\dot{r} - \dot{\xi})^2)^{-1/2} \\ &= mu + qA := p + qA, \end{aligned} \tag{13.49}$$

and the dynamical equation is given as

$$\frac{d}{d\tau}(p + qA) = -\nabla \bar{W}(1 + |\dot{r} - \dot{\xi}|^2)^{1/2}. \tag{13.50}$$

As $d\tau = dt(1-(u-u_f)^2)^{1/2}$ and $(1+(\dot{r}-\dot{\xi})^2)^{1/2} = (1-(u-u_f)^2)^{-1/2}$, we obtain finally from (13.50) precisely the dynamical equation (13.46), which leads to the next proposition.

Proposition 13.4. *The alternative classical relativistic electrodynamic model (13.44) allows the least action formulation (13.47) with respect to the "rest" reference system variables, where the Lagrangian function is given by the expression (13.48).*

13.3.3 A moving charged point particle formulation dual to the classical alternative electrodynamic model

It is easy to see that the action functional (13.47) is written utilizing the classical Galilean transformations. If we now consider the action functional (13.39) for a charged point particle moving with respect to the reference system $\mathcal{K}_r$, and take into account its interaction with an external magnetic field generated by the vector potential $A : M^4 \to \mathbb{E}^3$, it can be naturally generalized as

$$S := \int_{t_1}^{t_2} (-\bar{W}dt + q < A, dr >) = \int_{\tau_1}^{\tau_2} [-\bar{W}(1+\dot{r}^2)^{1/2} \ + q < A, \dot{r} >] d\tau, \tag{13.51}$$

where $d\tau = dt(1-u^2)^{1/2}$.

Thus, the corresponding common particle-field momentum takes the form

$$\begin{aligned} P &:= \partial\mathcal{L}/\partial\dot{r} = -\bar{W}\dot{r}(1+\dot{r}^2)^{-1/2} + qA \\ &= mu + qA := p + qA, \end{aligned} \tag{13.52}$$

and satisfies

$$\begin{aligned} \dot{P} &:= dP/d\tau = \partial\mathcal{L}/\partial r = -\nabla\bar{W}(1+\dot{r}^2)^{1/2} \ + q\nabla < A, \dot{r} > \\ &= -\nabla\bar{W}(1-u^2)^{-1/2} + q\nabla < A, u > (1-u^2)^{-1/2}, \end{aligned} \tag{13.53}$$

where

$$\mathcal{L} := -\bar{W}(1+\dot{r}^2)^{1/2} \ + q < A, \dot{r} > \tag{13.54}$$

is the corresponding Lagrangian function. Since $d\tau = dt(1-u^2)^{1/2}$, one easily finds from (13.53) that

$$dP/dt = -\nabla\bar{W} + q\nabla < A, u > . \tag{13.55}$$

Upon substituting (13.52) into (13.55) and making use of the well-known [225] identity

$$\nabla < a, b > = < a, \nabla > b + < b, \nabla > a + b \times (\nabla \times a) + a \times (\nabla \times b), \quad (13.56)$$

where $a, b \in \mathbb{E}^3$ are arbitrary vector functions, we obtain the classical expression for the Lorentz force F, acting on the moving charged point particle q :

$$dp/dt := F = qE + qu \times B, \quad (13.57)$$

where

$$E := -\nabla \bar{W} q^{-1} - \partial A/\partial t \quad (13.58)$$

is its associated electric field and

$$B := \nabla \times A \quad (13.59)$$

is the corresponding magnetic field. This result can be summarized as follows:

Proposition 13.5. *The classical relativistic Lorentz force (13.57) allows the least action formulation (13.51) with respect to the rest reference system variables, where the Lagrangian function is given by formula (13.54). Its electrodynamics, described by the Lorentz force (13.57) is completely equivalent to the classical relativistic moving point particle electrodynamics, described by means of the Lorentz force (13.35) above.*

As for the dynamical equation (13.50), it is easy to see that it is equivalent to

$$dp/dt = (-\nabla \bar{W} - q dA/dt + q\nabla < A, u >) - q\nabla < A, u >, \quad (13.60)$$

which, owing to (13.55) and (13.57), takes the following Lorentz type force form

$$dp/dt = qE + qu \times B - q\nabla < A, u >, \quad (13.61)$$

that can be found in [321, 322, 351].

Expressions (13.57) and (13.61) are equal up to the gradient term $F_c := -q\nabla < A, u >$, which reconciles the Lorentz forces acting on a charged moving particle q with respect to different reference systems. This fact is important for our vacuum field theory approach since it uses no special geometry and makes it possible to analyze both electromagnetic and gravitational fields simultaneously by employing the new definition of the dynamical mass by means of expression (13.42).

13.4 Vacuum field electrodynamics: Hamiltonian analysis

Any Lagrangian theory has an equivalent canonical Hamiltonian representation via the classical Legendre transformation [3, 14, 176, 177, 326, 387]. As we have already formulated our vacuum field theory of a moving charged particle q in Lagrangian form, we proceed now to its Hamiltonian analysis making use of the action functionals (13.39), (13.48) and (13.51).

Take, first, the Lagrangian function (13.41) and the momentum expression (13.40) for defining the corresponding Hamiltonian function

$$\begin{aligned} H &:=< p, \dot{r} > -\mathcal{L} \\ &= - < p, p > \bar{W}^{-1}(1 - p^2/\bar{W}^2)^{-1/2} + \bar{W}(1 - p^2/\bar{W}^2)^{-1/2} \\ &= -p^2\bar{W}^{-1}(1 - p^2/\bar{W}^2)^{-1/2} + \bar{W}^2\bar{W}^{-1}(1 - p^2/\bar{W}^2)^{-1/2} \\ &= -(\bar{W}^2 - p^2)(\bar{W}^2 - p^2)^{-1/2} = -(\bar{W}^2 - p^2)^{1/2}. \end{aligned} \tag{13.62}$$

Consequently, it is easy to show [3, 14, 326, 387] that the Hamiltonian function (13.62) is a conservation law of the dynamical field equation (13.38); that is, for all $\tau, t \in \mathbb{R}$

$$dH/dt = 0 = dH/d\tau, \tag{13.63}$$

which naturally leads to an energy interpretation of H. Thus, we can represent the particle energy as

$$\mathcal{E} = (\bar{W}^2 - p^2)^{1/2}. \tag{13.64}$$

Accordingly the Hamiltonian equivalent to the vacuum field equation (13.38) can be written as

$$\begin{aligned} \dot{r} &:= dr/d\tau = \partial H/\partial p = p(\bar{W}^2 - p^2)^{-1/2} \\ \dot{p} &:= dp/d\tau = -\partial H/\partial r = \bar{W}\nabla\bar{W}(\bar{W}^2 - p^2)^{-1/2}, \end{aligned} \tag{13.65}$$

and we have the following result.

Proposition 13.6. *The alternative freely moving point particle electrodynamic model (13.38) allows the canonical Hamiltonian formulation (13.65) with respect to the "rest" reference system variables, where the Hamiltonian function is given by expression (13.62). Its electrodynamics is completely equivalent to the classical relativistic freely moving point particle electrodynamics described above.*

In an analogous manner, one can now use the Lagrangian (13.48) to construct the Hamiltonian function for the dynamical field equation (13.46), describing the motion of charged particle q in an external electromagnetic field in the canonical Hamiltonian form

$$\dot{r} := dr/d\tau = \partial H/\partial P, \qquad \dot{P} := dP/d\tau = -\partial H/\partial r, \tag{13.66}$$

where

$$\begin{aligned} H &:= < P, \dot{r} > -\mathcal{L} \\ &= < P, \dot{\xi} - P\bar{W}^{-1}(1 - P^2/\bar{W}^2)^{-1/2} > + \bar{W}[\bar{W}^2(\bar{W}^2 - P^2)^{-1}]^{1/2} \\ &= < P, \dot{\xi} > + P^2(\bar{W}^2 - P^2)^{-1/2} - \bar{W}^2(\bar{W}^2 - P^2)^{-1/2} \\ &= -(\bar{W}^2 - P^2)(\bar{W}^2 - P^2)^{-1/2} + < P, \dot{\xi} > \\ &= -(\bar{W}^2 - P^2)^{1/2} - q < A, P > (\bar{W}^2 - P^2)^{-1/2}. \end{aligned} \tag{13.67}$$

Here we took into account that, owing to definitions (13.45) and (13.49),

$$\begin{aligned} qA &:= \bar{W}u_f = \bar{W}d\xi/dt \\ &= \bar{W}\frac{d\xi}{d\tau} \cdot \frac{d\tau}{dt} = \bar{W}\dot{\xi}(1 - (u - v))^{1/2} \\ &= \bar{W}\dot{\xi}(1 + (\dot{r} - \dot{\xi})^2)^{-1/2} \\ &= -\bar{W}\dot{\xi}(\bar{W}^2 - P^2)^{1/2}\bar{W}^{-1} = -\dot{\xi}(\bar{W}^2 - P^2)^{1/2}, \end{aligned} \tag{13.68}$$

or

$$\dot{\xi} = -qA(\bar{W}^2 - P^2)^{-1/2}, \tag{13.69}$$

where $A : M^4 \rightarrow \mathbb{R}^3$ is the related magnetic vector potential generated by the moving external charged particle. Equations (13.67) can be rewritten with respect to the laboratory reference system $\mathcal{K}$ in the form

$$dr/dt = u, \qquad dp/dt = qE + qu \times B - q\nabla < A, u >, \tag{13.70}$$

which coincides with the result (13.61).

Whence, we see that the Hamiltonian function (13.67) satisfies the energy conservation conditions

$$dH/dt = 0 = dH/d\tau, \tag{13.71}$$

for all $\tau, t \in \mathbb{R}$, and that the suitable energy expression is

$$\mathcal{E} = (\bar{W}^2 - P^2)^{1/2} + q < A, P > (\bar{W}^2 - P^2)^{-1/2}, \tag{13.72}$$

where the generalized momentum $P = p + qA$. The result (13.72) differs in an essential way from that obtained in [225], which makes use of the Einsteinian Lagrangian for a moving charged point particle q in an external electromagnetic field. Thus, we obtain the following result.

Proposition 13.7. *The alternative classical relativistic electrodynamic model (13.70), which is intrinsically compatible with the classical Maxwell equations (13.7), allows the Hamiltonian formulation (13.66) with respect to the rest reference system variables, where the Hamiltonian function is given by expression (13.67).*

The inference above is a natural candidate for experimental validation of the theory. It is strongly motivated by the following remark.

Remark 13.1. It is necessary to mention here that the Lorentz force expression (13.70) uses the particle momentum $p = mu$, where the dynamical "mass" $m := -\bar{W}$ satisfies condition (13.72). The latter gives rise to the following crucial relationship between the particle energy $\mathcal{E}_0$ and its rest mass m_0 (at the velocity $u := 0$ at the initial time moment $t = 0 \in \mathbb{R}$) :

$$\mathcal{E}_0 = m_0(1 - \frac{q^2}{m_0^2}A_0^2)^{-1/2}, \tag{13.73}$$

or, equivalently,

$$m_0 = \mathcal{E}_0(\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4q^2A_0^2}), \tag{13.74}$$

where $A_0 := A|_{t=0} \in \mathbb{E}^3$, which strongly differs from the classical formulation (13.34).

To make this difference clearer, we now analyze the Lorentz force (13.57) from the Hamiltonian point of view based on the Lagrangian function (13.54). Thus, we obtain that the corresponding Hamiltonian function

$$\begin{aligned} H &:=< P, \dot{r} > -\mathcal{L} =< P, \dot{r} > +\bar{W}(1 + \dot{r}^2)^{1/2} - q < A, \dot{r} > \\ &=< P - qA, \dot{r} > +\bar{W}(1 + \dot{r}^2)^{1/2} \\ &= - < p, p > \bar{W}^{-1}(1 - p^2/\bar{W}^2)^{-1/2} + \bar{W}(1 - p^2/\bar{W}^2)^{-1/2} \\ &= -(\bar{W}^2 - p^2)(\bar{W}^2 - p^2)^{-1/2} = -(\bar{W}^2 - p^2)^{1/2}. \end{aligned} \tag{13.75}$$

Since $p = P - qA$, expression (13.75) assumes the final *no interaction* [222, 225, 302, 339] form

$$H = -[\bar{W}^2 - (P - qA)^2]^{1/2}, \tag{13.76}$$

which is conserved with respect to the evolution equations (13.52) and (13.53), that is

$$dH/dt = 0 = dH/d\tau \tag{13.77}$$

for all $\tau, t \in \mathbb{R}$. These equations are equivalent to the following Hamiltonian system

$$\dot{r} = \partial H/\partial P = (P - qA)[\bar{W}^2 - (P - qA)^2]^{-1/2}, \tag{13.78}$$
$$\dot{P} = -\partial H/\partial r = (\bar{W}\nabla\bar{W} - \nabla < qA, (P - qA) >)[\bar{W}^2 - (P - qA)^2]^{-1/2},$$

as one can readily check by direct calculations. Actually, the first equation

$$\begin{aligned} \dot{r} &= (P - qA)[\bar{W}^2 - (P - qA)^2]^{-1/2} = p(\bar{W}^2 - p^2)^{-1/2} \\ &= mu(\bar{W}^2 - p^2)^{-1/2} = -\bar{W}u(\bar{W}^2 - p^2)^{-1/2} = u(1 - u^2)^{-1/2}, \end{aligned} \tag{13.79}$$

holds, owing to the condition $d\tau = dt(1 - u^2)^{1/2}$ and definitions $p := mu$, $m = -\bar{W}$, postulated from the start. Similarly, we find that

$$\begin{aligned} \dot{P} &= -\nabla\bar{W}(1 - p^2/\bar{W}^2)^{-1/2} + \nabla < qA, u > (1 - p^2/\bar{W}^2)^{-1/2} \\ &= -\nabla\bar{W}(1 - u^2)^{-1/2} + \nabla < qA, u > (1 - u^2)^{-1/2}, \end{aligned} \tag{13.80}$$

coincides with equation (13.55) in the evolution parameter $t \in \mathbb{R}$. This can be formulated as the next result.

Proposition 13.8. *The dual to the classical relativistic electrodynamic model (13.57) allows the canonical Hamiltonian formulation (13.78) with respect to the rest reference system variables, where the Hamiltonian function is given by expression (13.76). Moreover, this formulation circumvents the "mass-potential energy"controversy surrounding the classical electrodynamical model (13.32).*

The modified Lorentz force expression (13.57) and the related rest energy relationship are characterized by the following remark.

Remark 13.2. If we make use of the modified relativistic Lorentz force expression (13.57) as an alternative to the classical (13.35), the corresponding particle energy expression (13.76) also gives rise to a different energy expression (at the velocity $u := 0 \in \mathbb{E}^3$ at the initial time moment $t = 0$); namely, $\mathcal{E}_0 = m_0$ instead of $\mathcal{E}_0 = m_0 + q\varphi_0$, where $\varphi_0 := \varphi|_{t=0}$, in the the classical case (13.34).

13.5 Quantization of electrodynamics models in vacuum field theory: No-geometry approach

13.5.1 *The problem setting*

Recently [321, 322], we devised a new regular *no-geometry* approach to deriving the electrodynamics of a moving charged point particle q in an external electromagnetic field from first principles. This approach has, in part, reconciled the mass-energy controversy [75] in classical relativistic electrodynamics. Using the vacuum field theory approach proposed in [321, 322, 351], we reanalyzed this problem above both from the Lagrangian and Hamiltonian perspective and derived key expressions for the corresponding energy functions and Lorentz type forces acting on a moving charge point particle q.

Since all of our electrodynamics models have been represented here in canonical Hamiltonian form, they are well suited to the application of Dirac quantization [57, 58, 100] and the corresponding derivation of related Schrödinger type evolution equations. We describe these procedures in this section.

13.5.2 *Free point particle electrodynamics model and its quantization*

The charged point particle electrodynamics models discussed in detail above, were also considered in [322] from the dynamical point of view, where a Dirac quantization of the corresponding conserved energy expressions was attempted. However, from the canonical point of view, the true quantization procedure should be based on the relevant canonical Hamiltonian formulation of the models given in (13.65), (13.66) and (13.78).

In particular, consider a free charged point particle electrodynamics model characterized by (13.65) and having the Hamiltonian equations

$$
\begin{aligned}
dr/d\tau &:= \partial H/\partial p = -p(\bar{W}^2 - p^2)^{-1/2}, \\
dp/d\tau &:= -\partial H/\partial r = -\bar{W}\nabla\bar{W}(\bar{W}^2 - p^2)^{-1/2},
\end{aligned} \tag{13.81}
$$

where $\bar{W} : M^4 \rightarrow \mathbb{R}$ defined in the preceding sections is the corresponding vacuum field potential characterizing the medium field structure, $(r, p) \in T^*(\mathbb{E}^3) \simeq \mathbb{E}^3 \times \mathbb{E}^3$ are the standard canonical coordinate-momentum variables on the cotangent space $T^*(\mathbb{E}^3)$, $\tau \in \mathbb{R}$ is the proper rest reference system $\mathcal{K}_r$ time parameter of the moving particle, and $H : T^*(\mathbb{E}^3) \rightarrow \mathbb{R}$ is the Hamiltonian function

$$
H := -(\bar{W}^2 - p^2)^{1/2}, \tag{13.82}
$$

expressed here and hereafter in light speed units. The rest reference system $\mathcal{K}_r$, parametrized by variables $(\tau, r) \in \mathbb{E}^4$, is related to any other reference system $\mathcal{K}$ in which our charged point particle q moves with velocity vector $u \in \mathbb{E}^3$. The frame $\mathcal{K}$ is parametrized by variables $(t, r) \in M^4$ via the

Euclidean infinitesimal relationship

$$dt^2 = d\tau^2 + dr^2, \tag{13.83}$$

which is equivalent to the Minkowskian infinitesimal relationship

$$d\tau^2 = dt^2 - dr^2. \tag{13.84}$$

The Hamiltonian function (13.82) clearly satisfies the energy conservation conditions

$$dH/dt = 0 = dH/d\tau \tag{13.85}$$

for all $t, \tau \in \mathbb{R}$. Thus, the relevant energy

$$\mathcal{E} = (\bar{W}^2 - p^2)^{1/2} \tag{13.86}$$

can be treated by the Dirac quantization scheme [100, 104, 215] to obtain, as $\hbar \to 0$, (or the light speed $c \to \infty$) the governing Schrödinger type dynamical equation. To do this following the approach in [321, 322], we need to make canonical operator replacements $\mathcal{E} \to \hat{\mathcal{E}} := -\frac{\hbar}{i}\frac{\partial}{\partial \tau},\ p \to \hat{p} := \frac{\hbar}{i}\nabla$, as $\hbar \to 0$, in the energy expression

$$\mathcal{E}^2 := (\hat{\mathcal{E}}\psi, \hat{\mathcal{E}}\psi) = (\psi, \hat{\mathcal{E}}^2\psi) = (\psi, \hat{H}^+\hat{H}\psi), \tag{13.87}$$

where $(\cdot,\cdot)$ is the standard L^2 - inner product. It follows from (13.86) that

$$\hat{\mathcal{E}}^2 = \bar{W}^2 - \hat{p}^2 = \hat{H}^+\hat{H} \tag{13.88}$$

is a suitable operator factorization in the Hilbert space $\mathcal{H} := L^2(\mathbb{R}^3;\mathbb{C})$ and $\psi \in \mathcal{H}$ is the corresponding normalized quantum vector state. Since the following elementary identity

$$\bar{W}^2 - \hat{p}^2 = \bar{W}(1 - \bar{W}^{-1}\hat{p}^2\bar{W}^{-1})^{1/2}(1 - \bar{W}^{-1}\hat{p}^2\bar{W}^{-1})^{1/2}\bar{W} \tag{13.89}$$

holds, we can use (13.88) and (13.89) to define the operator

$$\hat{H} := (1 - \bar{W}^{-1}\hat{p}^2\bar{W}^{-1})^{1/2}\bar{W}. \tag{13.90}$$

Calculating the operator expression (13.90) as $\hbar \to 0$ up to accuracy $O(\hbar^4)$, it is easy see that

$$\hat{H} = \frac{\hat{p}^2}{2m(u)} + \bar{W} := -\frac{\hbar^2}{2m(u)}\nabla^2 + \bar{W}, \tag{13.91}$$

where we have taken into account the dynamical mass definition $m(u) := -\bar{W}$ (in the light speed units). Consequently, using (13.87) and (13.91), we obtain up to operator accuracy $O(\ \hbar^4)$ the following Schrödinger type evolution equation

$$i\hbar\frac{\partial\psi}{\partial\tau} := \hat{\mathcal{E}}\psi = \hat{H}\psi = -\frac{\hbar^2}{2m(u)}\nabla^2\psi + \bar{W}\psi \tag{13.92}$$

with respect to the rest reference system $\mathcal{K}_r$ evolution parameter $\tau \in \mathbb{R}$. For a related evolution parameter $t \in \mathbb{R}$ parametrizing a reference system $\mathcal{K}$, the equation (13.92) takes the form

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2 m_0}{2m(u)^2} \nabla^2 \psi - m_0 \psi. \tag{13.93}$$

Here we used the fact that (13.86) implies that the classical mass relationship

$$m(u) = m_0 (1 - u^2)^{-1/2} \tag{13.94}$$

holds, where $m_0 \in \mathbb{R}_+$ is the corresponding rest mass of the point particle q.

The linear Schrödinger equation (13.93) for the case $\hbar/c \to 0$ actually coincides with the well-known expression [100, 123, 225] from classical quantum mechanics.

13.5.3 *Classical charged point particle electrodynamics model and its quantization*

We start here from the first vacuum field theory reformulation of the classical charged point particle electrodynamics introduced above and based on the conserved Hamiltonian function (13.76)

$$H := -[\bar{W}^2 - (P - qA)^2]^{1/2}, \tag{13.95}$$

where $q \in \mathbb{R}$ is the particle charge, $(\bar{W}, A) \in \mathbb{R} \times \mathbb{E}^3$ is the corresponding representation of the electromagnetic field potentials and $P \in \mathbb{E}^3$ is the common generalized particle-field momentum

$$P := p + qA, \qquad p := mu, \tag{13.96}$$

which satisfies the classical Lorentz force equation. Here $m := -\bar{W}$ is the observable dynamical mass of our charged particle, and $u \in \mathbb{E}^3$ is its velocity vector with respect to a chosen reference system $\mathcal{K}$, all expressed in light speed units.

The electrodynamics based on (14.4) is canonically Hamiltonian, so the Dirac quantization scheme

$$P \to \hat{P} := \frac{\hbar}{i} \nabla, \qquad \mathcal{E} \to \hat{\mathcal{E}} := -\frac{\hbar}{i} \frac{\partial}{\partial \tau} \tag{13.97}$$

should be applied to the energy expression

$$\mathcal{E} := [\bar{W}^2 - (P - qA)^2]^{1/2}, \tag{13.98}$$

following from the conservation conditions

$$dH/dt = 0 = dH/d\tau, \tag{13.99}$$

satisfied for all $\tau, t \in \mathbb{R}$.

Proceeding as above, we factorize the operator $\hat{\mathcal{E}}^2$ as

$$\begin{gathered}\bar{W}^2 - (\hat{P} - qA)^2 = \bar{W}[1 - \bar{W}^{-1}(\hat{P} - qA)^2\bar{W}^{-1}]^{1/2} \\ \times[1 - \bar{W}^{-1}(\hat{P} - qA)^2\bar{W}^{-1}]^{1/2}\bar{W} := \hat{H}^+\hat{H},\end{gathered}$$

where (as $\hbar/c \to 0$, $\hbar c = const$)

$$\hat{H} := \frac{1}{2m(u)}(\frac{\hbar}{i}\nabla - qA)^2 + \bar{W} \tag{13.100}$$

up to operator accuracy $O(\hbar^4)$. Hence, the related Schrödinger type evolution equation in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3; \mathbb{C})$ is

$$i\hbar\frac{\partial\psi}{\partial\tau} := \hat{\mathcal{E}}\psi = \hat{H}\psi = \frac{1}{2m(u)}(\frac{\hbar}{i}\nabla - qA)^2\psi + \bar{W}\psi \tag{13.101}$$

with respect to the rest reference system $\mathcal{K}_r$ evolution parameter $\tau \in \mathbb{R}$, and corresponding Schrödinger type evolution equation with respect to the evolution parameter $t \in \mathbb{R}$ takes the form

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{m_0}{2m(u)^2}(\frac{\hbar}{i}\nabla - qA)^2\psi - m_0\psi, \tag{13.102}$$

which coincides (as $\hbar/c \to 0$, $\hbar c = const$) with the classical quantum mechanics version [103, 226].

13.5.4 *Modified charged point particle electrodynamics model and its quantization*

From the canonical viewpoint, we now turn to the true quantization procedure for the electrodynamics model, characterized by (13.50) and having the Hamiltonian function (13.67)

$$H := -(\bar{W}^2 - P^2)^{1/2} - q < A, P > (\bar{W}^2 - P^2)^{-1/2}. \tag{13.103}$$

Accordingly the suitable energy function is

$$\mathcal{E} := (\bar{W}^2 - P^2)^{1/2} + q < A, P > (\bar{W}^2 - P^2)^{-1/2}, \tag{13.104}$$

where, as before,

$$P := p + qA, \qquad p := mu, \quad m := -\bar{W}, \tag{13.105}$$

is a conserved quantity for (13.50), which we will canonically quantize via the Dirac procedure (13.97). Toward this end, we consider the quantum condition

$$\mathcal{E}^2 := (\hat{\mathcal{E}}\psi, \hat{\mathcal{E}}\psi) = (\psi, \hat{\mathcal{E}}^2\psi), \qquad (\psi,\psi) := 1, \tag{13.106}$$

where $\hat{\mathcal{E}} := -\frac{\hbar}{i}\frac{\partial}{\partial t}$ and $\psi \in \mathcal{H} = L^2(\mathbb{R}^3;\mathbb{C})$ is a normalized quantum state vector. Now making use of the energy function (13.104), one readily computes that

$$\mathcal{E}^2 = \bar{W}^2 - (P - qA)^2 + q^2[< A, A > + < A, P > (\bar{W}^2 - P^2)^{-1} < P, A >], \tag{13.107}$$

which transforms via canonical Dirac quantization $P \to \hat{P} := \frac{\hbar}{i}\nabla$ into the symmetrized operator expression

$$\hat{\mathcal{E}}^2 = \bar{W}^2 - (\hat{P} - qA)^2 + q^2 < A, A > + q^2 < A, \hat{P} > (\bar{W}^2 - \hat{P}^2)^{-1} < \hat{P}, A > . \tag{13.108}$$

Factoring the operator (13.108) in the form $\hat{\mathcal{E}}^2 = \hat{H}^+\hat{H}$, and retaining only terms up to $O(\hbar^4)$ (as $\hbar/c \to 0$, $\hbar c = const$), we compute that

$$\begin{aligned} \hat{H} :=& \frac{1}{2m(u)}(\frac{\hbar}{i}\nabla - qA)^2 - \frac{q^2}{2m(u)} < A, A > \\ & - \frac{q^2}{2m^3(u)} < A, \frac{\hbar}{i}\nabla >< \frac{\hbar}{i}\nabla, A >, \end{aligned} \tag{13.109}$$

where, as before, $m(u) = -\bar{W}$ in light speed units. Thus, owing to (13.106) and (13.109), the resulting Schrödinger evolution equation is

$$\begin{aligned} i\hbar\frac{\partial\psi}{\partial\tau} :=& \hat{H}\psi = \frac{1}{2m(u)}(\frac{\hbar}{i}\nabla - qA)^2\psi - \frac{q^2}{2m(u)} < A, A > \psi \\ & - \frac{q^2}{2m^3(u)} < A, \frac{\hbar}{i}\nabla >< \frac{\hbar}{i}\nabla, A > \psi \end{aligned} \tag{13.110}$$

with respect to the rest reference system proper evolution parameter $\tau \in \mathbb{R}$. This can be rewritten in the equivalent form

$$\begin{aligned} i\hbar\frac{\partial\psi}{\partial\tau} =& -\frac{\hbar^2}{2m(u)}\nabla^2\psi - \frac{1}{2m(u)} < [\frac{\hbar}{i}\nabla, qA]_+ > \psi \\ & - \frac{q^2}{2m^3(u)} < A, \frac{\hbar}{i}\nabla >< \frac{\hbar}{i}\nabla, A > \psi, \end{aligned} \tag{13.111}$$

where $[\cdot,\cdot]_+$ is the formal anti-commutator of operators. Similarly, one obtains the related Schrödinger equation with respect to the time parameter $t \in \mathbb{R}$, which we shall not dwell upon here. The result (13.110) only slightly differs from the classical Schrödinger evolution equation (13.101). Simultaneously, its form (13.111) almost completely coincides with the classical ones from [103, 226, 302] modulo the evolution considered with respect to the rest reference time parameter $\tau \in \mathbb{R}$. This suggests that we must more thoroughly reexamine the physical motivation of the principles underlying the classical electrodynamic models, described by the Hamiltonian functions (14.4) and (13.103), giving rise to different Lorentz type force expressions.

13.6 Some relevant observations

All of dynamical field equations discussed above are canonical Hamiltonian systems with respect to the corresponding proper rest reference systems $\mathcal{K}_r$, parametrized by suitable time parameters $\tau \in \mathbb{R}$. Upon passing to the basic laboratory reference system $\mathcal{K}$ with the time parameter $t \in \mathbb{R}$, naturally the related Hamiltonian structure is lost, giving rise to a new interpretation of the real particle motion. Namely, one that has an absolute sense only with respect to the proper rest reference system, and otherwise being completely relative with respect to all other reference systems. As for the Hamiltonian expressions (13.62), (13.67) and (13.76), one observes that they all depend strongly on the vacuum potential field function $\bar{W} : M^4 \to \mathbb{R}$, thereby avoiding the mass problem of the classical energy expression pointed out by L. Brillouin [75]. It should be noted that the canonical Dirac quantization procedure can be applied only to the corresponding dynamical field systems considered with respect to their proper rest reference systems.

Remark 13.3. Some comments are in order concerning the classical relativity principle. We have obtained our results without using the Lorentz transformations - relying only on the natural notion of the rest reference system and its suitable parametrization with respect to any other moving reference systems. It seems reasonable then that the true state changes of a moving charged particle q are exactly realized only with respect to its proper rest reference system. Then the only remaining question would be about the physical justification of the corresponding relationship between time parameters of moving and rest reference systems.

The relationship between reference frames that we have used is expressed as

$$d\tau = dt(1 - u^2)^{1/2}, \tag{13.112}$$

where $u := dr/dt \in \mathbb{E}^3$ is the velocity vector with which the rest reference system $\mathcal{K}_r$ moves with respect to another arbitrarily chosen reference system $\mathcal{K}$. Expression (13.112) implies, in particular, that

$$dt^2 - dr^2 = d\tau^2, \tag{13.113}$$

which is identical to the classical infinitesimal Lorentz invariant. This is not a coincidence, since all our dynamical vacuum field equations were derived in turn [321, 322] from the governing equations of the vacuum potential field function $W : M^4 \rightarrow \mathbb{R}$ in the form

$$\partial^2 W/\partial t^2 - \nabla^2 W = \rho, \ \partial W/\partial t + \nabla(vW) = 0, \ \partial \rho/\partial t + \nabla(v\rho) = 0, \tag{13.114}$$

which is a priori Lorentz invariant. Here $\rho \in \mathbb{R}$ is the charge density and $v := dr/dt$ the associated local velocity of the vacuum field potential evolution. Consequently, the dynamical infinitesimal Lorentz invariant (13.113) reflects this intrinsic structure of equations (13.114). If it is rewritten in the following nonstandard Euclidean form

$$dt^2 = d\tau^2 + dr^2 \tag{13.115}$$

it gives rise to a completely different relationship between the reference systems $\mathcal{K}$ and $\mathcal{K}_r$, namely

$$dt = d\tau(1 + \dot{r}^2)^{1/2}, \tag{13.116}$$

where $\dot{r} := dr/d\tau$ is the related particle velocity with respect to the rest reference system. Thus, we observe that all our Lagrangian analysis above is based on the corresponding functional expressions written in these "Euclidean"space-time coordinates and with respect to which the least action principle was applied. So there are two alternatives - the first is to apply the least action principle to the corresponding Lagrangian functions expressed in the Minkowski space-time variables with respect to an arbitrarily chosen reference system $\mathcal{K}$, and the second is to apply the least action principle to the corresponding Lagrangian functions expressed in Euclidean space-time variables with respect to the rest reference system $\mathcal{K}_r$.

This leads us to a slightly amusing but thought-provoking observation: It follows from our analysis that all of the results of classical special relativity related with the electrodynamics of charged point particles can be obtained (in a one-to-one correspondence) using our new definitions of the dynamical particle mass and the least action principle with respect to the associated Euclidean space-time variables in the rest reference system.

An additional remark concerning the quantization procedure of the proposed electrodynamics models is in order: If the dynamical vacuum field equations are expressed in canonical Hamiltonian form, as we have done here, only straightforward technical details are required to quantize the equations and obtain the corresponding Schrödinger evolution equations in suitable Hilbert spaces of quantum states. There is another striking implication from our approach: the Einsteinian equivalence principle [123, 124, 225, 302, 196] is rendered superfluous for our vacuum field theory of electromagnetism and gravity.

Using the canonical Hamiltonian formalism devised here for the alternative charged point particle electrodynamics models, we found it rather easy to treat the Dirac quantization. The results obtained compared favorably with classical quantization, but it must be admitted that we still have not given a compelling physical motivation for our new models. This is something that needs to be revisited in future investigations. Another important aspect of our vacuum field theory no-geometry (geometry-free) approach to combining the electrodynamics with the gravity, is the manner in which it singles out the decisive role of the rest reference system $\mathcal{K}_r$. More precisely, all of our electrodynamics models allow both the Lagrangian and Hamiltonian formulations with respect to the rest reference system evolution parameter $\tau \in \mathbb{R}$, which are well suited to canonical quantization. The physical nature of this fact is as yet not quite clear. In fact, it appears [168, 196, 225, 234, 236, 302] that there is no physically reasonable explanation of this decisive role of the rest reference system, except for that given by R. Feynman who argued in [123] that the relativistic expression for the classical Lorentz force (13.35) has physical sense only with respect to the rest reference system variables $(\tau, r) \in \mathbb{E}^4$.

13.7 Introduction to further analysis

We shall next delve more deeply into an analysis of the results above in order to better understand the significance of the approach as well as to perceive possible additional applications.

13.7.1 *The classical relativistic electrodynamics backgrounds: A charged point particle analysis*

It is well known [24, 123, 225, 302] that the relativistic least action principle for a point charged particle q in Minkowski space-time $M^4 \simeq \mathbb{E}^3 \times \mathbb{R}$

can be formulated on a time interval $[t_1, t_2] \subset \mathbb{R}$ (in light speed units) as

$$\begin{aligned} \delta S^{(t)} = 0, \quad S^{(t)} &:= \int_{\tau(t_1)}^{\tau(t_2)} (-m_0 d\tau - q < \mathcal{A}, dx >_{M^4}) \\ &= \int_{s(t_1)}^{s(t_2)} (-m_0 < \dot{x}, \dot{x} >_{M^4}^{1/2} - q < \mathcal{A}, \dot{x} >_{M^4}) ds. \end{aligned} \tag{13.117}$$

Here $\delta x(s(t_1)) = 0 = \delta x(s(t_2))$ are the boundary constraints, $m_0 \in \mathbb{R}_+$ is the so-called particle rest mass, the 4-vector $x := (r, t) \in M^4$ is the particle location in M^4, and $\dot{x} := dx/ds \in T(M^4)$ is the particle 4-vector velocity with respect to a laboratory reference system $\mathcal{K}$ parametrized by means of the Minkowski space-time parameters $(r, s(t)) \in M^4$ and related to each other by means of the infinitesimal Lorentz interval relationship

$$d\tau :=< dx, dx >_{M^4}^{1/2} := ds < \dot{x}, \dot{x} >_{M^4}^{1/2} . \tag{13.118}$$

In addition, $\mathcal{A} \in T^*(M^4)$ is an external electromagnetic 4-vector potential satisfying the classical Maxwell equations [126, 225, 302], the symbol $< \cdot, \cdot >_{\mathcal{H}}$ is, in general, the corresponding scalar product in a finite-dimensional vector space $\mathcal{H}$ and $T(M^4), T^*(M^4)$ are, respectively, the tangent and cotangent spaces [3, 14, 112, 173, 387] of M^4. In particular, $< x, x >_{M^4} := t^2 - < r, r >_{\mathbb{E}^3}$ for any $x := (r, t) \in M^4$.

The subintegral expression in (13.117)

$$\mathcal{L}^{(t)} := -m_0 < \dot{x}, \dot{x} >_{M^4}^{1/2} - q < \mathcal{A}, \dot{x} >_{M^4} \tag{13.119}$$

is the related Lagrangian function, whose first part is proportional to the particle world line length with respect to the proper rest reference system $\mathcal{K}_r$ and the second part is proportional to the pure electromagnetic particle-field interaction with respect to the Minkowski laboratory reference system $\mathcal{K}$. Moreover, the positive rest mass parameter $m_0 \in \mathbb{R}_+$ is introduced into (13.119) as an external physical ingredient, also describing the point particle with respect to the proper rest reference system $\mathcal{K}_r$. The electromagnetic 4-vector potential $\mathcal{A} \in T^*(M^4)$ is at the same time expressed as a 4-vector, constructed and measured with respect to $\mathcal{K}$ that from physical point of view is somewhat controversial, since the action functional (13.117) is forced to be extremal with respect to the laboratory reference system $\mathcal{K}$. This, in particular, means that the real physical motion of our charged point particle, realized with respect to the proper rest reference system $\mathcal{K}_r$, somehow depends on external observation data [75, 119, 122, 125, 234, 236, 235, 237] with respect to the occasionally chosen laboratory reference system $\mathcal{K}$. This aspect was never discussed in the physical literature except by Feynman in [126], who argued that the relativistic expression for the classical Lorentz force has a physical sense only with respect to the Euclidean rest reference system $\mathcal{K}_r$ variables $(r, \tau) \in \mathbb{E}^4$ related to the Minkowski laboratory reference system $\mathcal{K}$ parameters $(r, t) \in M^4$ by means of the infinitesimal relationship

$$d\tau := < dx, dx >_{M^4}^{1/2} = dt(1 - u^2)^{1/2}, \tag{13.120}$$

where $u := dr/dt \in T(\mathbb{E}^3)$ is the point particle velocity with respect to $\mathcal{K}$.

It is worth pointing out that to be correct, it would be necessary to include into the action functional the part describing the electromagnetic field itself. But this part is not taken into account, since it is generally assumed [34, 79, 157, 195, 196, 256, 255, 280, 395, 399] that the charged particle influence on the electromagnetic field is negligible. This is true if the particle charge value q is very small but the support $\operatorname{supp} \mathcal{A} \subset M^4$ of the electromagnetic 4-vector potential is compact. It is clear that in the case of two interacting charged particles the above assumption cannot be applied, as it is necessary to take into account the relative motion of two particles and the varying interaction energy. This aspect of the action functional choice for the problem appears to be very important when one tries to analyze the related Lorentz forces exerted by charged particles on each other. We will return to this problem in the next section.

Upon calculating the least action condition (13.117), we easily obtain from (13.119) the classical relativistic dynamical equations

$$\begin{gathered} dP/ds := -\partial \mathcal{L}^{(t)}/\partial x = -q\nabla_x < \mathcal{A}, \dot{x} >_{M^4}, \\ P := -\partial \mathcal{L}^{(t)}/\partial \dot{x} = m_0 \dot{x} < \dot{x}, \dot{x} >_{M^4}^{-1/2} + q\mathcal{A}, \end{gathered} \tag{13.121}$$

where $P \in T^*(M^4)$ denotes the common particle-field momentum of the interacting system.

Now at $s = t \in \mathbb{R}$, using the standard infinitesimal change of variables (13.120) we readily obtain from (13.121) the classical Lorentz force expression

$$dp/dt = qE + qu \times B \tag{13.122}$$

with the relativistic particle momentum and mass

$$p := mu, \quad m := m_0(1 - u^2)^{-1/2}, \tag{13.123}$$

respectively, the electric field

$$E := -\partial A/\partial t - \nabla \varphi \tag{13.124}$$

and the magnetic field

$$B := \nabla \times A, \tag{13.125}$$

where we have expressed the electromagnetic 4-vector potential as $\mathcal{A} := (A, \varphi) \in T^*(M^4)$.

The Lorentz force (13.122), owing to our preceding assumption, is the force exerted by the external electromagnetic field on the charged point particle, whose charge q is so small that it does not influence the field. This fact becomes very important if we try to make use of the Lorentz force expression (13.122) for the case of two interacting charged particles, since then one cannot assume that the charge q exerts a negligible influence on other charged particle. Thus, the corresponding Lorentz force between two charged particles should be suitably altered. Nonetheless, modern physics [22, 57, 25, 58, 35, 74, 100, 98, 185, 186, 225] seems to have largely ignored this needed Lorentz force modification and the classical expression (13.122) seems to be used almost everywhere. This situation was observed and analyzed in [351], where it is shown that the electromagnetic Lorentz force between two moving charged particles can be modified in such a way that it ceases to be dependent on their relative motion, contrary to the classical relativistic case.

Unfortunately, the least action principle approach to analyzing the Lorentz force structure was ignored in [351], which has led to some incorrect and physically unmotivated statements concerning the role of mathematical physics in modern electrodynamics. To make the problem more transparent we analyze it in the next section using the vacuum field theory approach recently devised in [60, 321, 322].

13.7.2 *Least action principle analysis*

Consider the least action principle (13.117) and observe that the extremal condition

$$\delta S^{(t)} = 0, \qquad \delta x(s(t_1)) = 0 = \delta x(s(t_2)), \tag{13.126}$$

is calculated with respect to the laboratory reference system $\mathcal{K}$, whose point particle coordinates $(r, t) \in M^4$ are parametrized by means of an arbitrary parameter $s \in \mathbb{R}$ owing to expression (13.118). Recalling the definition of the invariant proper rest reference system $\mathcal{K}_r$ time parameter (13.120), we find that at the critical parameter value $s = \tau \in \mathbb{R}$ the action functional

(13.117) on the fixed interval $[\tau_1, \tau_2] \subset \mathbb{R}$ turns into

$$S^{(t)} = \int_{\tau_1}^{\tau_2} (-m_0 - q < \mathcal{A}, \dot{x} >_{M^4}) d\tau \tag{13.127}$$

subject to the constraint

$$< \dot{x}, \dot{x} >_{M^4}^{1/2} = 1, \tag{13.128}$$

where, $\dot{x} := dx/d\tau$, $\tau \in \mathbb{R}$.

The expressions (13.127) and (13.128) require comment since the Lagrangian function corresponding to (13.127) is

$$\mathcal{L}^{(t)} := -m_0 - q < \mathcal{A}, \dot{x} >_{M^4} \tag{13.129}$$

and this depends only virtually on the unobservable rest mass parameter $m_0 \in \mathbb{R}$ and, evidently, it has no direct impact on the resulting particle dynamical equations following from the condition $\delta S^{(t)} = 0$. Nonetheless, the rest mass arises as a suitable Lagrangian multiplier owing to the imposed constraint (13.128). To demonstrate this, consider the extended Lagrangian function (13.129)

$$\mathcal{L}_{\lambda}^{(t)} := -m_0 - q < \mathcal{A}, \dot{x} >_{M^4} - \lambda(< \dot{x}, \dot{x} >_{M^4}^{1/2} - 1), \tag{13.130}$$

where $\lambda \in \mathbb{R}$ is a suitable Lagrangian multiplier. The resulting Euler–Lagrange equations are

$$\begin{gathered} P_r := \partial\mathcal{L}_{\lambda}^{(t)}/\partial\dot{r} = qA + \lambda\dot{r}, \quad P_t := \partial\mathcal{L}_{\lambda}^{(t)}/\partial\dot{t} = -q\varphi - \lambda\dot{t}, \\ \partial\mathcal{L}_{\lambda}^{(t)}/\partial\lambda = < \dot{x}, \dot{x} >_{M^4}^{1/2} - 1 = 0, \quad dP_r/d\tau = q\nabla_r < A, \dot{r} >_{\mathbb{E}^3} - q\dot{t}\nabla_r\varphi, \\ dP_t/d\tau = q < \partial A/\partial t, \dot{r} >_{\mathbb{E}^3} - q\dot{t}\partial\varphi/\partial t, \end{gathered} \tag{13.131}$$

and, owing to relationship (13.120), these give rise to the dynamical equations

$$\frac{d}{dt}(\lambda u\dot{t}) = qE + qu \times B, \quad \frac{d}{dt}(\lambda\dot{t}) = q < E, u >_{\mathbb{E}^3}, \tag{13.132}$$

where

$$E := -\partial A/\partial t - \nabla\varphi, \quad B = \nabla \times A \tag{13.133}$$

are the corresponding electric and magnetic fields. As a simple consequence of (13.132), one obtains

$$\frac{d}{dt}\log(\lambda\dot{t}) + \frac{d}{dt}\log(1 - u^2)^{1/2} = 0, \tag{13.134}$$

which, owing to the relationship (13.120), is equivalent for all $t \in \mathbb{R}$, to

$$\lambda \dot{t}(1 - u^2)^{1/2} = \lambda := m_0, \tag{13.135}$$

where $m_0 \in \mathbb{R}_+$ is a constant, which could be interpreted as the rest mass of the charged point particle q. In fact, the first equation of (13.132) can be rewritten as

$$dp/dt = qE + qu \times B, \tag{13.136}$$

where

$$p := mu, \; m := \lambda \dot{t} = m_0(1 - u^2)^{-1/2}, \tag{13.137}$$

and so coincides exactly with that of (13.120).

Thus, we have retrieved all of the results obtained in the preceding section, making use of the action functional (13.127), represented with respect to the rest reference system $\mathcal{K}_r$ under the constraint (13.128). During these derivations, we faced a very delicate inconsistency property of the definition of the action functional $S^{(t)}$: It is defined with respect to the rest reference system $\mathcal{K}_r$, but depends on the external electromagnetic potential function $\mathcal{A} : M^4 \to T^*(M^4)$, constructed in an exceptional manner with respect to the laboratory reference system $\mathcal{K}$. Namely, this potential function, as a physical observable quantity, is defined and, respectively, measurable only with respect to the fixed laboratory reference system $\mathcal{K}$. This, in particular, means that a physically reasonable action functional should be constructed by means of an expression primarily calculated in the laboratory reference system $\mathcal{K}$ by means of coordinates $(r, t) \in M^4$ and later suitably transformed subject to the rest reference system $\mathcal{K}_r$ coordinates $(r, \tau) \in \mathbb{E}^4$, depending on the charged point particle q motion. Thus, the corresponding action functional should from the beginning be written as

$$S^{(\tau)} = \int_{t(\tau_1)}^{t(\tau_2)} (-q < \mathcal{A}, \dot{x} >_{\mathbb{E}^3}) dt, \tag{13.138}$$

where $\dot{x} := dx/dt$, $t \in \mathbb{R}$, calculated on a time interval $[t(\tau_1), t(\tau_2)] \subset \mathbb{R}$, suitably related with the proper motion of the charged point particle q on the true time interval $[\tau_1, \tau_2] \subset \mathbb{R}$ with respect to the rest reference system $\mathcal{K}_r$ and whose charge is assumed so small that it exerts no influence on the external electromagnetic field. The problem now arises: how does one correctly compute the variation $\delta S^{(\tau)} = 0$ of the action functional (13.138)?

For an answer we turn to the Feynman, who argued in [126] that when deriving the relativistic Lorentz force expression, that the real charged particle dynamics can only be unambiguously determined with respect to the rest reference system time parameter. Namely, Feynman wrote: "...we calculate a growth Δx for a small time interval Δt. But in the other reference system the interval Δt may correspond to changing both t' and x', thereby at the change of the only t' the suitable change of x will be other... Making use of the quantity $d\tau$ one can determine a good differential operator $d/d\tau$, as it is invariant with respect to the Lorentz reference systems transformations". This means that if the charged particle q moves in the Minkowski space M^4 during the time interval $[t_1, t_2] \subset \mathbb{R}$ with respect to the laboratory reference system $\mathcal{K}$, its proper real and invariant duration of motion with respect to the rest reference system $\mathcal{K}_r$ will be $[\tau_1, \tau_2] \subset \mathbb{R}$.

As a corollary of Feynman's reasoning, we conclude that it is necessary to rewrite the action functional (13.138) as

$$S^{(\tau)} = \int_{\tau_1}^{\tau_2} (-q < \mathcal{A}, \dot{x} >_{M^4}) d\tau, \quad \delta x(\tau_1) = 0 = \delta x(\tau_2), \tag{13.139}$$

where $\dot{x} := dx/d\tau$, $\tau \in \mathbb{R}$, subject to the constraint

$$< \dot{x}, \dot{x} >_{M^4}^{1/2} = 1, \tag{13.140}$$

as it is equivalent to the infinitesimal transformation (13.120). Simultaneously, the proper time interval $[\tau_1, \tau_2] \subset \mathbb{R}$ is mapped on the time interval $[t_1, t_2] \subset \mathbb{R}$ by means of the infinitesimal transformation

$$dt = d\tau (1 + \dot{r}^2)^{1/2}, \tag{13.141}$$

where $\dot{r} := dr/d\tau$, $\tau \in \mathbb{R}$. Thus, we can now postulate what appears to be the true least action problem equivalent to (13.139) as

$$\delta S^{(\tau)} = 0, \qquad \delta r(\tau_1) = 0 = \delta r(\tau_2), \tag{13.142}$$

where the functional

$$S^{(\tau)} = \int_{\tau_1}^{\tau_2} [-\bar{W}(1 + \dot{r}^2)^{1/2} + q < A, \dot{r} >_{\mathbb{E}^3}] d\tau \tag{13.143}$$

is characterized by the Lagrangian function

$$\mathcal{L}^{(\tau)} := -\bar{W}(1 + \dot{r}^2)^{1/2} + q < A, \dot{r} >_{\mathbb{E}^3} . \tag{13.144}$$

Here we have denoted, for convenience, $\bar{W} := q\varphi$, as a suitable vacuum field [321–323, 351] potential function. The resulting Euler-Lagrange equation yields the relationships

$$P := \partial\mathcal{L}^{(\tau)}/\partial\dot{r} = -\bar{W}\dot{r}(1+\dot{r}^2)^{-1/2} + qA, \tag{13.145}$$
$$dP/d\tau := \partial\mathcal{L}^{(\tau)}/\partial r = -\nabla\bar{W}(1+\dot{r}^2)^{1/2} + q\nabla < A, \dot{r} >_{\mathbb{E}^3} .$$

Once again using the infinitesimal transformation (13.141) and the crucial dynamical particle mass definition [321, 323, 351] (in the light speed units)

$$m := -\bar{W}, \tag{13.146}$$

we can easily rewrite equations (13.145) with respect to the parameter $t \in \mathbb{R}$ as the classical relativistic Lorentz force

$$dp/dt = qE + qu \times B, \tag{13.147}$$

where

$$p := mu, \qquad u := dr/dt, \tag{13.148}$$
$$B := \nabla \times A, \quad E := -q^{-1}\nabla\bar{W} - \partial A/\partial t.$$

Thus, once again we have obtained the relativistic Lorentz force expression (13.147), but in a form slightly different from (13.122), since the classical relativistic momentum expression of (13.123) does not completely coincide with our modified relativistic momentum expression

$$p = -\bar{W}u, \tag{13.149}$$

which depends strongly on the scalar vacuum field potential function $\bar{W} : M^4 \to \mathbb{R}$. However, recalling here that our action functional (13.139) was derived under the assumption that the particle charge q is negligible and does not exert an essential influence on the electromagnetic field source, we can make use of a prior result [60, 321, 351] that the vacuum field potential function $\bar{W} : M^4 \to \mathbb{R}$, owing to (13.147)-(13.149), satisfies as $q \to 0$ the dynamical equation

$$d(-\bar{W}u)/dt \simeq -\nabla\bar{W}, \tag{13.150}$$

with solution

$$-\bar{W} = m_0(1-u^2)^{-1/2}, \; m_0 = -\left.\bar{W}\right|_{u=0}. \tag{13.151}$$

Consequently, owing to (13.151) and (13.149), we see that our result (13.147) for the relativistic Lorentz force coincides almost completely with that of (13.122) as $q \to 0$.

The results obtained so far prove the following statement.

Proposition 13.9. *Under the assumption of negligible influence of a charged point particle q on an external electromagnetic field source, a physically reasonable action functional is given by expression (13.138), which is equivalently defined with respect to the rest reference system $\mathcal{K}_r$ in the form (13.139),(13.140). The resulting relativistic Lorentz force (13.147) coincides almost exactly with that of (13.122) obtained from the classical Einstein type action functional (13.117), but the momentum expression (13.149) differs from the classical expression (13.123), as it takes into account the related vacuum field potential interaction energy impact.*

An obvious but important consequence of the above is the following.

Corollary 13.1. *The Lorentz force expression (13.147) should be corrected when the weak charge q influence assumption made above does not hold.*

Remark 13.4. Concerning the infinitesimal relationship (13.141), one observes that it reflects the Euclidean nature of transformations $\mathbb{R} \ni t \rightleftharpoons \tau \in \mathbb{R}$.

In spite of the results obtained above by means of two different least action principles (13.117) and (13.139), we must admit that the first is based to an extent on what might be considered controversial logic, which may give rise to unpredictable, unexplainable and even nonphysical effects. Amongst these controversies we mention: i) the definition of the Lagrangian function (13.119) as an expression depending on the external and undefined rest mass parameter with respect to the rest reference system $\mathcal{K}_r$ time $\tau \in \mathbb{R}$, but serving as a variational integrand with respect to the laboratory reference system $\mathcal{K}$ time $t \in \mathbb{R}$; ii) the least action condition (13.117) is calculated with respect to fixed boundary conditions at the ends of a time interval $[t_1, t_2] \subset \mathbb{R}$, so the resulting dynamics is strongly dependent on the chosen laboratory reference system $\mathcal{K}$ which, according to Feynman [126, 125], is physically unreasonable; iii) the resulting relativistic particle mass and its energy depend only on the particle velocity in the laboratory reference system $\mathcal{K}$, and do not take into account the vacuum field potential energy, which exerts a significant action on the particle motion; and iv) the assumption concerning the negligible influence of a charged point particle on the external electromagnetic field source is also physically inconsistent.

13.8 The charged point particle least action principle revisited: Vacuum field theory approach

13.8.1 *A free charged point particle in a vacuum*

Here we start from the following action functional for a charged point particle q moving with velocity $u := dr/dt \in \mathbb{E}^3$ with respect to a laboratory reference system $\mathcal{K}$:

$$S^{(\tau)} := -\int_{t(\tau_2)}^{t(\tau_1)} \bar{W} dt, \tag{13.152}$$

defined on the time interval $[t(\tau_1), t(\tau_2)] \subset \mathbb{R}$ by means of a vacuum field potential function $\bar{W} : M^4 \to \mathbb{R}$ characterizing the intrinsic properties of the vacuum medium and its interaction with a charged point particle q, subject to the constraint

$$< \dot{\xi}, \dot{\xi} >_{\mathbb{E}^4}^{1/2} = 1, \tag{13.153}$$

where $\xi := (r, \tau) \in \mathbb{E}^4$ is a charged point particle position 4-vector with respect to the proper rest reference system $\mathcal{K}_r$, $\dot{\xi} := d\xi/dt$, $t \in \mathbb{R}$. As the dynamics of the charged point particle q depends strongly only on the time interval $[\tau_1, \tau_2] \subset \mathbb{R}$ of its own motion in the rest reference system $\mathcal{K}_r$, we need to calculate the extremal condition

$$\delta S^{(\tau)} = 0, \qquad \delta r(\tau_1) = 0 = \delta r(\tau_2). \tag{13.154}$$

It follows from (13.153) or (13.141) that the action functional (13.152) is equivalent to

$$S^{(\tau)} := -\int_{\tau_2}^{\tau_1} \bar{W}(1 + \dot{r}^2)^{1/2} d\tau, \tag{13.155}$$

where $\dot{r} := dr/d\tau$, $\dot{r}^2 :=< \dot{r}, \dot{r} >_{\mathbb{E}^3}$, $\tau \in \mathbb{R}$. Hence, from (13.155) and (13.154) one readily concludes that

$$p := -\bar{W}\dot{r}(1 + \dot{r}^2)^{-1/2},\ dp/d\tau = -\nabla\bar{W}(1 + \dot{r}^2)^{1/2}. \tag{13.156}$$

Taking into account once more relationship (13.141), we can rewrite (13.156) as

$$dp/dt = -\nabla\bar{W}, \qquad p := -\bar{W}u. \tag{13.157}$$

Recalling the dynamic mass definition (13.146), it is clear that equation (13.157) turns into the Newtonian dynamical expression

$$dp/dt = -\nabla\bar{W},\ p = mu. \tag{13.158}$$

Since equation (13.158) is completely equivalent to equation (13.150), we obtain right away from (13.151) that the particle mass is

$$m = m_0(1-u^2)^{-1/2}, \tag{13.159}$$

where

$$m_0 := -\left.\bar{W}\right|_{u=0} \tag{13.160}$$

is the so-called particle rest mass. Moreover, since the Lagrangian function corresponding to (13.155) and given as

$$\mathcal{L}^{(\tau)} := -\bar{W}(1+\dot{r}^2)^{1/2} \tag{13.161}$$

is not degenerate, we can easily construct [3, 14, 112, 173, 323] the related conservative Hamiltonian function

$$\mathcal{H}^{(\tau)} = -(\bar{W}^2 - p^2)^{1/2}, \tag{13.162}$$

where $p^2 :=< p, p >_{\mathbb{E}^3}$, satisfying the canonical Hamiltonian equations

$$dr/d\tau = \partial\mathcal{H}^{(\tau)}/\partial p, \;\; dp/d\tau = -\partial\mathcal{H}^{(\tau)}/\partial r \tag{13.163}$$

and conservation conditions

$$d\mathcal{H}^{(\tau)}/dt = 0 = d\mathcal{H}^{(\tau)}/d\tau \tag{13.164}$$

for all $\tau, t \in \mathbb{R}$. Whence, the quantity

$$\mathcal{E} := (\bar{W}^2 - p^2)^{1/2} \tag{13.165}$$

can be naturally interpreted as the total energy of the particle.

It is important to note here that energy expression (13.165) takes into account both kinetic and potential energies, but the particle dynamic mass (13.159) depends only on its velocity, reflecting its free motion in a vacuum. Moreover, since the vacuum potential function $\bar{W} : M^4 \to \mathbb{R}$ is not, in general, constant, the motion of the particle q with respect to the laboratory reference system $\mathcal{K}$ is not typically linear with constant velocity. This situation was discussed by Feynman in [125]. Thus, we obtained the classical relativistic mass dependence on the freely moving particle velocity (13.159), taking into account both the nonconstant vacuum potential function $\bar{W} : M^4 \to \mathbb{R}$ and the particle velocity $u \in \mathbb{E}^3$.

We also mention that the vacuum potential function $\bar{W} : M^4 \to \mathbb{R}$ itself should be simultaneously found via a suitable solution to the Maxwell equation $\partial^2 W/\partial t^2 - \Delta W = \rho$, where $\rho \in \mathbb{R}$ is an ambient charge density and $\bar{W}(r(t)) := \lim_{r\to r(t)} W(r,t)|$, with $r(t) \in \mathbb{E}^3$ the position of the charged point particle at time $t \in \mathbb{R}$. A more detail description [321] of the vacuum field potential $W : M^4 \to \mathbb{R}$, characterizing the vacuum medium structure is provided in the next section.

Now we rewrite expression (13.152) in the invariant form

$$S^{(\tau)} = -\int_{s(\tau_1)}^{s(\tau_2)} \bar{W} < \dot{\xi}, \dot{\xi} >_{\mathbb{E}^4}^{1/2} ds, \tag{13.166}$$

where $s \in \mathbb{R}$ parametrizes the particle world line related with the laboratory reference system $\mathcal{K}$ time parameter $t \in \mathbb{R}$ by means of the Euclidean infinitesimal relationship

$$dt := < \dot{\xi}, \dot{\xi} >_{\mathbb{E}^4}^{1/2} ds. \tag{13.167}$$

Clearly, at $s = t \in \mathbb{R}$ functional (13.166) turns into (13.152) and (13.153). The action functional (13.166) is to be supplemented with the boundary conditions

$$\delta\xi(s(\tau_1)) = 0 = \delta\xi(s(\tau_2)), \tag{13.168}$$

which are, obviously, completely equivalent to those of (13.154), since the map $\mathbb{R} \ni s \rightleftharpoons t \in \mathbb{R}$ is injective by virtue of its definition (13.167).

Calculating the least action condition $\delta S^{(\tau)} = 0$ subject to the constraints (13.168), one easily obtains the same equation (13.157) and relationships (13.159), (13.165) for the particle dynamical mass and its conserved energy.

13.8.2 *Charged point particle electrodynamics*

Next we generalize the results obtained above for a free point particle in the vacuum medium on the case of a charged point particle q interacting with external charged point particle q_f, both moving with respect to a laboratory reference system $\mathcal{K}$. In the vacuum field theory approach devised in [60, 321, 322], it is natural to reduce the problem to that considered above by introducing the reference system $\mathcal{K}_f$ moving with respect to the reference system $\mathcal{K}$ with the same velocity as that of the external charged point particle q_f. Thus, if considered with respect to the laboratory reference $\mathcal{K}_f$, the external charged particle q_f will be at rest, but the test charged point particle q will be moving with the velocity $u - u_f \in T(\mathbb{E}^3)$, where $u := dr/dt$, $u_f := dr_f/dt$, $t \in \mathbb{R}$, are the corresponding velocities of these charged point particles q and q_f with respect to $\mathcal{K}$. Accordingly, we have the action functional expression

$$S^{(\tau)} = -\int_{s(\tau_1)}^{s(\tau_2)} \bar{W} < \dot{\eta}_f, \dot{\eta}_f >_{\mathbb{E}^4}^{1/2} ds, \tag{13.169}$$

where $\eta_f := (\tau, r - r_f) \in \mathbb{E}^4$ represents the charged point particle q position coordinates with respect to the rest reference system $\mathcal{K}_r$, calculated subject to the introduced laboratory reference system $\mathcal{K}_f$, where $s \in \mathbb{R}$ parametrizes the corresponding point particle world line, which is infinitesimally related to the time parameter $t \in \mathbb{R}$ according to

$$dt :=< \dot{\eta}_f, \dot{\eta}_f >_{\mathbb{E}^4}^{1/2} ds. \tag{13.170}$$

The boundary conditions for the functional (13.169) assume the natural form

$$\delta\xi(s(\tau_1)) = 0 = \delta\xi(s(\tau_2)), \tag{13.171}$$

where $\xi = (\tau, r) \in \mathbb{E}^4$. The least action condition $\delta S^{(\tau)} = 0$ together with (13.171) gives rise to the equations

$$\begin{aligned} P &:= \partial\mathcal{L}^{(\tau)}/\partial\dot{\xi} = -\bar{W}\dot{\eta}_f < \dot{\eta}_f, \dot{\eta}_f >_{\mathbb{E}^4}^{-1/2}, \\ dP/ds &:= \partial\mathcal{L}^{(\tau)}/\partial\xi = -\nabla_\xi\bar{W} < \dot{\eta}_f, \dot{\eta}_f >_{\mathbb{E}^4}^{1/2}, \end{aligned} \tag{13.172}$$

where the Lagrangian function is

$$\mathcal{L}^{(\tau)} := -\bar{W} < \dot{\eta}_f, \dot{\eta}_f >_{\mathbb{E}^4}^{1/2}. \tag{13.173}$$

Defining the charged point particle q momentum $p \in T^*(\mathbb{E}^3)$ as

$$p := -\bar{W}\dot{r} < \dot{\eta}_f, \dot{\eta}_f >_{\mathbb{E}^4}^{-1/2} = -\bar{W}u \tag{13.174}$$

and the induced external magnetic vector potential $A \in T^*(\mathbb{E}^3)$ as

$$qA := \bar{W}\dot{r}_f < \dot{\eta}_f, \dot{\eta}_f >_{\mathbb{E}^4}^{-1/2} = \bar{W}u_f, \tag{13.175}$$

we obtain, owing to relationship (13.170), the relativistic Lorentz type force expression

$$dp/dt = qE + qu \times B - q\nabla < u, A >_{\mathbb{E}^3}, \tag{13.176}$$

where

$$E := -q^{-1}\nabla\bar{W} - \partial A/\partial t, \qquad B = \nabla \times A, \tag{13.177}$$

are, respectively, the external electric and magnetic fields acting on the charged point particle q.

The result (13.176) includes the additional Lorentz force component

$$F_c := -q\nabla < u, A >_{\mathbb{E}^3}, \tag{13.178}$$

not present in the classical relativistic Lorentz force expressions (13.122) and (13.147). Moreover, from (13.174) one finds that the point particle q momentum is

$$p = -\bar{W}u := mu, \tag{13.179}$$

where the particle mass

$$m := -\bar{W} \tag{13.180}$$

already does not coincide with the corresponding classical relativistic relationship of (13.123).

Now consider the least action condition for functional (13.169) at the critical parameter value $s = \tau \in \mathbb{R}$:

$$\delta S^{(\tau)} = 0, \qquad \delta r(\tau_1) = 0 = \delta r(\tau_2), \tag{13.181}$$
$$S^{(\tau)} := -\int_{\tau_1}^{\tau_2} \bar{W}(1 + |\dot{r} - \dot{r}_f|^2_{\mathbb{E}^3})^{1/2} d\tau.$$

The resulting Lagrangian function

$$\mathcal{L}^{(\tau)} := -\bar{W}(1 + |\dot{r} - \dot{r}_f|^2_{\mathbb{E}^3})^{1/2} \tag{13.182}$$

gives rise to the generalized momentum expression

$$P := \partial\mathcal{L}^{(\tau)}/\partial\dot{r} = -\bar{W}(\dot{r} - \dot{r}_f)(1 + |\dot{r} - \dot{r}_f|^2_{\mathbb{E}^3})^{-1/2} := p + qA, \tag{13.183}$$

which makes it possible to construct [3, 14, 112, 173, 326] the corresponding Hamiltonian function as

$$\begin{aligned} \mathcal{H} :=&< P, \dot{r} >_{\mathbb{E}^3} - \mathcal{L}^{(\tau)} = -(\bar{W}^2 - |p + qA|^2_{\mathbb{E}^3})^{1/2} \\ &- < p + qA, qA >_{\mathbb{E}^3} (\bar{W}^2 - |p + qA|^2_{\mathbb{E}^3})^{-1/2}, \end{aligned} \tag{13.184}$$

satisfying the canonical Hamiltonian equations

$$dP/d\tau := \partial\mathcal{H}/\partial r, \quad dr/d\tau := -\partial\mathcal{H}/\partial r, \tag{13.185}$$

evolving with respect to the proper rest reference system time parameter $\tau \in \mathbb{R}$. In deriving (13.184) we made use of the relationship (13.170) at $s = \tau \in \mathbb{R}$ together with definitions (13.174) and (13.175). Since the Hamiltonian function (13.184) is conserved with respect to the evolution parameter $\tau \in \mathbb{R}$, owing to relationship (13.170) at $s = \tau \in \mathbb{R}$, one computes that

$$d\mathcal{H}/d\tau = 0 = d\mathcal{H}/dt \tag{13.186}$$

for all $t, \tau \in \mathbb{R}$. These results can be formulated as follows.

Proposition 13.10. *The charged point particle electrodynamics determined by the least action principle (13.169) and (13.171), reduces to the modified Lorentz force equation (13.176), and is equivalent to the canonical Hamilton system (13.185) with respect to the proper rest reference system time parameter* $\tau \in \mathbb{R}$. *The corresponding Hamiltonian function (13.184) is a conservation law for the Lorentz dynamics (13.176) satisfying the conditions (13.186) with respect to both reference systems parameters* $t, \tau \in \mathbb{R}$.

As a corollary, the corresponding energy expression for electrodynamical model (13.176) can be defined as

$$\mathcal{E} := (\bar{W}^2 - |p+qA|^2_{\mathbb{E}^3})^{1/2} + < p+qA, qA >_{\mathbb{E}^3} (\bar{W}^2 - |p+qA|^2_{\mathbb{E}^3})^{-1/2}. \tag{13.187}$$

This formula is a necessary ingredient for quantizing the relativistic electrodynamics (13.176) of the charged point particle q under the influence of the external electromagnetic field.

13.9 A new hadronic string model: Least action principle and relativistic electrodynamics analysis in the vacuum field theory approach

13.9.1 *A new hadronic string model: Least action formulation*

A classical relativistic hadronic string model was first proposed in [23, 154, 275] and studied in [24] using the principle of least action and the related Lagrangian and Hamiltonian formalisms. We shall not discuss this classical string model and not comment on the physical problems accompanying it, especially those related to its diverse quantization versions. Instead, we formulate a new relativistic hadronic string model, constructed by means of the vacuum field theory approach, as devised in [60, 321, 322]. The corresponding least action principle is [24] expressed as

$$\delta S^{(\tau)} = 0, \quad S^{(\tau)} := \int_{s(\tau_1)}^{s(\tau_2)} ds \int_{\sigma_1(s)}^{\sigma_2(s)} \bar{W}(x(\xi))(\dot{\xi}^2\xi'^2 - < \dot{\xi}, \xi' >^2_{\mathbb{E}^4})^{1/2} d\sigma \wedge ds, \tag{13.188}$$

where $\bar{W} : M^4 \rightarrow \mathbb{R}$ is a vacuum field potential function characterizing the interaction of the vacuum medium with the string. The differential 2-form $d\Sigma^{(2)} := (\dot{\xi}^2\xi'^2 - < \dot{\xi}, \xi' >^2_{\mathbb{E}^4})^{1/2} d\sigma \wedge ds = \sqrt[2]{g(\xi)} d\sigma \wedge ds$, $g(\xi) := \det(g_{ij}(\xi)|_{i,j=\overline{1,2}})$, $\dot{\xi}^2 := < \dot{\xi}, \dot{\xi} >_{\mathbb{E}^4}$, $\xi^{\prime,2} := < \xi', \xi' >_{\mathbb{E}^4}$ is related to the Euclidean infinitesimal metrics $dz^2 :=< d\xi, d\xi >_{\mathbb{E}^4} = g_{11}(\xi)d\sigma^2 + g_{12}(\xi)d\sigma ds + g_{21}(\xi)ds d\sigma + g_{22}(\xi)ds^2$ on the string world surface. Thus, in view of [3, 24, 112, 387] the infinitesimal two-dimensional world surface element is parametrized by variables $(\sigma, s) \in \mathbb{E}^2$ and embedded in the 4-dimensional Euclidean space-time with coordinates $\xi := (r, \tau(\sigma, s)) \in \mathbb{E}^4$ subject to the proper rest reference system $\mathcal{K}$, and $\dot{\xi} := \partial\xi/\partial s$, $\xi' := \partial\xi/\partial\sigma$ are the corresponding partial derivatives. The related boundary conditions are chosen as

$$\delta\xi(\sigma(s), s) = 0 \tag{13.189}$$

at string parameter $\sigma(s) \in \mathbb{R}$ for all $s \in \mathbb{R}$. The action functional expression is strongly motivated by that constructed for the point particle action functional (13.152):

$$S^{(\tau)} := -\int_{\sigma_1}^{\sigma_2} dl(\sigma) \int_{t(\sigma,\tau_1)}^{t(\sigma,\tau_2)} \bar{W} dt(\tau,\sigma), \tag{13.190}$$

where the laboratory reference time parameter $t(\tau,\sigma) \in \mathbb{R}$ is related to the proper rest string reference system time parameter $\tau \in \mathbb{R}$ by means of the standard Euclidean infinitesimal relationship

$$dt(\tau,\sigma) := (1 + \dot{r}_\perp^2(\tau,\sigma))^{1/2} d\tau, \qquad \dot{r}_\perp^2 :=< \dot{r}_\perp, \dot{r}_\perp >_{\mathbb{E}^3}, \tag{13.191}$$

with $\sigma \in [\sigma_1, \sigma_2] \subset \mathbb{R}$ a spatial variable parametrizing the string length measure $dl(\sigma)$ on the real axis $\mathbb{R}$, $\dot{r}_\perp(\tau,\sigma) := \hat{N}$, and $\dot{r}(\tau,\sigma) \in \mathbb{E}^3$ are components orthogonal to the string velocity. In addition,

$$\hat{N} := (1 - r^{\prime,-2} r' \otimes r'), \quad r^{\prime,-2} :=< r', r' >_{\mathbb{E}^3}^{-1}, \tag{13.192}$$

is the corresponding projector operator in $\mathbb{E}^3$ orthogonal to the string direction, expressed for brevity by means of the standard tensor product "$\otimes$"in the Euclidean space $\mathbb{E}^3$. A simple calculation of (13.190) yields

$$S^{(\tau)} = -\int_{\tau_1}^{\tau_2} d\tau \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} \bar{W}[(r^{\prime,2}(1+\dot{r}^2) - < \dot{r}, r' >_{\mathbb{E}^3}]^{1/2} d\,\sigma, \tag{13.193}$$

where we made use of the infinitesimal measure representation $dl(\sigma) =< r', r' >_{\mathbb{E}^3}^{1/2} d\sigma$, $\sigma \in [\sigma_1, \sigma_2]$. If we introduce on the string world surface local coordinates $(\sigma, s(\tau,\sigma)) \in \mathbb{E}^2$ and the related Euclidean string position vector $\xi := (r(\sigma,s), \tau) \in \mathbb{E}^4$, the string action functional reduces equivalently to that of (13.188).

Next we proceed to a Lagrangian and Hamiltonian analysis of the least action conditions for expressions (13.188) and (13.193).

13.9.2 *Lagrangian and Hamiltonian analysis*

First we will obtain the Euler–Lagrange equations corresponding to (13.188) with respect to the special [24, 112] internal conformal variables $(\sigma, s) \in \mathbb{E}^2$ on the world string surface, for which the metrics on it become equal to $dz^2 = \xi^{\prime,2} d\sigma^2 + \dot{\xi}^2 ds^2$, where $< \xi', \dot{\xi} >_{\mathbb{E}^4} = 0 = \xi^{\prime,2} - \dot{\xi}^2$, and the corresponding infinitesimal world surface measure $d\Sigma^{(2)}$ becomes $d\Sigma^{(2)} =$

$(\xi^{\prime,2}\dot{\xi}^2)^{1/2}d\sigma \wedge ds$. As a result of simple calculations, one obtains the linear second order partial differential equation

$$\partial(\bar{W}\dot{\xi})/\partial s + \partial(\bar{W}\xi')/\partial\sigma = (\xi^{\prime,2}\dot{\xi}^2)^{1/2}\partial\bar{W}/\partial\sigma \tag{13.194}$$

subject to the suitably chosen boundary conditions

$$\xi' - \dot{\xi}\ \dot{\sigma} = 0 \tag{13.195}$$

for all $s \in \mathbb{R}$. Note that the equation (13.194) is elliptic in contradistinction to the case considered in [24]. Evidently, this results from the fact that the metric on the string world surface is Euclidean. This follows because we took into account that the string motion is realized with respect to its proper rest reference system $\mathcal{K}_r$, which is not dependent on string motion observation data measured with respect to any external laboratory reference system $\mathcal{K}$.

The differential equation (13.194) depends on the vacuum field potential function $\bar{W} : M^4 \to \mathbb{R}$, which, as a function of the Minkowski 4-vector variable $x := (r, t(\sigma, s)) \in M^4$ of the laboratory reference system $\mathcal{K}$, should be expressed using the infinitesimal relationship (13.191) as

$$dt =< \hat{N}\partial\xi/\partial\tau, \hat{N}\partial\xi/\partial\tau >^{1/2} (\frac{\partial\tau}{\partial s}ds + \frac{\partial\tau}{\partial\sigma}d\sigma), \tag{13.196}$$

defined on the string world surface. The function $\bar{W} : M^4 \to \mathbb{R}$ should be simultaneously determined as a suitable solution to the Maxwell equation $\partial^2 W/\partial t^2 - \Delta W = \rho$, where $\rho \in \mathbb{R}$ is an ambient charge density and by definition, $\bar{W}(r(t)) := \lim_{r\to r(t)} W(r,t)|$, with $r(t) \in \mathbb{E}^3$ the position of the string element with coordinates $(\sigma, \tau) \in \mathbb{E}^2$ at time $t = t(\sigma, \tau) \in \mathbb{R}$.

We now construct the dynamical Euler equations for our string model, making use of the action functional (13.193). It is easy to calculate that the generalized momentum

$$\begin{aligned} p := \partial\mathcal{L}^{(\tau)}/\partial\dot{r} &= \frac{-\bar{W}(r^{\prime 2}\dot{r} - r' < r', \dot{r} >_{\mathbb{E}^3})}{[r^{\prime,2}(\dot{r}^2+1)- < r', \dot{r} >^2_{\mathbb{E}^3}]^{1/2}} \\ &= \frac{-\bar{W}(r^{\prime 2}\hat{N}\dot{r})}{[r^{\prime,2}(\dot{r}^2+1)- < r', \dot{r} >^2_{\mathbb{E}^3}]^{1/2}} \end{aligned} \tag{13.197}$$

satisfies the dynamical equation

$$\begin{aligned} dp/d\tau := \delta\mathcal{L}^{(\tau)}/\delta r = [r^{\prime,2}(\dot{r}^2+1)- < r', \dot{r} >^2_{\mathbb{E}^3}]^{1/2}\nabla\bar{W} \\ - \frac{\partial}{\partial\sigma}\{\bar{W}[(1+\dot{r}^2\hat{T})r']^{-1/2}\}, \end{aligned} \tag{13.198}$$

where

$$\mathcal{L}^{(\tau)} := -\bar{W}[(r^{\prime,2}(1+\dot{r}^2)- < \dot{r}, r' >_{\mathbb{E}^3}]^{1/2} \tag{13.199}$$

is the corresponding Lagrangian and

$$\hat{T} := 1 - \dot{r}^{-2}\ \dot{r} \otimes \dot{r}, \qquad \dot{r}^{-2} :=< \dot{r}, \dot{r} >_{\mathbb{E}^3}^{-2}, \tag{13.200}$$

represent the related dynamic projector operator in $\mathbb{E}^3$. The Lagrangian function is degenerate [24, 112], and it satisfies the obvious identity

$$< p, r' >_{\mathbb{E}^3} = 0 \tag{13.201}$$

for all $\tau \in \mathbb{R}$. For the Hamiltonian formulation of the dynamics (13.198), we construct the corresponding Hamiltonian functional as

$$\begin{aligned} \mathcal{H} &:= \int_{\sigma_1}^{\sigma_2} (< p, \dot{r} >_{\mathbb{E}^3} - \mathcal{L}^{(\tau)}) d\sigma \\ &= \int_{\sigma_1}^{\sigma_2} \{\bar{W} r'^{,2} [r'^{,2}(\dot{r}^2 + 1) - < r', \dot{r} >_{\mathbb{E}^3}^2]^{-1/2} d\sigma \\ &= \int_{\sigma_1}^{\sigma_2} [(\bar{W} r')^2 - p^2]^{1/2} d\sigma, \end{aligned} \tag{13.202}$$

which satisfy the canonical equations

$$dr/d\tau := \delta\mathcal{H}/\delta p, \quad dp/d\tau := -\delta\mathcal{H}/\delta r, \tag{13.203}$$

where

$$d\mathcal{H}/d\tau = 0, \tag{13.204}$$

holds only with respect to the proper rest reference system $\mathcal{K}_r$ time parameter $\tau \in \mathbb{R}$. The identity (13.201) can be used to show that the Hamiltonian functional (13.202) is equivalently represented in symbolic form as

$$\mathcal{H} = \int_{\sigma_1}^{\sigma_2} |\bar{W} r' \pm ip|_{\mathbb{E}^3} d\sigma, \tag{13.205}$$

where $i := \sqrt{-1}$. It should be noted here that one cannot construct a suitable Hamiltonian function expression and relationship of type (13.204) with respect to the laboratory reference system $\mathcal{K}$, since the expression (13.205) is not completely defined for a separate laboratory time parameter $t \in \mathbb{R}$ that is locally dependent both on the spatial parameter $\sigma \in \mathbb{R}$ and the proper rest reference system time parameter $\tau \in \mathbb{R}$.

Accordingly we have the following result.

Proposition 13.11. *The hadronic string model (13.188) allows on the related world surface the conformal local coordinates, with respect to which the resulting dynamics is described by means of the linear second order elliptic equation (13.194). Subject to the proper rest reference system Euclidean coordinates, the corresponding dynamics is equivalent to the canonical Hamiltonian equations (13.203) with Hamiltonian functional (13.202).*

Let us now construct the action functional expression for a charged string under an external magnetic field generated by a point velocity charged particle q_f, moving with velocity $u_f := dr_f/dt \in \mathbb{E}^3$ with respect to a laboratory reference system $\mathcal{K}$. To solve this problem, we make use of the trick above by passing to the proper rest reference system $\mathcal{K}_r$ with respect to the relative reference system $\mathcal{K}_f$, moving with velocity $u_f \in \mathbb{E}^3$. As a result of this reasoning, we can write down the action functional as

$$S^{(\tau)} = -\int_{\tau_1}^{\tau_2} d\tau \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} \bar{W}[r^{\prime,2}(1+(\dot{r}-\dot{r}_f)^2) - <\dot{r}-\dot{r}_f, r'>^2_{\mathbb{E}^3}]^{1/2} d\ \sigma, \tag{13.206}$$

which gives rise to the dynamical equation

$$\begin{aligned} dP/d\tau := \delta\mathcal{L}^{(\tau)}/\delta r = -[r^{\prime,2}(1+(\dot{r}-\dot{r}_f)^2) - <\dot{r}-\dot{r}_f, r'>^2_{\mathbb{E}^3}]^{1/2}\nabla\bar{W} \\ + \frac{\partial}{\partial\sigma}\left\{\frac{\bar{W}[1+(\dot{r}-\dot{r}_f)^2\hat{T}_f]r'}{[r^{\prime,2}(1+(\dot{r}-\dot{r}_f)^2) - <\dot{r}-\dot{r}_f, r'>^2_{\mathbb{E}^3}]^{1/2}}\right\}, \end{aligned} \tag{13.207}$$

where the generalized momentum

$$P := \frac{-\bar{W}[r'^2\hat{N}(\dot{r}-\dot{r}_f)]}{[r^{\prime,2}(1+(\dot{r}-\dot{r}_f)^2) - <\dot{r}-\dot{r}_f, r'>^2_{\mathbb{E}^3}]^{1/2}} \tag{13.208}$$

and the projection operator in $\mathbb{E}^3$ is

$$\hat{T}_f := 1 - (\dot{r}-\dot{r}_f)^{-2}\ (\dot{r}-\dot{r}_f)\otimes(\dot{r}-\dot{r}_f). \tag{13.209}$$

If we define

$$p := \frac{-\bar{W}(r'^2\hat{N}\dot{r})}{[r^{\prime,2}(1+(\dot{r}-\dot{r}_f)^2) - <\dot{r}-\dot{r}_f, r'>^2_{\mathbb{E}^3}]^{1/2}} \tag{13.210}$$

to be the local string momentum and

$$qA := \frac{\bar{W}(r'^2\hat{N}\dot{r}_f)}{[r^{\prime,2}(1+(\dot{r}-\dot{r}_f)^2) - <\dot{r}-\dot{r}_f, r'>^2_{\mathbb{E}^3}]^{1/2}} \tag{13.211}$$

to be the external vector magnetic potential, equation (13.207) reduces to

$$\begin{gathered} dp/d\tau = q\dot{r}\times B - q\nabla <A,\dot{r}>_{\mathbb{E}^3} - q\frac{\partial A}{\partial\tau} \\ -[r^{\prime,2}(1+(\dot{r}-\dot{r}_f)^2) - <\dot{r}-\dot{r}_f, r'>^2_{\mathbb{E}^3}]^{1/2}\nabla\bar{W} \\ + \frac{\partial}{\partial\sigma}\left\{\frac{\bar{W}[1+(\dot{r}-\dot{r}_f)^2\hat{T}_f]r'}{[r^{\prime,2}(1+(\dot{r}-\dot{r}_f)^2) - <\dot{r}-\dot{r}_f, r'>^2_{\mathbb{E}^3}]^{1/2}}\right\}, \end{gathered} \tag{13.212}$$

where $q \in \mathbb{R}$ is a charge density distributed along the string length, and $B := \nabla\times A$ is the external magnetic field acting on the string. The expression

$$\begin{aligned} E := -q\frac{\partial A}{\partial\tau} - [r^{\prime,2}(1+(\dot{r}-\dot{r}_f)^2) - <\dot{r}-\dot{r}_f, r'>^2_{\mathbb{E}^3}]^{1/2}\nabla\bar{W} \\ + \frac{\partial}{\partial\sigma}\left\{\frac{\bar{W}[1+(\dot{r}-\dot{r}_f)^2\hat{T}_f]r'}{[r^{\prime,2}(1+(\dot{r}-\dot{r}_f)^2) - <\dot{r}-\dot{r}_f, r'>^2_{\mathbb{E}^3}]^{1/2}}\right\}, \end{aligned} \tag{13.213}$$

in a manner similar to that for the charged point particle case, models a related electric field, exerted on the string by the external electric charge q_f. Making use of the standard scheme, one can also derive, as above, the Hamiltonian interpretation of the dynamical equations (13.207).

13.9.3 *Some conclusions*

Based on the vacuum field theory approach devised recently in [60, 321, 322], we revisited the alternative charged point particle and hadronic string electrodynamics models, and succeeded in treating their Lagrangian and Hamiltonian properties. The results were compared with those obtained classically. An important aspect of the vacuum field theory approach consists in singling out the decisive role of the related rest reference system $\mathcal{K}_r$, with respect to which the relativistic object motion is realized. Namely, with respect to the proper rest reference system evolution parameter $\tau \in \mathbb{R}$ all of our electrodynamics models allow both (physically reasonable) Lagrangian and Hamiltonian formulations suitable for the canonical procedure. The deeper physical nature of this fact appears to still be elusive, but it is related to some observations of Feynman who argued in [125, 126] that the relativistic expressions have physical sense only with respect to the proper rest reference systems.

13.9.4 *Maxwell's electromagnetism theory from the vacuum field theory perspective*

We start from the following field theoretical model [60] of the microscopic vacuum medium structure: It is considered as physical reality embedded in the standard three-dimensional Euclidean space reference system marked by three spatial coordinates $r \in \mathbb{E}^3$, endowed, as usual, with the standard scalar product $<\cdot,\cdot>_{\mathbb{E}^3}$, and parametrized by means of the scalar temporal parameter $t \in \mathbb{R}$. We first describe the physical vacuum medium, endowing it with an everywhere sufficiently smooth four-vector potential function $(W, A) : M^4 \to \mathbb{R} \times \mathbb{E}^3$, defined in the Minkowski space M^4 and naturally related to light propagation properties. The material objects embedded in the vacuum medium, will be model (classically here) by means of the scalar charge density function $\rho : M^4 \to \mathbb{R}$ and the vector current density $J : M^4 \to \mathbb{E}^3$, also assumed to sufficiently smooth and globally defined functions.

(1) The *first* field theory principle regarding the vacuum is formulated as follows: the four-vector function $(W, A) : M^4 \to \mathbb{R} \times \mathbb{E}^3$ satisfies the standard Lorentz continuity equation

$$\frac{1}{c}\frac{\partial W}{\partial t} + < \nabla, A >_{\mathbb{E}^3} = 0, \tag{13.214}$$

where $\nabla := \partial/\partial r$ is the usual gradient operator.

(2) The *second* field theory principle is a dynamical relationship on the scalar potential component $W : M^4 \to \mathbb{R}$:

$$\frac{1}{c^2}\frac{\partial^2 W}{\partial t^2} - \nabla^2 W = \rho, \tag{13.215}$$

assuming the linear law of the small vacuum uniform and isotropic perturbation propagations in space-time, understood here as a first (linear) approximation in the case of sufficiently weak fields.

(3) The *third* principle is similar to the first one and simply establishes the continuity condition for the density and current density functions as

$$\partial\rho/\partial t + < \nabla, J >_{\mathbb{E}^3} = 0. \tag{13.216}$$

We need to note here that the vacuum field perturbation velocity parameter $c > 0$, used above, coincides with the vacuum light velocity, as we are trying to derive our results from first principles. For example, Maxwell's electromagnetic field equations shall be employed to analyze the Lorentz forces and special relativity relationships. To do this, we first combine equations (13.214) and (13.215):

$$\frac{1}{c^2}\frac{\partial^2 W}{\partial t^2} = - < \nabla, \frac{1}{c}\frac{\partial A}{\partial t} >_{\mathbb{E}^3} = < \nabla, \nabla W >_{\mathbb{E}^3} + \rho,$$

from which we deduce

$$< \nabla, -\frac{1}{c}\frac{\partial A}{\partial t} - \nabla W >_{\mathbb{E}^3} = \rho. \tag{13.217}$$

Whence, recalling the definition

$$E := -\frac{1}{c}\frac{\partial A}{\partial t} - \nabla W, \tag{13.218}$$

we obtain the first material Maxwell equation

$$< \nabla, E >_{\mathbb{E}^3} = \rho \tag{13.219}$$

for the electric field $E : M^4 \to \mathbb{E}^3$. Upon applying $\nabla\times$ to expression (13.218) we obtain the first Maxwell field equation

$$\frac{1}{c}\frac{\partial B}{\partial t} - \nabla \times E = 0 \tag{13.220}$$

for the magnetic field vector function $B : M^4 \to \mathbb{E}^3$, defined as

$$B := \nabla \times A. \tag{13.221}$$

To derive the second Maxwell field equation, we make use of (13.221), (13.214) and (13.218):

$$\begin{aligned} \nabla \times B &= \nabla \times (\nabla \times A) = \nabla < \nabla, A >_{\mathbb{E}^3} - \nabla^2 A \\ &= \nabla(-\frac{1}{c}\frac{\partial W}{\partial t}) - \nabla^2 A = \frac{1}{c}\frac{\partial}{\partial t}(-\nabla W - \frac{1}{c}\frac{\partial A}{\partial t} + \frac{1}{c}\frac{\partial A}{\partial t}) - \nabla^2 A \\ &= \frac{1}{c}\frac{\partial E}{\partial t} + (\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \nabla^2 A). \end{aligned} \tag{13.222}$$

We have from (13.218), (13.219) and (13.216) that

$$< \nabla, \frac{1}{c}\frac{\partial E}{\partial t} >_{\mathbb{E}^3} = \frac{1}{c}\frac{\partial \rho}{\partial t} = -\frac{1}{c} < \nabla, J >_{\mathbb{E}^3},$$

or

$$< \nabla, -\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \nabla(\frac{1}{c}\frac{\partial W}{\partial t}) + \frac{1}{c}J >_{\mathbb{E}^3} = 0. \tag{13.223}$$

Now making use of (13.214), from (13.223), we find that

$$\begin{aligned} &< \nabla, -\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \nabla(\frac{1}{c}\frac{\partial W}{\partial t}) + \frac{1}{c}J >_{\mathbb{E}^3} = < \nabla, -\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} + \nabla < \nabla, A >_{\mathbb{E}^3} + \frac{1}{c}J \\ &= < \nabla, -\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} + \nabla^2 A + \nabla \times (\nabla \times A) + \frac{1}{c}J >_{\mathbb{E}^3} \\ &= < \nabla, -\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} + \nabla^2 A + \frac{1}{c}J >_{\mathbb{E}^3} = 0. \end{aligned} \tag{13.224}$$

Whence, equation (13.224) yields

$$\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \nabla^2 A = \frac{1}{c}(J + \nabla \times S) \tag{13.225}$$

for a smooth vector function $S : M^4 \to \mathbb{E}^3$. Here we need to note that continuity equation (13.216) is defined, concerning the current density vector $J : M^4 \to \mathbb{R}^3$, up to a vorticity expression: that is, $J \simeq J + \nabla \times S$ and equation (13.225) can finally be rewritten as

$$\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \nabla^2 A = \frac{1}{c}J. \tag{13.226}$$

Then upon substitution of (13.226) in (13.222), we obtain the second Maxwell field equation

$$\nabla \times B - \frac{1}{c}\frac{\partial E}{\partial t} = \frac{1}{c}J. \tag{13.227}$$

In addition, from (13.221) one also finds the magnetic no-charge relationship

$$< \nabla, B >_{\mathbb{E}^3}= 0. \tag{13.228}$$

Thus, we have derived all the Maxwell electromagnetic field equations from our three main principles (13.214), (13.215) and (13.216). The success of our undertaking will be more impressive if we adapt our results to those following from relativity theory in the case of point charges or masses. We shall demonstrate these corresponding derivations based on some completely new physical conceptions of the vacuum medium first discussed in [60, 351].

Remark 13.5. It is interesting to analyze a partial case of the first field theory vacuum principle (13.214) when the following local conservation law for the scalar potential field function $W : M^4 \to \mathbb{R}$ holds:

$$\frac{d}{dt}\int_{\Omega_t} W d^3r = 0, \tag{13.229}$$

where $\Omega_t \subset \mathbb{E}^3$ is any open domain in space $\mathbb{E}^3$ with a smooth boundary $\partial\Omega_t$ for all $t \in \mathbb{R}$ and d^3r is the standard volume measure in $\mathbb{E}^3$ in a vicinity of the point $r \in \Omega_t$. A simple calculation using the expression (13.229) yields the following equivalent continuity equation

$$\frac{1}{c}\frac{\partial W}{\partial t} + < \nabla, \frac{v}{c} W >_{\mathbb{E}^3}= 0, \tag{13.230}$$

where $\nabla := \nabla_r$ is, as above, the gradient operator and $v := dr/dt$ is the velocity vector of a vacuum medium perturbation at point $r \in \mathbb{E}^3$ carrying the field potential quantity W. Comparing equations (13.214), (13.230) and using equation (13.216), we can make the very important identifications

$$A = \frac{v}{c} W, \qquad J = \rho v, \tag{13.231}$$

which are well known from classical electrodynamics [225] and superconductivity theory [126, 195]. Thus, we have a new physical interpretation of the conservative electromagnetic field theory when the vector potential $A : M^4 \to \mathbb{E}^3$ is completely determined via expression (13.231) by the scalar field potential function $W : M^4 \to \mathbb{R}$. It is also evident that all of Maxwell's field equations derived above hold as well in the case (13.231), as was first demonstrated in [60].

We now consider the conservation equation (13.229) together with the related integral *vacuum momentum* conservation condition

$$\frac{d}{dt}\int_{\Omega_t} (\frac{Wv}{c^2}) d^3r = 0, \quad \Omega_t|_{t=0} = \Omega_0, \tag{13.232}$$

where, as above, $\Omega_t \subset \mathbb{E}^3$ is for every time $t \in \mathbb{R}$ an open domain with smooth boundary $\partial\Omega_t$, whose evolution is governed by the equation

$$dr/dt = v(r,t) \tag{13.233}$$

for all $x \in \Omega_t$ and $t \in \mathbb{R}$, as well as by the initial state of the boundary $\partial\Omega_0$. It then follows from relation (13.232) that one has the new continuity equation

$$\frac{d(vW)}{dt} + vW < \nabla, v >_{\mathbb{E}^3} = 0. \tag{13.234}$$

Then, making use of (13.230) in the equivalent form

$$\frac{dW}{dt} + W < \nabla, v >_{\mathbb{E}^3} = 0,$$

we finally obtain the very interesting local conservation relationship

$$dv/dt = 0 \tag{13.235}$$

for the vacuum matter perturbation velocity $v = dr/dt$, which holds for all values of the time parameter $t \in \mathbb{R}$. It is easy to see that the relationship thus obtained coincides with the well-known hydrodynamic equation [88] for an ideal compressible liquid without external exertion (that is, there are no external forces and the field pressure is zero). The above analysis produces a very natural result when the propagation velocity of the vacuum field matter is constant and equals $v = c$. In particular, the wave equation (13.215) shows that small vacuum field matter perturbations propagate with the speed of light.

Chapter 14

SUPPLEMENT: Basics of Differential Geometry for Dynamical Systems

Our intention here is to provide a fairly complete treatment of the basics of differential geometry that are employed throughout the book, so as to render the overall exposition essentially complete.

14.1 General setting

We first introduce some of the fundamentals of linear algebra, differential geometry and topology.

Definition 14.1. Let A be an algebra with unit over a field K. A *grading* of A is a countable sum $\{A_p\}$, $p \in \mathbb{Z}_+$, of subspaces of A, such that

$$1) \quad A = \sum_{p \geq 0} A_p = \oplus_{p\geq 0} A_p; \qquad 2) \quad A_p A_q \subset A_{p+q}$$

with $p, q \in \mathbb{Z}_+$.

It is evident that $K \simeq K \cdot 1 \subset A_0$. The algebra A is called a *graded algebra* and $\{A_p\}$ is the family of *homogeneous elements* of A of degree p. Let $B = \oplus_{p\geq 0} B_p$, $p \in \mathbb{Z}_+$ be another graded algebra. We say that a homomorphism $h \colon B \to A$ *preserves the grading,* if $h(B_p) \subset A_p$, for all $p \in \mathbb{Z}_+$.

Definition 14.2. A graded algebra $A = \oplus_{p\geq 0} A_p$ is called non-commutative or a *Grassmann algebra* if $x_p x_q = (-1)^{pq} x_q x_p$ for all $x_p \in A_p$, $x_q \in A_q$.

Definition 14.3. Let $p = 2n$, $n \in \mathbb{Z}_+$. A *differentiation (derivation)* of a graded algebra A of degree p is an endomorphism (of modules) $d \colon A \to A$ such that

$$1) \quad dA_q \subset A_{q+p}; \qquad 2) \ \forall\, x, y \in A \quad d(xy) = (dx)y + x(dy),$$

where $q \in \mathbb{Z}_+$.

Definition 14.4. Let $p = 2n + 1,\ \ n \in \mathbb{Z}_+$. An *anti-differentiation (anti-derivation)* of a graded algebra A of degree p is an endomorphism $d' \colon A \to A$ such that

1) $d'A_q \subset A_{p+q}$; 2) $\forall\, x \in A_q, \forall\, y \in A$: $d'(xy) = (d'x)y + (-1)^q x(d'y)$,

where $q \in \mathbb{Z}_+$.

As a general result describing differentiations (derivations) and anti-differentiations (anti-derivations), we have the following:

Remark 14.1. If d_1 and d_2 are differentiations of degree p_1 and p_2, respectively, and a_1 and a_2 are anti-differentiations of degree q_1 and q_2, respectively, then 1) $a_1a_2 + a_2a_1$ is a differentiation of degree $q_1 + q_2$; 2) $[d_1, d_2]$ is a differentiation of degree $p_1 + p_2$; and 3) $[a_1, d_1]$ is an anti-differentiation of degree $q_1 + p_1$.

Let $A(E) = \sum_{p \geq 0} A^p(E)$ be a Grassmann algebra on the space of exterior forms on a linear space E and $A^p(E)$ be a module on $A = A^0(E)$. Then the structure of an algebra is provided by the wedge product: if $\alpha \in A^p(E)$, $\beta \in A^q(E)$, then

$$\alpha \wedge \beta = (p!q!)^{-1} \sum_{\sigma \in S_{p+q}} \varepsilon(\sigma)\alpha(e_{\sigma(1)}, \ldots, e_{\sigma(p)})\beta(e_{\sigma(p+1)}, \ldots, e_{\sigma(p+q)}),$$

where S_{p+q} is the symmetric group of permutations of a set of $(p + q)$ elements and $\varepsilon(\sigma)$ is the permutation parity (defined as $\alpha \in A^q(E) \Leftrightarrow \alpha(X_{\sigma(1)}, \ldots, X_{\sigma(q)}) = \varepsilon(\sigma)\alpha(X_1, \ldots, X_q)$).

We now introduce a *contraction operation*: $i(x) : A^p \to A^{p-1}$:

$$i(x)\alpha(e_1, \ldots, e_{p-1}) = \alpha(x, e_1, \ldots, e_{p-1})$$

for all $\alpha \in A^p$. It is easy to check that this operation is an anti-differentiation of degree -1. Let H be a subspace of E. Then $\alpha \in A^p(E/H) \Leftrightarrow \alpha(e_1, \ldots, e_p) \equiv 0$, if there exists at least one $e_j \in H,\ \ 1 \leq j \leq p$.

Definition 14.5. A submodule J in $A(E)$ is called an *ideal* in a Grassmann algebra A, if $J \wedge \alpha \subset J$ for all $\alpha \in A(E)$.

Definition 14.6. An ideal J is *homogeneous* if $J = \sum_p J \cap A^p(E)$.

Lemma 14.1. *Let J be a homogeneous ideal in a Grassmann algebra $A(E)$ and $H \subset E$. The ideal J is generated by forms from $A(E/H)$ if and only if $i(x)J \subset J$ for all $x \in H$.*

The ideal J is generated by forms $\alpha^{(j)}$ from $A(E)$ if every form $\alpha \in J$ can be written as $\alpha = \sum_j \alpha^{(j)} \wedge \beta_j$, for some $\beta_j \in A(E)$. Since the generators $\alpha^{(j)}$ of the ideal J lie in $A(E/H)$, it follows from the definitions that $i(x)\alpha^{(j)} = 0$ for all $x \in H$. Then $\forall\, \alpha: \ \alpha = \sum \alpha^{(j)} \wedge \beta_j$ and $i(x)\alpha \in J$.

We now prove the converse: the space $J \cap A^p(E)$ is generated by elements of $A(E/H)$. According to the convention $i(x)J \subset J$, if every $\beta^1 \in J \cap A^1(E)$, then $i(x)\beta^{(1)} = 0$ for $x \in H$. So every $\beta^{(1)} \in A^1(E/H)$. Let $\{e_1, \dots, e_p\}$ be a basis in H, $\{e_1, \dots, e_p, \dots, e_n\}$ a basis in E, and $\{\varepsilon_1, \dots, \varepsilon_n\}$ be the dual basis in $E^* = A^1(E)$. Then $\{\varepsilon_{p+1}, \dots, \varepsilon_n\}$ is the dual basis in $A^1(E/H)$. If $\beta^{(2)} \in J \cap A^2(E)$; $\quad i(e_j)\beta^{(2)} \in J$, $1 \le j \le p$, the element $\beta_1^{(2)} := \beta^{(2)} - \varepsilon_1 \wedge i(e_1)\beta^{(2)}$ does not include ε_1: $i(e_j)\alpha_1^{(2)} = 0$.

Furthermore, the element $\beta_2^{(2)} := \beta_1^{(2)} - \varepsilon_2 \wedge i(e_2)\beta_1^{(2)}$ does not include ε_1 and ε_2. Then $\beta_p^{(2)} := \beta_{p-1}^{(2)} - \varepsilon_p \wedge i(e_p)\beta_{p-1}^{(2)} \in A^2(E/H) \cap J$. Hence, $\beta_p^{(2)} + \omega^{(2)} = \beta^{(2)}$, where $\beta_p^{(2)} \in A^2(E/H)$ and $\omega^{(2)} \in J \cap A^2(E)$, and $\omega^{(2)}$ is a sum of elements all proportional to elements from $J \cap A^1(E)$. Moreover, by induction, all subspaces $J \cap A^{q-1}(E)$ are generated by elements of $A^{q-1}(E/H)$. For all $\beta^{(q)} \in A^q(E) \cap J$ the element $\beta^{(q)}$ is in the form $\beta_p^{(q)} + \omega^{(q)} = \beta^{(q)}$, where $\beta_p^{(q)} \in A^q(E/H)$ and $\omega^{(q)}$ is generated by elements of $A^{q-1}(E) \cap J$. Whence, by induction, $\beta^{(q)}$ is generated by elements of $A^q(E/H) \cap J$. $\quad\square$

Lemma 14.2 (Cartan's Lemma). *Let $\beta_j, \alpha_j \in A^1(E)$, $1 \le j \le p$, and*

$$\sum_{j=1}^{p} \alpha_j \wedge \beta_j = 0, \tag{14.1}$$

where α_j are linearly independent forms. Then $\beta_j = \sum_{k=1}^{p} A_{jk}\alpha_k$, where $A_{ij} = A_{ji}$.

Let $\alpha_{p+1}, \dots, \alpha_n$ be 1-forms, chosen so that $\{\alpha_j,\ 1 \le j \le n\}$ is a basis of the space $A^1(E)$. Then $\forall\, \beta_j = \sum_{k=1}^{p} A_{jk}\alpha_k + \sum_{k=p+1}^{n} B_{jk}\alpha_k, j = 1, \dots, n$. Substituting this in (14.1), we obtain

$$\sum_{i<j\le p} (A_{ij} - A_{ji})\alpha_i \wedge \alpha_j + \sum_{\substack{i\le k\\ j>p}} (B_{ij}\alpha_i \wedge \alpha_j) = 0.$$

Since the products $\alpha_i \wedge \alpha_j \in A^2(E)$ are linearly independent, $A_{ij} = A_{ji}, B_{ij} = 0$. $\square$

Definition 14.7. Let J be a homogeneous ideal in the Grassmann algebra of exterior forms on E, $\dim E = n$. A subspace $A(J) \subset E$ is *associated* to the ideal J, if $A(J) = \max H$, where the ideal J is generated by forms from $A(E/H)$.

It is easy to check that the maximum exists since

$$A(E/(H_1 + H_2)) = A(E/H_1) \cap A(E/H_2),$$

or which is the same $A(J) = \max\, H$, where $H := \{x \in E : \quad i(x)J \subset J\}$.

Definition 14.8. Let J be an ideal. The subspace $A^*(J) := A(J)^{\perp} \subset A^1(E)$ is called the *system of 1-forms*, associated to the ideal J.

Definition 14.9. The *rank* of an ideal J is $\dim A^*(J)$.

Lemma 14.3. *Let an ideal J be generated by a form $\alpha^{(p)}$. Then:*

$$1)\quad A(J) = \Big\{x \in E : \quad i(x)\alpha^{(p)} = 0, \quad \alpha^{(p)} \in A^p(E)\Big\};$$

$$2)\quad A^*(J) = \sum_{x_j \in E,\ 1 \le j \le p-1} i(x_1)i(x_2)\dots i(x_{p-1})\alpha^{(p)}.$$

1) Since $\alpha^{(p)} \in J$, from the definition of $A(J)$ it follows that $i(x)\alpha^{(p)} \in J \Rightarrow i(x)\alpha^{(p)} = 0$, so $x \in A(J)$.

2) If $x \in A(J)$, then $i(x_1)\dots i(x_{p-1})i(x)\alpha^{(p)} \equiv 0$, for all $x_j \in E$. Hence

$$\sum_{x_j \in E;\, 1 \le j \le p-1} i(x_1)\dots i(x_{p-1})\alpha^{(p)} = \tilde{A}^*(J) \subset A^*(J).$$

Now, we prove that $\tilde{A}^* = A^*$. Let $\{e_1, \dots, e_n\}$ be a basis of E such that the dual basis $\{\varepsilon_1, \dots, \varepsilon_n\}$ has the following properties:

1) $\varepsilon_1, \dots, \varepsilon_r$ is a basis of $\tilde{A}^*(J)$;

2) $\varepsilon_1, \dots, \varepsilon_r, \dots, \varepsilon_s$ is a basis of $A^*(J)$.

If $\varepsilon_s \in \tilde{A}^*(J)$, one can write $\alpha = \alpha^{(1)} \wedge \varepsilon_s + \beta$, where the forms $\alpha^{(1)}$, β are generated by $\varepsilon_1, \dots, \varepsilon_{s-1}$.

Let an element $(x_1, \dots, x_{p-1}) \in E^{p-1}$ be such that $\alpha^{(1)}(x_1, \dots, x_{p-1}) = a \neq 0$. Since $i(x)\alpha^{(1)} = i(x - \varepsilon_s(x)e_s)\alpha^{(1)}$, we can assume that $\varepsilon_s(x_i) = 0$, $i = 1, \dots, p-1$. Whence, $i(x_1)\dots i(x_{p-1})\alpha := i(x_1)\cdots(x_{p-1})(\alpha^{(1)} \wedge \varepsilon_s + \beta) = a\varepsilon_s + \sum_{i<s} a_i\varepsilon_i$, which is impossible because of $\varepsilon_s \not\subset \tilde{A}^*(J)$. $\square$

Exercise. Verify that if $\dim E = n$, $\alpha \in A^{n-1}(E)$, then $\dim A^*(I(\alpha)) = n - 1$. If $\alpha \in A^{n-2}(E)$, then $\dim A^*(I(\alpha)) = n - 2$ or n.

14.2 Differential-geometric structures related to dynamical systems on manifolds

Next we briefly treat some topics in differential geometry that are fundamental in dynamical systems theory.

14.2.1 *Exterior forms of degree 2 and their canonical representation*

Theorem 14.1. *If $\alpha \in A^2(E)$, there exists a number $2s \leq n$ and a basis $\{e_i\}_1^n$ of E, such that* 1) $\alpha(e_{2i-1}, e_{2i}) = -\alpha(e_{2i}, e_{2i-1}) = 1, \quad 1 \leq i \leq s$; 2) $\alpha(e_i, e_j) = 0$ *for all other vectors* e_j.

This can be readily proved by induction. For $n = 1$ it is obvious. Let $\alpha \neq 0$ and $e_1, e_2 \in E$ be such that $\alpha(e_1, e_2) = 1$; e_1 and e_2 generate the subspace $F \subset E$, with $\dim F = 2$. Consider the following hypersurfaces in E: $H_1 \colon i(e_1)\alpha = 0$ and $H_2 \colon i(e_2)\alpha = 0$. Their intersection $H_1 \cap H_2$ is the subspace G, such that $G \cap F = 0$ and $G \oplus F = E$. So G is the complement of F in E. From induction we deduce that there exists a basis $\{e_j\}_3^n$ of G such that the statements of theorem are true. Proceeding from G to E, we obtain the desired result. $\square$

If $\alpha \in A^2(E)$ then there exists a number $2s \leq n$ and $2s$ independent forms $\{\varepsilon_i\}_1^{2s}$ on E such that $\alpha = \varepsilon_1 \wedge \varepsilon_2 + \varepsilon_3 \wedge \varepsilon_4 + \cdots + \varepsilon_{2s-1} \wedge \varepsilon_{2s}$, where the form ε_1 can be chosen from $A^*(\alpha) = A^*(I(\alpha))$.

Let $\gamma \in A^*(I(\alpha))$. There exist e_1, e_2 such that $i(e_2)\alpha = -\gamma \in A^*(I(\alpha))\gamma(e_1) = 1$. By Theorem 14.1, there exists a basis $\{e_i\}_1^n$ satisfying conditions 1 and 2. Since $< e_1, \gamma >= 1$, $< e_i, \gamma >= 0$ for all $2 \leq i \leq n$; hence, the vector γ is from the dual basis $\{\varepsilon_i\}_1^n$ of the space E^* to the basis $\{e_i\}_1^n$. For this basis we obtain $\alpha = \varepsilon_1 \wedge \varepsilon_2 + \varepsilon_3 \wedge \varepsilon_4 + \cdots + \varepsilon_{2s-1} \wedge \varepsilon_{2s}$. $\square$

We note the following consequences of the above theorem: The rank of an ideal $I(\alpha) \in A^2(E)$ is always even and equal to $2s$. An ideal $I(\alpha)$ has the rank $2s$ iff $\alpha^s \neq 0$, $\alpha^{s+1} = 0$, $\alpha \in A^2(E)$. All ideals $I(\alpha)$, $I(\alpha^2), \ldots, I(\alpha^s)$ possess the same associated system $A^*(I(\alpha))$.

Since $I(\alpha^k)$ is generated (Lemma 14.13) by $(2k-1)$ anti-differentiations $i(x_j)$, $j = 1, \ldots, 2k-1$, it follows that every $A^*(I(\alpha^k))$ is likewise generated from $A^*(I(\alpha))$ by 1-forms $\varepsilon_1, \ldots, \varepsilon_{2s}$. $\square$

Definition 14.10. A *symplectic structure* on E is an exterior 2-form $\omega^{(2)}$ of the maximal rank $\dim I(\omega^{(2)}) = n = \dim E$ on E. A pair $(E, \omega^{(2)})$ is called a *symplectic vector space* (and obviously $\dim E = 2s$).

Exercise. Let E be a vector space of dimension $2s$ and $\omega^{(2)}$ be an exterior 2-form on E. Verify that the following conditions are equivalent:

1) $(E, \omega^{(2)})$ is a symplectic vector space;
2) the mapping $x \to i(x)\omega^{(2)}$ is an isomorphism of E on $E^* = A^1(E)$.

Definition 14.11. Let (E, α) and (F, β) be symplectic vector spaces. A *symplectic isomorphism* or *symplectomorphism* of E on F is a linear mapping $h\colon E \to F$ such that $h^*\beta = \alpha$.

Exercise. Let $(E, \omega^{(2)})$ be a symplectic vector space. Verify that the set $Sp(E, \omega^{(2)})$ of all symplectic automorphisms of $(E, \omega^{(2)})$ is a subgroup of $SL(E; \mathbb{R})$– the group of all automorphisms of E with determinants equal to 1.

Exercise. Let $(E, \omega^{(2)})$ be a symplectic vector space and $\omega^{(2)}$ be $\omega^{(2)} = \varepsilon_1 \wedge \varepsilon_2 + \varepsilon_3 \wedge \varepsilon_4 + \cdots + \varepsilon_{2s-1} \wedge \varepsilon_{2s}$. Let $\{e_i\}$ be the dual basis to $\{\varepsilon_j\}\colon \varepsilon_j(e_i) = \delta_{ij}$ and $J = \omega^{(2)}(e_i, e_j)$. Show that the matrix of the form $\omega^{(2)}$ with respect to the basis $\{e_i\}$ is

$$J = \begin{pmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & & 0 \\ & \ddots & \\ 0 & & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{pmatrix}.$$

Verify that an endomorphism $h\colon E \to E$ is a symplectic automorphism of $(E, \omega^{(2)})$, iff $M^\top JM = J$, where $M^\top$ is the transpose of the matrix of the map h with respect to the basis $\{e_j\}$.

Definition 14.12. An *orientation* of a vector space E is an equivalence class in $A^n(E)\backslash\{0\}$ of the equivalence relation $\omega \sim v\colon \omega = \lambda v$, $\lambda > 0$.

Thus, there are two different orientations.

Definition 14.13. An automorphism $h\colon E \to E$ is *compatible* with an orientation or *orientation preserving*, if for all $v \in A^n(E)\backslash\{0\}$ the n-forms v and h^*v determine the same orientation ($\dim E = n$).

It is obvious that an automorphism h is orientation preserving iff $\det\ h > 0$.

Let $(E, \omega^{(2)})$ be a symplectic vector space. The *canonical orientation* of a symplectic vector space $(E, \omega^{(2)})$ is determined by the form $(\omega^{(2)})^s \in A^{2s}(E)$. We note that a symplectic automorphism is compatible with the canonical orientation if and only if $\det\ h = 1 > 0$.

14.2.2 *Locally trivial fiber bundles and their structures*

Definition 14.14. Let E be a topological space. A *locally trivial fiber bundle* with a fiber F is the quadruple $\eta = (E, p,\ B;\ F)$, where E and B are topological spaces, $p\colon E \to B$ is a continuous surjection, and for all $b \in B$ there exists $U \ni b$ and a homeomorphism $\Phi\colon p^{-1}(U) \rightleftarrows U \times F$ such that $p_1 \circ \Phi = p$ (p_1 is the projection on the first set of the product $U \times F$). The space E is called the *total space* of the fiber bundle; B is the *base space*; p is the *projection* and $F_b = p^{-1}(b)$ is the fiber over $b \in B$. A pair (U, Φ) is called a *chart* of the fiber bundle η.

Let G be the family of all charts (U, Φ). If (V, Ψ) and (U, Φ) are charts such that $U \cap V \neq \emptyset$, then

$$\Psi \circ \Phi^{-1}(b, f) = (b, g(b)f), \quad (b, f) \in (U \cap V) \times F,$$

$g\colon B \to Aut\ F := \{the\ group\ of\ homeomorphisms\ on\ F\}$.

If $A \subset B$, then $\eta|_A = \{p^{-1}(A), p, A; F\}$ is a locally trivial fiber bundle with the base space A called the *restriction* of η to A.

It is easy to check that :

1) the fiber F_b over b in E is homeomorphic to F ;

2) the projection p is an open mapping;

3) the base space B is the quotient space of E with respect to the equivalence relation with equivalence classes equal to the fibers of η.

Exercise. Verify that if B and F are paracompact (i.e. are locally compact with countable bases) topological spaces, then so is E.

Definition 14.15. Let $\eta = (E, p, B)$ be a locally trivial fiber bundle and $A \subset B$. A *cross-section* η over A is a continuous mapping $s\colon A \to E$ such that $p \circ s = 1_A$ (so that s is a homeomorphism of A onto $s(A)$).

Definition 14.16. Let $\eta = (E, p, B)$ and $\eta' = (E', p', B')$ be fiber bundles with different fiber structures. A homeomorphism of the fiber bundle η into η' is a pair (H, h) of continuous mappings $H\colon E \to E'$, $h\colon B \to B'$, such that $p' \circ H = h \circ p$; i.e. the diagram

$$\begin{array}{ccc} E & \xrightarrow{H} & E' \\ p \downarrow & & \downarrow p' \\ B & \xrightarrow{h} & B' \end{array}$$

Figure 14.1

commutes (and h is uniquely determined by H).

Definition 14.17. If (U, Φ) and (U', Φ') are charts of η and η', $h(U) \cap U \neq \emptyset$, then $\Phi \circ H \circ \Phi^{-1}(b, f) = (h(b), l(b)f)$, $(b, f) \in U \cap h^{-1}(U') \times F$, $l \colon U \cap h^{-1}(U') \to \mathcal{B}(F; F') :=$ {the family of continuous mappings of F into F'}.

Definition 14.18. An *isomorphism* of η and η' is a pair (H, h), such that H and h are homeomorphisms. Isomorphic fiber bundles are also called *equivalent.*

Definition 14.19. Let $\vartheta = (B \times F, p, B)$. A fiber bundle η is *trivial* if there exists an isomorphism $H \colon \eta \to \vartheta$ over B. The isomorphism H is called a *trivialization* of η.

Example. The cylinder $\coprod = \mathbb{S}^1 \times [-1, 1]$ is a trivial fiber bundle with the base space $\mathbb{S}^1$ and fiber $[-1, 1]$.

Let $(E', p', B'; F')$ be a locally trivial fiber bundle and $h \colon B \to B'$ be a continuous mapping. Let E be the family of pairs $(b, e') \in B \times E'$ such that $h(b) = p'(e')$ and $p(b, e') = b$ be the projection E on B. Let $b \in B$; $\Phi' \colon (p'^{-1}(V)) \to V \times F$ be the trivialization $\eta'|_V$, where $V \ni h(b)$.

The set $U = h^{-1}(V)$ is an open in B, and the map $\Phi \colon \ (b, e') \to (b, p_2\ \Phi'(e))$ is a homeomorphism of $p^{-1}(U)$ onto $U \times F$. Then, (U, Φ) is a chart in the fibre bundle $(E, p, B; F)$. A continuous mapping $H \colon (b, e') \to e'$ from E to E' is a homomorphism η into η' over h.

The fiber bundle η is said to be *induced* from the fiber bundle η' by means of the (pullback) map $h \colon B \to B'$ and is denoted by $h^*(\eta') = \eta$.

Definition 14.20. Let $\eta := (E, p, B; F)$ be a locally trivial fiber bundle, where F is a vector space. A *vector bundle structure* on η is defined by a family of charts $\mathcal{A} = \{(U_\alpha, \Phi_\alpha)\}$ of the fiber bundle η such that the following properties hold : 1) $\{U_\alpha\}$ is a covering of B; 2) $\forall\ \alpha, \beta\ :\ \ (U_\alpha \cap U_\beta \neq \emptyset) \Rightarrow \Phi_\beta \circ \Phi_\alpha^{-1}(b, f) = (b, g_{\beta\alpha}(b)f)$, $g_{\beta\alpha} \colon B \to GL(F)$, $b \in U_\alpha \cap U_\beta$; 3) if $\mathcal{B} \supset \mathcal{A}$

is a family of charts of η which satisfies conditions 1 and 2, then $\mathcal{B} = \mathcal{A}$, that is $\mathcal{A}$ is the maximal set of charts. The family $\mathcal{A}$ is called an *atlas* of the vector bundle η and elements of $\mathcal{A}$ are called *charts.* The functions $g_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(F)$ are called the *transition functions* of the atlas $\mathcal{A}$, or *cocycles.*

Exercise. Verify that $g_{\gamma\beta}(b)g_{\beta\alpha}(b) = g_{\gamma\alpha}(b)$ hold for all $b \in U_\alpha \cap U_\beta \cap U_\gamma$.

It is easy to check that if there exists an atlas satisfying relations 1 and 2, then there is a maximal atlas. Different atlases can determine different vector bundle structures. Let s be a cross-section of η. Then $s(b) = \Phi_\alpha^{-1}(b, s_\alpha(b))$, for all $b \in U_\alpha,\ s_\alpha \colon U_\alpha \to F$ and $s_\beta(b) = g_{\beta\alpha}(b)s_\alpha(b)$, $b \in U_\alpha \cap U_\beta$.

Definition 14.21. A pair (H, h) is called a *homomorphism* from a vector bundle η into η', if for all $(U_\alpha, \Phi_\alpha) \subset \mathcal{A}$ and $(U_\gamma, \Phi_\gamma) \subset \mathcal{A}$, such that $h(U_\alpha) \cap U'_\gamma \neq \emptyset$, the following conditions hold:

$$\begin{gathered} \Phi'_\gamma \circ H \circ \Phi_\alpha^{-1}(b, f) = (h(b), h_{\gamma\alpha}(b)f), \\ (b, f) \in U_\alpha \cap h^{-1}(U'_\gamma) \times F, \\ h_{\gamma\alpha} \colon h^{-1}(U'_\gamma) \cap U_\alpha \to Hom(F, F'). \end{gathered} \tag{14.2}$$

Exercise. Verify that the following:

$$\begin{gathered} h_{\gamma\beta}(b)g_{\beta\alpha}(b) = h_{\gamma\alpha}(b),\ b \in h^{-1}(U'_\gamma) \cap U_\alpha \cap U_\beta, \\ g'_{\delta\gamma}(h(b))h_{\gamma\alpha}(b) = h_{\delta\alpha}(b),\ b \in h^{-1}(U'_\delta \cap U'_\gamma) \cap U_\alpha. \end{gathered} \tag{14.3}$$

The proofs of the next two results follow readily from the definitions,

Theorem 14.2. *Let η and η' are vector fibre bundles with fibres F, F' and with atlases $\mathcal{A}$, $\mathcal{A}'$, respectively, $h \colon B \to B'$ be a continuous mapping and $h_{\gamma\alpha} : h^{-1}(U'_\gamma) \cap U_\alpha \to Hom(F; F')$ be a family satisfying relationships (14.3). Then there exists the homomorphism $H : \eta \to \eta'$ over h which satisfies the first relationship in (14.2).*

Theorem 14.3. *Let $\mathcal{U} = \{U_\alpha\}$ be an open covering of the space B and F be a vector space ($dimF < \infty$). Let $g_{\beta\alpha}(U_\alpha \cap U_\beta) \to GL(F) \simeq Aut(F)$ satisfy conditions*

$$g_{\gamma\beta}(x)g_{\beta\alpha}(x) = g_{\gamma\alpha}(x), \quad \forall\ x \in U_\alpha \cap U_\beta \cap U_\gamma.$$

Then there exists a unique (up to equivalence) vector bundle $\eta = (E, p, B; \mathcal{A})$ with the fiber F, such that $g_{\beta\alpha}(b)$ are the transition functions of the atlas $\mathcal{A}$.

The family of pairs $(U_\alpha, g_{\beta\alpha})$ is called a *cocycle* on B taking values in *Aut* F, corresponding to the covering $\{U_\alpha\}$ of the atlas $\mathcal{A}$. Let Σ be the topological sum of sets $U_\alpha \times F$, and ρ be the equivalence relation on Σ, defined as follows: $(x, e) \in U_\alpha \times F$ is in relation ρ with $(y, f) \in U_\beta \times F$ if $x = y$ and $f = g_{\beta\alpha}(x)e$.

Then ρ is really an equivalence relationship and it is *open.* Let $\pi\colon \Sigma \to \Sigma/\rho$, the continuous mapping from Σ onto B, determined by the projection on the first set in $U_\alpha \times F$ is *compatible* with ρ. Thus, it determines the continuous surjection $p\colon E \to B$. The quadruple $(E, p, B; F)$ is a locally trivial fiber bundle with the fiber F. If π_α is a cross-section of π on $U_\alpha \times F$, then π_α is a homeomorphism of $U_\alpha \times F$ onto $p^{-1}(U_\alpha)$ such that $p \circ \pi_\alpha(x, f) = x$, so $p(U_\alpha, \pi_\alpha^{-1})$ is a chart of η. Since for all $b \in U_\alpha \cap U_\beta \neq \varnothing$,

$$\pi_\beta^{-1} \circ \pi_\alpha(b, f) = (b, g_{\beta\alpha}(b)f), \quad (b, f) \in (U_\alpha \cap U_\beta) \times F,$$

the family $\mathcal{A} = \{(U_\alpha, \pi_\alpha^{-1})\}$ is an atlas on η. Thus, η is a vector bundle. $\triangleright$

Let $\eta = (E, p, B; F)$ be a vector bundle with a fiber F and $(U_\alpha, g_{\beta\alpha})$ be a cocycle on B corresponding to a maximal atlas $\mathcal{A}$. Let λ be a continuous homomorphism from $GL(F)$ onto $GL(F')$ (the groups of automorphisms on F and F' respectively). The continuous maps $g'_{\beta\alpha} = \lambda \circ g_{\beta\alpha}$ from intersections $U_\alpha \cap U_\beta$ into $GL(F')$ satisfy the following:

$$g'_{\gamma\beta} \circ g'_{\beta\alpha} = g'_{\gamma\alpha}, \quad \forall\, b \in U_\alpha \cap U_\beta \cap U_\gamma.$$

Thus, $(U_\alpha, g'_{\gamma\alpha})$ is a cocycle on B.

Definition 14.22. A vector bundle η with a base B and a fiber F' determined by cocycle $(U_\alpha, \lambda \circ g_{\beta\alpha})$, is called a vector bundle *associated to* η by means of a homomorphism λ.

Example 14.1. Vector bundles of exterior forms $A^p(\eta)$: Let $A^p(F)$ be the vector space of exterior p-forms on F. Let $a \in GL(F)$. Then the mapping $a \to (a^*)^{-1} \in GL(A^p(F))$ defines the associated vector bundle $A^p(\eta) = \eta_{\lambda p}$, called the *bundle of exterior forms* on η. The vector bundle $A^1(\eta) = \eta^*$ is called the *dual bundle* to η.

Definition 14.23. A vector bundle η of dimension n is *oriented* if the bundle $A^n(\eta)$ is trivial.

Exercise. Verify that a vector bundle η is oriented iff there exists a non-vanishing cross-section s of $A^n(\eta)$.

Definition 14.24. Let $\eta = (E, p, B)$ be a vector bundle with fiber F, $\dim F = n < \infty$, and Γ_0 be a set of all non-vanishing cross-sections of

$A^n(\eta)$. The *orientation* of η is an equivalence class in Γ_0 with respect to the relation $s_1 \sim s_2 \colon s_1 = \lambda s_2$, where $\lambda > 0$ is a continuous positive function.

Definition 14.25. Let η and η' be two vector bundles of dimension n which are oriented with respect to non-vanishing cross-sections of $A^n(\eta)$ and $A^n(\eta')$. An isomorphism $(H,h) : \eta \to \eta'$ is called *compatible* with these orientations (or *orientable*), if the volume forms v and $(H,h)^* v'$ give the same orientation of η.

Theorem 14.4. *Let B be a paracompact, a locally connected topological space, η be a vector bundle over B with fiber F, which is an oriented vector space, $\dim F = n < \infty$. The bundle η is oriented if and only if there exists an atlas $\mathcal{A}$ with a cocycle $(U_\alpha, g_{\beta\alpha})$ on B such that for all (α,β) and for all $b \in U_\alpha \cap U_\beta$ the automorphisms $g_{\beta\alpha}(b) \in GL(F)$ are compatible with an orientation of F.*

Proof. Let $\mathcal{A} = \{(U_\alpha, \Phi_\alpha)\}$ be an atlas for η with a cocycle $(U_\alpha, g_{\beta\alpha})$ on B. Since B is locally connected, every connected component of any U_α is open in B. So we can assume that all open U_α are connected. Let s be a non-vanishing cross section $A^n(\eta)$ over B that is determined by a set of continuous mappings $s_\alpha : U_\alpha \to A^n(F)$. For every chart $(U_\alpha, \Phi_\alpha) \in \mathcal{A}$ we can write

$$(\Phi_\alpha^{-1})^\star (s|_{U_\alpha})(b) = s_\alpha(b) = \lambda_\alpha(b) v,$$

where v is a volume form giving an orientation of F and $\lambda_\alpha \colon U_\alpha \to \mathbf{R}$ is a non-vanishing continuous function.

We can suppose that every $\lambda_\alpha > 0$. Thus, $s_\alpha(b) = \lambda_\beta(b) v = \det(g_{\beta\alpha}(b)) s_\alpha(b) = \det(g_{\beta\alpha}(b)) \lambda_\alpha(b) v$, when $\det(g_{\beta\alpha}(b)) = \lambda_\beta / \lambda_\alpha > 0$, establishing necessity.

To prove sufficiency, we assume that $\mathcal{A} = \{(U_\alpha, \Phi_\alpha)\}$ is an atlas such that all det $(g_{\beta\alpha}) > 0$. Since B is paracompact, we can assume that $\{U_\alpha\}$ is a locally finite open covering. Thus there exists a partition of unity (φ_α) subordinate to the covering $\{U_\alpha\}$.

For any chart (U_α, Φ_α) of the bundle η the mapping $\sigma_\alpha = \Phi{*}_\alpha v$ is a cross-section of $A^n(\eta)$ over U_α (with v a cross-section of the trivial bundle $U_\alpha \times A^n(F)$). A cross-section $s_\alpha := (\varphi_\alpha|_{U_\alpha}) \sigma_\alpha$ is continuously prolonged by zero on $B \backslash U_\alpha$ to a cross-section of $A^n(\eta)$ over B. Since the open covering $\{U_\alpha\}$ is locally finite, the sum $s = \sum_\alpha s_\alpha$ is well defined and is a cross-section of $A^n(\eta)$ over B. We now show that this cross-section is non-vanishing. Let $b \in B$ be an arbitrary point belonging to $U_{\alpha_1}, \ldots, U_{\alpha_r}$.

Then

$$(\Phi_{\alpha_1}^{-1})^*(s(b)) = (b, \sum_{i=1}^{r} \lambda_{\alpha_i}(b)\det(g_{\alpha_1\alpha_i}(b))v),$$

and thus $s(b) \neq 0$ for all $b \in B$. $\square$

Definition 14.26. Let $\eta = (E, p, B)$ and $\eta' = (E', p', B)$ be two vector bundles with a common base space B. The *Whitney sum* of η and η' is the vector bundle $\eta \oplus \eta'$, induced from $\eta \times \eta'$ by the diagonal mapping $d : B \longrightarrow B \times B$ (Fig. 14.2). It is denoted by $\eta \oplus \eta' := (E \oplus E', p \oplus p', B)$.

$$\begin{array}{ccc} E \oplus E' & \xrightarrow{D} & E \times E' \\ p \oplus p' \downarrow & & \downarrow p \otimes p' \\ B & \xrightarrow{d} & B \times B \end{array}$$

Figure 14.2

The total space $E \oplus E'$ consists of triples $(b, e, e') \in B \times E \times E'$, such that $p(e) = p'(e') = b$.

Theorem 14.5. *Let $\eta = (E, p, B)$ be a vector bundle with a fiber F and $\eta_p := \oplus_p \eta$ be the p-multiple Whitney sum of η. Then the vector space of cross-sections of the bundle $A^p(\eta)$ over B is isomorphic to the set of continuous functions $\sigma\colon \oplus_p E \to \mathbb{R}$ such that sections of every fiber $(p^{-1}(b) \times \cdots \times p^{-1}(b))$, $b \in B$, are exterior p-forms on $p^{-1}(b) = F_b$.*

Proof. Let $\{(U_\alpha, \Phi_\alpha)\} = \mathcal{A}$ be a maximal atlas and $(U_\alpha, g_{\beta\alpha})$ its cocycle. A cross-section s of a bundle $A^m(\eta)$ over B is given by a set of continuous mappings $s_\alpha\colon U_\alpha \to A^m(F)$, such that

$$s_\beta(b) = (g_{\beta\alpha}^*)^{-1} s_\alpha(b),\ b \in U_\alpha \cap U_\beta.$$

For an arbitrary chart (U_α, Φ_α) of the bundle η_p one can determine continuous mappings σ_α of the open set $(\oplus^p E)_\alpha = \cup_{b \in U_\alpha}(F_b) \subset \oplus^p E$ into $\mathbb{R}$ so that

$$\sigma_\alpha(e_1, \ldots, e_m) = s_\alpha(b)(p_2\Phi_\alpha(e_1), \ldots, p_2\Phi_\alpha(e_m)).$$

It is evident that the set $\{\sigma_\alpha\} := \sigma$ defines a continuous function on $\oplus_p E$, whose restrictions σ_α on $(F_b)^p$ are exterior forms on F_b of degree p. $\square$

14.2.3 Subbundles and factor bundles

Theorem 14.6. *Let F be a finite-dimensional vector space and F' be a subspace of F. Let $\eta := (E, p, B; F)$ be a vector bundle with the fiber F and $\eta' := (E', p', B, F')$ be a vector bundle with the base space B and fiber F'. Let H be an injective homomorphism η' into η over B. Then there exists an atlas $\mathcal{A} = \{(U_\alpha, \Phi_\alpha)\}$ of η such that for all α the following properties hold: 1) there exists a homeomorphism $\Phi'_\alpha : p^{'-1}(U_\alpha) \to U_\alpha \times F'$ such that (U_α, Φ'_α) is a chart of η'; 2) $\Phi_\alpha \circ H \circ \Phi_\alpha^{-1}(b, f) = (b, f) \quad \forall\, (b, f) \in U_\alpha \times F'$; 3) $\{(U_\alpha, \Phi'_\alpha)\}$ is an atlas $\mathcal{A}'$ of η'; 4) transition functions $g_{\beta\alpha}(b)$ of the atlas $\mathcal{A}$ transform F' into F'; 5) functions $g'_{\beta\alpha}$ of the atlas $\mathcal{A}'$ are restrictions of $g_{\beta\alpha}(b)$ on F. The bundle η' is called a fiber subbundle of η.*

Proof. Let $\mathcal{A} = \{(U_\alpha, \Psi_\alpha)\}$ be an atlas of η such that for every α there exists a *trivialization* Φ'_α of the fiber bundle $\eta'|_{U_\alpha}$. Then $\Psi_\alpha \circ H \circ \Phi_\alpha^{'-1}(b, f) = (b, h_\alpha(b)f); (b, f) \in U_\alpha \times F'$, where $h_\alpha : U_\alpha \to Hom(F, F')$. Taking, if it is necessary a partition of $\{U_\alpha\}$, we can find for every α a continuous function $g : U_\alpha \to GL(F)$ such that $\forall\, b \in U_\alpha \;\; g_\alpha(b)h_\alpha(b)$ is a canonical embedding of F' into F.

The function $\Phi_\alpha : e \to (p(e), g_\alpha(p(e))p_2\Psi_\alpha(e))$ determines the vector bundle chart (U_α, Φ_α) of the fiber bundle η, and the set of these charts (U_α, Φ_α) provides the atlas $\mathcal{A}$ such, that properties 1–3 hold. Properties 4–5 are also easy to verify. □

The following result, which is essentially the converse of this theorem, can be readily verified using the rather obvious elements of the above proof.

Theorem 14.7. *Let $(U_\alpha, g_{\beta\alpha})$ be a cocycle on B with values in $GL(F)$ such that $F' \subset F$ is invariant with respect to $g_{\beta\alpha}$. Let $\eta = (E, p, B)$ and $\eta' = (E', p', B')$ be vector fiber bundles with fibers F and F', respectively. Then there exists an injective homomorphism $H : \eta' \to \eta$ (constructed above), such that*

$$h_{\alpha\alpha}(b)f = f, \quad b \in U_\alpha, \quad f \in F'.$$

In this vein, we also have the following related factor (quotient) construction and its converse, which we state without proof.

Theorem 14.8. *Let F be a vector space $dimF < \infty$, F' be a factor space for F and q be the projection F onto F'. Let $(E, p, B) = \eta$ and $(E', p', B) = \eta'$ be vector bundles with fibers F and F', respectively. Let K be a surjective homomorphism η onto η' over B. Then there exists an atlas*

$\mathcal{A} = \{(U_\alpha, \Phi_\alpha)\}$ *of the bundle* η*, such that the following properties hold: 1) there exists a homeomorphism* $\Phi'_\alpha\colon p'^{-1}(U_\alpha) \rightleftarrows U_\alpha \times F'$ *such that a pair* (U_α, Φ'_α) *is a chart in* η'*; 2)* $\Phi'_\alpha \circ K \circ \Phi_\alpha^{-1}(b, f) = (b, q(f)),\ b,\ f \in U_\alpha \times F$*; 3)* $\{(U_\alpha, \Phi'_\alpha)\}$ *is an atlas* $\mathcal{A}'$ *of the bundle* η'*; 4) functions* $g_{\beta\alpha}$ *of the atlas* $\mathcal{A}$ *are compatible and commute with* q*; 5) functions* $g'_{\beta\alpha}$ *of the atlas* $\mathcal{A}'$ *are factor functions of* $g_{\beta\alpha}$*.*
The bundle η' *is called a* ***factor bundle*** *or* ***quotient bundle*** *of* η*.*

Theorem 14.9. *Let* $(U_\alpha, g_{\beta\alpha})$ *be a cocycle of* B *with values in* $GL(F)$*, such that the transition functions* $g_{\beta\alpha}$ *commute with the projection* $q\colon F \to F'$*. Let* $\eta = (E, p, B; F)$ *and* $\eta' = (E, p, B; F')$ *be two fiber bundles with given cocycles. Then there exists a unique surjective homomorphism* $K\colon \eta \to \eta'$ *over* B*, such that*

$$h_{\alpha\alpha}(b)f = q(f) \quad \forall\ b \in U_\alpha, \quad f \in F.$$

14.2.4 *Manifolds. Tangent and cotangent bundles*

Let M^m be a smooth manifold of dimension m with a maximal atlas $\mathcal{A} = \{(U_i, \varphi_i)\}$. The functions $g_{ji}\colon x \to D(\varphi_j\varphi_i^{-1})|_{\varphi_i(x)}$ (where D denotes the derivative) satisfy

1) $g_{ji}\colon U_i \cap U_j \to GL(m; \mathbb{R}) \quad \forall\ x \in M^m$;
2) $g_{kj}g_{js} = g_{ks} \quad \forall\ x \in U_i \cap U_k \cap U_j$,

and these functions give rise to the differential cocycle (U_i, g_{ij}) on M^m with values in $GL(m; \mathbb{R})$.

Definition 14.27. Let M^m be an m-dimensional smooth manifold with a maximal atlas $\{(U_i, \varphi_i)\} = \mathcal{A}$ and cocycle (U_i, g_{ki}). The *tangent bundle* to manifold M^m is a differentiable vector bundle with base space M^m and fiber $\mathbb{R}^m$, which is determined by the cocycle (U_i, g_{ki}). It is denoted by $\tau(M) = (T(M), p, M^m; \mathbb{R}^m)$.

The fiber $T_x(M)$ over $x \in M^m$ is called the *tangent space* at a point $x \in M$, and $T(M)$ is the tangent space over M: $T(M) = \bigcup_{x\in M} T_x(M)$. The associated bundle $A^p(\tau(M))$ of forms on $\tau(M)$ is also a differentiable vector bundle, and

$$A^1(\tau(M)) = \tau^*(M) = (T^*(M), q_M, M^m)$$

is called the *cotangent bundle.* $T^*(M)$ is the cotangent space to M^m.

Let $h\colon M^m \to N^n$ be a smooth map. For every (U, φ) on M^m and (V, ψ) on N^n, such that $h(U) \cap V \neq \emptyset$ there is the function $g\colon x \to D(\psi \circ h \circ$

$\varphi^{-1})|_{\varphi(x)}$, which is a smooth mapping from $h^{-1}(V) \cap U$ into $Hom(\mathbb{R}^m, \mathbb{R}^n)$ and satisfies

$$h_{\gamma\beta} g_{\beta\alpha} = h_{\gamma\alpha}, \quad g'_{\delta\gamma} h_{\gamma\alpha} = h_{\delta\alpha},$$

where $\{g_{\alpha\beta}\}$ is a cocycle on M^m and $\{g'_{\alpha\beta}\}$ is a cocycle on N^n, determining a homomorphism $(h_*, h)\colon \tau(M) \to \tau(N)$, called the *tangent map* of h. We note that if $h\colon M \to N, k\colon N \to V$, then $(k \circ h)_* = k_* \circ h_*$.

If $c\colon \mathbb{R} \to M^m$ is a smooth curve, then the vector $c_*(t, 1) \in T_{c(t)}(M)$ is called a *tangent vector* to the curve c at the point $c(t)$. Let M be a manifold, and $\mathcal{D}(M)$ be the set of all differentiable functions on M. Let $df \in \mathcal{D}(T(M))$ be the second component of the tangent map $f_*\colon T(M) \to T(\mathbb{R}) \approx \mathbb{R} \times \mathbb{R}\colon p_2 \circ f_* = df$. We say that df is the *differential* of f.

Definition 14.28. A manifold M^m is *oriented* if the bundle $A^m(\tau(M))$ is trivial.

The next result follows directly from the above theorems on oriented bundles.

Theorem 14.10. *A manifold M^m is oriented if: 1) there exists a non-vanishing cross-section $A^m(\tau(M))$ over M^m; 2) there exists an atlas $\mathcal{A} = \{(U_i, \varphi_i)\}$ on M^m such that for all functions $\varphi_j \circ \varphi_i^{-1}\colon U_i \cap U_j \to \mathbb{R}^m$, $\det(\varphi_j \circ \varphi_i^{-1})$ is positive.*

Example 14.2. Let $N = T(M)$ be the tangent space to M. It is easy to see that $T(M)$ is an oriented manifold. Indeed if $\mathcal{A} = \{(U_i, \varphi_i)\}$ is an atlas of M, then $B = \{(p_M^{-1}(U_i), \varphi_{i,*})\}$ is an atlas of $T(M)$, and $\det[D(\varphi_{j,*} \circ \varphi_{i,*}^{-1})|_{\varphi_{i,*}(x)}] = (\det[D(\varphi_j \circ \varphi_i^{-1})])^2 > 0$.

Theorem 14.11. *If M^m is an oriented manifold with boundary ∂M, then ∂M is also oriented and orientation of M determines the orientation of ∂M.*

Proof. An orientation of M^m makes it possible to choose an atlas $\mathcal{A} = \{(U_i, \varphi_i)\}$ having the property $\det[D(\varphi_j \circ \varphi_i^{-1})|_{\varphi_i(x)}] > 0$. Let B be the set of all charts (V, ψ) of the manifold ∂M^m for which there exists a chart $(U, \varphi) \in \mathcal{A}$ such that $V = U \cap \partial M$ and $\psi = \varphi|_V$. Let (U_i, φ_i), $(U_j, \varphi_j) \in \mathcal{A}$ and $U_j \cap U_i \cap \partial M \neq \emptyset$. Denoting $\varphi_j \circ \varphi_i^{-1}(x_1, \ldots, x_m) = (h_1(x_1, \ldots, x_m), \ldots, h_m(x_1, \ldots, x_m))$, we find that

$$\frac{\partial h_m}{\partial x_i}(x_1, \ldots, x_{m-1}, 0) = 0, \quad 0 \leq i \leq m-1;$$

$$\frac{\partial h_m}{\partial x_m}(x_1, \ldots, x_{m-1}, 0) = a(x_1, \ldots, x_{m-1}) > 0$$

(since $h_m \geq 0$, $\frac{\partial h_m}{\partial x_m} \neq 0$).

Hence for all $x \in V_i \cap V_j = U_i \cap U_j \cap \partial M$

$$\det[D(\psi_j \circ \psi_i^{-1})|_{\psi_i(x)}] = \frac{1}{a(\varphi_i(x))} \det[D(\varphi_j \circ \varphi_i^{-1})|_{\varphi_i(x)}] > 0.$$

Thus, the atlas B of ∂M^m has property 2) of Theorem 14.10. □

14.2.5 *The rank theorem for a differential map and its corollaries*

Theorem 14.12. *Let M^m, N^n be differentiable manifolds without boundaries, $h\colon M \to N$ be a differentiable map of constant rank $p = \operatorname{rank} h_x$, $h_x\colon T_x(M) \to T_{h(x)}(N)$ (if $p = n$ $\forall\ x \in M$: h is a submersion, if $p = m$ $\forall\ x \in M$: h is an immersion and* rank $h_x =$ rank $D[\psi \circ h \circ \varphi^{-1}]$ *and the rank of h is a semi-continuous function with nonnegative integer values). Then for all $x \in M$ there exists a system of differentiable local coordinates $(y_1, \ldots, y_m)$ in an open neighborhood of the point x and a system of local coordinates $(z_1, \ldots, z_n)$ in an open neighborhood of the point $h(x)$, such that in these coordinates the map h has the form*

$$h(y_1, \ldots, y_m) = (z_1 = y_1, z_2 = y_2, \ldots, z_p = y_p, 0 \ldots 0).$$

This can be proved by a straightforward application of the inverse mapping theorem (as in [144]).

Definition 14.29. Differentiable functions $f_1, \ldots, f_p$ on a manifold M^m are called *independent* at $y \in M^m$, if the mapping $f\ :\ \ y \to (f_1(y), \ldots, f_p(y))$, $f\ :\ \ M^m \xrightarrow{f} \mathbb{R}^p$ has the rank p at $y \in M^m$ (that is $df_1, \ldots, df_m$ are linearly independent forms on $T_y(M)$).

Definition 14.30. Let M^m be an m-dimensional differentiable manifold without boundary. A *submanifold* of M^m, of $n \leq m$ dimension is a subspace $N \subset M^m$ which has the following property: $\forall\ \ x \in M^m$ $\exists\ (y_1(x), \ldots, y_m(x))$ on $U \ni x$, such that $U \cap N$ is determined by the equalities $y_{n+1}(x) = \cdots = y_m(x) = 0$, or $y_{n+1} = y_{n+2} = \cdots = y_m = 0$.

Definition 14.31. When an immersion $h\colon N^n \to M^m$ is injective, then it is called an *embedding* if $h(N^n) \subset M^m$ is a submanifold.

Definition 14.32. If $h\colon M^m \to N^n$, a point $c \in N^n$ is called a *regular value* of h if the rank of h is equal to n at every point $h^{-1}(c)$ (in particular, if the set $h^{-1}(c)$ is empty).

As a consequence of Theorem 14.10, we have the following results.

Theorem 14.13. *If $h\colon M^m \to N^n$ is a differentiable map of manifolds without boundaries and $c \in N^n$ is a regular value of h, then $h^{-1}(c)$ is $(m-n)$-dimensional submanifold of M^m.*

Theorem 14.14. *If $f\colon M^m \to \mathbb{R}$ is a differentiable function on M^m, then for a regular value $c \in h(M^m)$ the set $h^{-1}((-\infty, c])$ is a submanifold of M^m having a boundary that is a submanifold of $h^{-1}(c)$.*

14.2.6 *Vector fields*

Definition 14.33. Let M^m be a differentiable manifold. A *vector field* on M^m is a differentiable cross-section of the tangent bundle $\tau(M)$.

A vector field X on M for an atlas $\mathcal{A} = \{(U_i, \varphi_i)\}$ determines a family of differentiable maps $X_i\colon U_i \to \mathbb{R}^m$, such that

$$X_j(y) = [D(\varphi_j \circ \varphi_i^{-1})|_{\varphi_i(y)}]X_i(y) \ \forall\ y \in U_i \cap U_j.$$

The set $\mathcal{T}(M)$ of vector fields on M^m is a module over the ring $\mathcal{D}(M)$ of differentiable functions on M^m, and one can readily verify the following result.

Theorem 14.15. *If $h\colon M^m \to N^n$ is a diffeomorphism, then $h_*X \circ h^{-1} = Y$ is a vector field on N^m for every $X \in \mathcal{T}(M)$.*

Let $f \in \mathcal{D}(M), X \in \mathcal{T}(M)$. We can define an operation $f \to Xf \in D(M)$ as $Xf(y) = df(X(y))$; Xf, which is sometimes called the *directional derivative* of f in the direction X.

Theorem 14.16. *For every differentiation D of the ring $\mathcal{D}(M)$, there is a unique $X \in \mathcal{T}(M)$ such that $Xf = Df$ for all $f \in \mathcal{D}(M)$.*

Proof. For all $f \in \mathcal{D}(M)$ we have

$$\begin{aligned} f(x) &= f(y) + \sum_i (x_i - y_i) \int_0^t \frac{\partial f(tx + (1-t)y)}{\partial x_i} dt \\ &= f(y) + \sum_i (x_i - y_i) g_i(x, y). \end{aligned}$$

Let D be differentiation of the ring $\mathcal{D}(M)$, then

$$Df(x) = \sum_i (Dx_i) g_i(x, y) + \sum_i (x_i - y_i) D g_i(x, y). \tag{14.4}$$

Now we find functions $a_i = Dx_i$ and construct a vector field $\tilde{X}$, that has coordinates $a_i = \tilde{X}(x_i)$ in a chart (U, φ). Hence $\tilde{X}f = \sum_i a_i \cdot \frac{\partial f}{\partial x_i}$. We note in (14.4), with $y \to x$, that the left-hand side is independent of y: $Df = \sum_i a_i g_i(x, x)$, but if $g_i(x, x) = \frac{\partial f(x)}{\partial x_i}$, we find $X = D$. $\square$

As a direct corollary of the rank theorem, we have the following result.

Theorem 14.17. *Let $h\colon N^n \to M^m$ be an embedding (injective immersion) N^n into M^m, and $X \in \mathcal{T}(M)$ be such that $\forall\ y \in N^n$ we have $X \circ h(y) \subset h_\star T_y(N)$. Then there exists a unique vector field Y on N^m such that $h_\star Y = X \circ h$.*

Definition 14.34. Let $i\colon N^n \to M^m$ be including into M^m. The vector field $X \in \mathcal{T}(M)$ is tangent to N, if $X \circ i(y) \in i_* T_y(N)\ \forall\ y \in N,\ \forall\ y \in M.$

Let N be a submanifold of M^m, where M^m is without boundary, and suppose that X, Y are two vector fields on M^m that are tangent to N. Then the commutator (Lie) bracket $[X, Y]$ is also tangent to N, and we have the following two results that follow easily from the above definitions and the rank theorem.

Theorem 14.18. *Let M^m be a differentiable manifold without boundary, and $h = (h_1, \ldots, h_m)$ a differentiable mapping $M^m \to \mathbb{R}^n$. Let c be a regular value of h such that $N = h^{-1}(c) \neq 0$. Then the vector field X on M is tangent to N if and only if $Xh_1 = Xh_2 = \cdots = Xh_n = 0$ on N.*

Theorem 14.19. *If $X_1, X_2 \in \mathcal{T}(M)$, then $[X_1, X_2] = X_1 \circ X_2 - X_2 \circ X_1\colon \mathcal{D}(M) \to \mathcal{D}(M)$ is also a differentiation of the ring $\mathcal{D}(M)$ of functions on M^m.*

14.2.7 *Differential forms*

Definition 14.35. A *differential form of degree p* on a differentiable manifold M^m is a cross-section of the bundle $A^p(\tau(M))$ of exterior differential p-forms on $\tau(M)$.

The set $\Lambda^p(M)$ of differential forms of degree p on M^m is a module over the ring $\mathcal{D}(M)$, and we have the following direct consequence of our more general theorem for vector bundles.

Theorem 14.20. *Let M^m be a (differentiable) manifold and $\varepsilon_p = (T^p, \pi, M^m) = \oplus\tau(M)$ the Whitney sum of p copies of the tangent fibre bundle $\tau(M)$. The module $\Lambda^p(M)$ of differentiable forms of degree p on M^m is isomorphic to the module of differentiable functions $\sigma\colon T^p \to \mathbb{R}$ having restrictions on every $(T_y(M))^p$, $y \in M^m$, that are exterior p-form on $T_y(M)$.*

Accordingly we use the same symbol for such a function on T^p and a differential p-form on M^m.

Let $X_1, \ldots, X_p \in \mathcal{T}(M)$. Then $\forall\ x \in M^m \quad \alpha(X_1, \ldots, X_p)\colon x \to \alpha(X_1(x), \ldots, X_p(x))$ determine a function on M. Hence, α may be assigned to a form on $\mathcal{D}(M)$ – a module $\mathcal{T}(M)$ of vector fields. This assignment is evidently an isomorphism. Hence, $\Lambda^p(M) \simeq A^p(\mathcal{T}(M))$.

Let $\alpha \in \Lambda^p(M)$. Then every $h\colon N \to M$ induces a mapping

$$h^*\colon \Lambda^p(M) \to \Lambda^p(N)\colon (h^*\alpha)(X_1, \ldots, X_p) = \alpha(h_*X_1, \ldots, h_*X_p),$$

and $(h^*\alpha)$ is called the *pullback* of the form α by h.

For $\alpha \in \Lambda^p(M)$ the map defined as

$$(d\alpha)(X_1, \ldots, X_{p+1}) = \sum_{i=1}^{p+1} X_i\alpha(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1})(-1)^{i+1}$$

$$+ \sum_{i<j}(-1)^{i+j}\alpha([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+1}),$$

where a circumflex (hat) over a variable means that it is excluded. This is obviously a differential form of degree $p+1$, so

$$d\colon \Lambda^p(M) \to \Lambda^{p+1}(M).$$

Since $d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^{deg\alpha}\alpha\wedge d\beta$, d is an anti-derivation of degree $+1$ in the Grassmann algebra of differential forms $\Lambda(M)$ on M^m. If $f \in \Lambda^0(M)$, then $d \cdot f = df$. It is obvious that $dd\alpha = 0\ \forall\ \alpha \in \Lambda(M)$ and it suffices to define d on the elements of $\mathcal{D}(M)$ and their derivatives df:

1) $d \cdot f = df$;
2) $d(df) = ddf \equiv 0 \quad \forall\ f \in \mathcal{D}(M)$.

Definition 14.36. $d\alpha$ is called the *exterior derivative* of the form α.

With these definitions, the next result follows easily from the classical theory of differential equations.

Theorem 14.21. *If $X \in \mathcal{T}(M)$ then there exists an open set $U \supset \{0\} \times M^m$ in $\mathbb{R}^m \times M^m$ and a differentiable mapping $\Phi\colon (t,y) \to \varphi_t(y)$ in a neighborhood U in M^m, such that $\forall\ y \in M^m$ we have: 1) $\mathbb{R} \times \{y\} \cap U$ is connected; 2) $t \to \varphi_t(y)$ is the integral curve of X through y, so $d/dt\varphi_t(y) = X(\varphi_t(y))$; and $\varphi_0(y) = y$; 3) $(t', y), (t'+t, y)$ and $(t, \varphi_{t'}(y)) \subset U$, then $\varphi_{t+t'}(y) = \varphi_t(\varphi_{t'}(y))$; 4) φ_t is unique and defines the* ***local*** *one-parameter group of X.*

Every vector field X on a compact manifold M is complete, that is φ_t is a global one-parameter group generated by X. The next three lemmas are easily proved from the definitions.

Lemma 14.4. *For all $X \in \mathcal{T}(M)$ and $\forall\ f \in \mathcal{D}(M)$*

$$(Xf)(y) = \lim_{t \to 0} \frac{f(\varphi_t(y)) - f(y)}{t} = \frac{d}{dt} f(\varphi_t(y))|_{t=0}.$$

Lemma 14.5. *The Lie derivative of a form satisfies*

$$\mathcal{L}_X \alpha = \frac{d}{dt} \varphi_t^* \alpha|_{t=0}$$

for $\forall\ X \in \mathcal{T}(M),\ \alpha \in \Lambda^p(M)$.

Lemma 14.6. *The Lie bracket of vector fields can be computed as*

$$[X, Y] = \frac{d}{dt}(\varphi_{-t^*} Y \circ \varphi_t)|_{t=0}$$

Definition 14.37. Let $\alpha \in \Lambda^1(M)$ and $d\alpha = 0$. Such an α is called a *first integral* of X if $i_X\alpha = \alpha(X) = 0$.

If $\alpha = df$, then $f = const$ on the trajectories (orbits) of the field X. Moreover, if φ_t and ψ_t are one-parameter groups of the fields X and Y, then $[X, Y] = 0 \Leftrightarrow \varphi_t \circ \psi_\tau = \psi_\tau \circ \varphi_t,\ t, \tau \in \mathbb{R}$. We also note that $X, Y \in \mathcal{T}(M)$ and $h\colon M \to N$ imply that $h_*[X, Y] = [h_* X, h_* Y]$. The next result is very useful and easy to prove.

Lemma 14.7. (Poincaré) *Let M^m be a manifold and $U \subset M^m$ an open set such that any point of U can be connected with a point $x_0 \in U$ by an integral curve of a Liouville field expressed in local coordinates as $V = \sum_i x_i \frac{\partial}{\partial x_i}$. Then for any form $\alpha \in \Lambda^p(M)$ we have $\alpha = i_V d\tilde{\alpha} + d(i_V \tilde{\alpha})$, where $\tilde{\alpha} = \int_0^\infty \varphi_t^* \alpha dt$, $[\varphi_{t,*} X = X \circ \varphi_t]$.*

Definition 14.38. If $d\alpha = 0$, then the form α is *closed*, and if $\alpha = d\beta$, then the form α is *exact*.

Definition 14.39. For all $X \in \mathcal{T}(M)$ and $\forall\ \alpha \in \Lambda(M)$ one defines $(i_X \alpha)(y) = \alpha(y)(X(y))$.

Definition 14.40. Let $X \in \mathcal{T}(M)$. As shown by Cartan, the operation defined as $L_X = i_X d + di_X$ represents the Lie derivative of degree 0 of the relevant Grassmann algebra.

It is evident that

1) $L_X d = dL_X \quad \forall\ X \in \mathcal{T}(M)$;
2) $L_{X+Y} = L_X + L_Y$;
3) $L_{fX}\alpha = fL_X\alpha + df \wedge i_X\alpha$;
4) $i_{[X,Y]} = [L_X, i_Y]$;
5) $L_{[X,Y]} = [L_X, L_Y]$.

Theorem 14.22. *Let $X \in \mathcal{T}(M)$ and suppose $X(y) \neq 0$. Then there exists a system of local coordinates $(z_1, \ldots, z_m)$ for y, such that in an open neighborhood of this point the field X is $\partial/\partial z_1$.*

Proof. Let X be a field in $\mathbb{R}^m$ that is not zero at the origin. It is clear that we can choose the coordinate system so that $X(0) = \partial/\partial x_1$. Let (U, Φ) be a local one-parameter group of diffeomorphisms of $\mathbb{R}^m$, determined by the field X:

$$\Phi(t, x_1, \ldots, x_m) = (h_1(t, x_1, \ldots, x_m), \ldots, h_m(t, x_1, \ldots, x_m)).$$

Let $K(k_1, \ldots, k_m)$ denote a differentiable mapping, determined in a neighborhood of 0 by the following condition:

$$K_i(x_1, \ldots, x_m) = h_i(x_1, 0, x_2, \ldots, x_m).$$

Since $X(0) = \partial/\partial x_1$, the Jacobi matrix $\left(\frac{\partial h_i}{\partial x_j}(0,0)\right)$ is invertible. Hence, K has inverse mapping $\ell = (\ell_1, \ldots, \ell_m)$, in any neighborhood of the origin that determines the local coordinates $y_i = \ell_i(x_1, \ldots, x_m), \quad i = 1, \ldots, m$. In these coordinates the curves $t \to (t + y_1, y_2, \ldots, y_m)$ are trajectories of the field X, so locally $\partial/\partial y_1$. □

The following result is useful, and its proof follows directly from the definitions.

Theorem 14.23. *Let $X \in \mathcal{T}(M)$ be tangent to $N \subset M$. Then the integral curve passing through $y \in N$ belongs to M $\forall\ t \in \mathbb{R}$: $\varphi^t(y) \in N \quad \forall\ y \in N$.*

14.2.8 *Differential systems*

Definition 14.41. A *differential system* of degree p (or a *p-dimensional distribution*) on a manifold M^m is a submodule $\mathcal{X}$ of the module $\mathcal{T}(M)$ satisfying the following: 1) the module $\mathcal{X}$ is stable with respect to locally finite sums; 2) $\forall\ y \in M^m, \quad \mathcal{X}_y = \{X(y)\colon X \in \mathcal{X}\} \subset T_y(M)$ is p-dimensional.

Example 14.3. A non-vanishing vector field X on M, determines a differential system $\mathcal{X} \ni X$, and $\dim \mathcal{X} = 1$.

The localization of the system concept is expressed in the following easily proved result.

Lemma 14.8. *The submodule $\mathcal{X}|_U$ is a p-dimensional differential system on U.*

Definition 14.42. Let $\mathcal{X}$ be a p-dimensional differential system on a manifold U. An *integral manifold* of the system $\mathcal{X}$ is a pair (V^p, h), where V^p is a p-dimensional manifold and h is an injective immersion of V^p into M^m such that $h_* T_y(V) = \mathcal{X}_{h(y)}$ for all $y \in V^p$.

Definition 14.43. A differential system $\mathcal{X}$ on a manifold M is *integrable*, if for all $y \in M$ there exists an integral manifold of the system $\mathcal{X}$ such that its image includes y, $y \in h(V^p)$.

Theorem 14.24. *A differential system $\mathcal{X}$ on a p-dimensional manifold M^m is integrable if and only if $\mathcal{X}$ is closed with respect to the Lie bracket $[X, Y]$(i.e $X, Y \in \mathcal{X}$ implies that $[X, Y] \in \mathcal{X}$).*

Proof. **Necessity.** Let there exist a pair (h, V^p) such that $h_* T_y(V) = \mathcal{X}_{h(y)}$, where h is an injective immersion V^p into M^m. It follows from the rank Theorem 14.12 for h, $\forall\ x \in \mathcal{X}$ on the manifold V^p that there exists a vector field Y such that $h_* Y(y) = X \circ h(y)$, $y \in V^p$, $Y \in \mathcal{T}(V)$. Now, let $[Y_1, Y_2]$ be a vector field on V^p.

On $\mathcal{T}(M)$, the vector field

$$h_*[Y_1, Y_2] = [h_* Y_1, h_* Y_2] = [X_1 \circ h, X_2 \circ h]$$

corresponds to $[Y_1, Y_2]$. Since $[Y_1, Y_2](y) \in T_y(V)$, we have $[X_1 \circ h, X_2 \circ h] \in \mathcal{X}_{h(y)}$, which we need to prove. Let $h(V^p) = N^p \subset M^m$ be p-dimensional submanifold of M^m. If $X \in \mathcal{X}$, then $\tilde{Y}$ on N is locally determined by: $\{\tilde{Y} = \{a_j\}, a_j = Y z_j$, where z_j are local coordinates about the point $x \in N^p$ like those for a submanifold of M^m, $j = 1, \dots, p\}$. Let $h^{-1}\colon N^p \to V^p$ be the inverse map of the image N^p, which exists owing to the injectivity. Thus, Y on N^p can be constructed as follows:

$$Y \circ h^{-1}(h(y)) = (h^{-1})_* \tilde{Y}(h(y)) \ \forall \ y \in V^p.$$

Sufficiency. To prove this, we use the following lemma.

Lemma 14.9. (Frobenius) *Let $\mathcal{X}$ be p-dimensional differential system on M^m. If the system $\mathcal{X}$ is closed with respect to the Lie bracket, then $\forall \ y \in M^m$ and there exist local coordinates $z_1, \dots, z_m$ in any neighborhood $U \in y$ such that the module $\mathcal{X}|_U$ is generated by the fields $\partial/\partial z_1, \dots, \partial/\partial z_p$.*

Let $X_1, \dots, X_p$ be p vector fields of $\mathcal{X}$ such that $X_1(y), \dots, X_p(y)$ generate $\mathcal{X}|_y$. Then there exists an open set $V \in y$ and local coordinates $(y_1, \dots, y_m)$ on V such that the following conditions hold:

1) $X_1, \dots, X_p$ generate $\mathcal{X}_V$;

2) $y_1(y) = 0$;

3) a local expression of X_1 in V is $\partial/\partial y_1$.

Whence, Frobenius' lemma for $p = 1$ is obvious. Next, we use the mathematical induction on p. We determine new vector fields $Y_1, \dots, Y_p \in \mathcal{X}_V$, such that $Y_1 = X_1$, and

$$Y_2 = X_2 - (X_i, y_1)X_1, \dots, Y_i = X_i - (X_i, y_1)X_1.$$

These fields have the following properties:

1) $Y_1, \dots, Y_p$ generate $\mathcal{X}_V$;

2) $[Y_i, Y_j] \in \mathcal{X}_V$;

3) $Y_i y_1 = 0 \ \ \forall \ i \geq 2$. Let N^{m-1} be the submanifold of V defined by $y_1 = 0$. Then the vector fields $Y_i, i \geq 2$, are tangent to N^{p+1} and generate on N^{m-1} a $(p-1)$-dimensional differential system $\mathcal{X}$ that is closed with respect to the Lie bracket. Hence, there exist coordinates $(\xi_2, \dots, \xi_m)$ in a neighborhood $W \ni y$ in N^{m-1} such that $\mathcal{X}'_W$ is generated (inductively) by $\partial/\partial\xi_2, \dots, \partial/\partial\xi_p$. Let us consider the following differentiable functions in in a neighborhood of a point $y \in M^m$:

$$z_1 = y_1, \quad z_i = \xi_i(y_2, \dots, y_m), \quad 1 \leq i \leq m.$$

Since these functions are independent at $y \in M$, they form a system of local coordinates. Moreover, we have

$$Y_1 = \frac{\partial}{\partial z_1}, \quad \frac{\partial}{\partial z_1}(Y_i z_j) = [Y_1, Y_i] z_j = \sum_k a^k_{ij} Y_k z_j, \quad j \geq 2,$$

where a^k_{ij} is a set of structural constants of the Lie algebra $\mathcal{X}$. Thus, for all $j > p$ the functions $(Y_i z_j)$ are solutions of a system of ordinary linear equations. Since these functions vanish when $z_1 = 0$ (because they are generated by $\partial/\partial\xi_j$), they are equal to zero in a neighborhood of the point z_1, $z_1 \in U$.

Thus, on U the equalities $Y_i = \sum_{j\le p} b_{ij}(\partial/\partial z_j)$ hold, and this means that $\mathcal{X}$ is generated by $\partial/\partial z_1, \dots, \partial/\partial z_p$ from $\mathcal{X}_U$.

Now, the remainder of the proof follows directly: Let V^p be submanifolds of U such that $z_j = const$, $j = p+1, \dots, m$. It is obvious that they are integrable manifolds given by the image of $h = i$, where $i : V^p \to U$ are embeddings, so the whole system $\mathcal{X}_V$ is integrable. □

We note that it follows from the above theorem that the Pfaffian system $\mathcal{X}^\perp = \{\alpha \in \Lambda^1(M)\colon \alpha(X) = 0 \text{ for all } X \in \mathcal{X}\}$ is of constant dimension $m - p$.

14.2.9 *The class of an ideal. Darboux's theorem*

Let M^m be an m-dimensional differentiable manifold. We define on M^m a homogeneous ideal of differential forms in the Grassmann algebra $\Lambda(M)$. Let $d\colon \Lambda(M) \to \Lambda(M)$ be an anti-derivation of degree $+1$.

Definition 14.44. A *characteristic subspace* of a homogeneous ideal I in a neighborhood $U \ni y \in M^m$ is a subspace $C_y(I) \subset \mathcal{T}(y)$ that is the intersection of associated subspaces $A(I(y))$ and $A_L A(I(y))$, where

$$A(I(y)) = \{X(y) \in \mathcal{T}(y)\colon i(X(y))I(y) \subset I(y)\},$$

$$A_L(I(y)) = \{X(y) \in \mathcal{T}(y)\colon i(X(y))dI(y) + di(X(y))L_X I(y) \subset I(y)\},$$

$$C_y(I) = A(I(y)) \cap A_L(I(y)).$$

Definition 14.45. A *characteristic system* of an ideal I at a point $y \in M^m$ is a subspace $C_y^*(I) = C_y^\perp(I) \subset \Lambda(y)$ that is orthogonal to $C_y(I)$.

Definition 14.46. The *class* of an ideal I at $y \in M^m$ is $\dim C_y^*(I)$.

Definition 14.47. A *characteristic vector field* of an ideal I is a vector field $X \in \mathcal{T}(M)$ such that $X(y) \in C_y(I)$ for all $y \in M^m$. The set $\mathcal{L}(I)$ of all characteristic vector fields is a submodule of $\mathcal{T}(M)$ that is stable with respect to locally finite sums.

The next result follows right from the definitions.

Lemma 14.10. *A vector field $X \in \mathcal{T}(M)$ is characteristic if and only if $i_X I \subset I$, $L_X I \subset I$.*

Vector fields $X \in \mathcal{L}(I)$ generate a Lie algebra over the ring $\mathcal{D}(M)$ via

$$\forall\ X, Y \in \mathcal{L}(I)\ \ fX \in \mathcal{L}(I),\ fX \in \mathcal{L}(I)\ \ \forall\ f \in \mathcal{D}(M),$$

as is evident from the following identities :

$$i_{[X,Y]} = [L_X, i_Y], \quad L_{[X,Y]} = [L_X, L_Y].$$

Definition 14.48. A *characteristic Pfaffian form* of a homogeneous ideal I is a form $\omega \in \Lambda^1(M)$ such that

$$\omega(y) \in C_y^*(I)\ \ \forall\ y \in M^m.$$

Observe that the set $\mathcal{L}^*(I)$ of all characteristic Pfaffian forms is a submodule of $\Lambda^1(M)$ that is stable with respect to locally finite sums.

Definition 14.49. A homogeneous ideal is called *regular*, if $\forall\ y \in M^m,\ \ C_y^*(I) = \mathcal{L}_y^*(I)$.

Remark 14.2. It is evident that $C_y^*(I) \supset \mathcal{L}^*(I)$.

Theorem 14.25. *Let α be a differential form in $\Lambda^p(M)$ and $I(\alpha)$ be an ideal generated by a form α: $I(\alpha) = \sum_{p\geq 0} \alpha \wedge \Lambda^p(M)$. Then the ideal $I(\alpha)$ is regular.*

Proof. We need to show that $C_y^*(I(\alpha)) = \mathcal{L}_y^*(I(\alpha))\ \ \forall y \in M^m$. To this end, we first construct the following subspace in $T_y^*(M)$:

$$\Gamma_y = \{\alpha(y)\} + \{i(\overline{X}(y) d\alpha(y))\}, \quad \overline{X}(y) \in A(I(\alpha)(y)).$$

It is obvious that $\Gamma_y = C_y^*(I(\alpha))$; thus, $\dim C_y^*(I(\alpha))$ is continuous from below and takes integer values. This shows that $\forall\ \omega_y \in C_y^*(I(\alpha))$ there exists a 1-form $\omega \in \Lambda^1(U)$ in a neighborhood $U \ni y$ such that: $\omega(y) = \omega_y$, $\forall\ z \in U\ \ \omega(z) \in C_z^*(I(\alpha))$. Whence, using the partition of unity $\{\vartheta_1, \vartheta_2\}$ for $V \ni y$, $V \subset U$ and $M \backslash U$, it follows that the form $\vartheta_1 \omega$ can be extended to 1-form (zero on $M \backslash U$) on M, where $\vartheta_1 \omega \in \mathcal{L}^*(I(\alpha))$. Therefore, since $(\vartheta_1 \omega)(y) = \omega_y \in C_y^*(I(\alpha))$, we infer that $C_y^*(I(\alpha)) = \mathcal{L}^*(I(\alpha))$. □

Remark 14.3. For a form $\alpha \in \Lambda^p(M)$, $p \geq 2$, an analogous proof of regularity fails.

Definition 14.50. An *integral manifold* of a regular ideal I of constant class $(m - p)$, $m = \dim M^m$, is a pair (V^p, f), where f is an injective immersion of the manifold V^p such that $f^* I = 0$.

Definition 14.51. An ideal I is called *integrable* if for all $y \in M$ there exists an integral manifold (V^p, f) such that $f(V^p) \ni y$.

Theorem 14.26. *Let an ideal I be regular of constant class $m-p$. Then the characteristic system $\mathcal{L}(I)$ of vector fields defines a p-dimensional integrable differential system. Moreover, the ideal I is integrable.*

Proof. Owing to the regularity of the ideal $\mathcal{L}^*(I) = \mathcal{L}^\perp(I)$, the desired result is a direct consequence of the following lemma.

Lemma 14.11. *If $\mathcal{X}$ be p-dimensional differential system on M^m, $\mathcal{X}^\perp = \{\alpha \in \Lambda^1(M)\colon \alpha(X) = 0 \ \ \forall\, X \in \mathcal{X}\}$ generates a $(n-p)$-dimensional system of Pfaffian forms on M^m and conversely.*

Let $\{\vartheta_i\}_1^\infty$ be a partition of unity on M^m, subordinate to the atlas $\{(U_i, \varphi_i)\}$. Then $\sum_i \mathcal{X}\vartheta_i$ generates differential systems on every $U_i \subset M^m(\operatorname{supp} \vartheta_i \subset U_i)$, where there exists a basis of vectors that generate $\sum_i \mathcal{X}\vartheta_i|_{U_i}$. Owing to the local finiteness of $\{\vartheta_i\}_1^\infty$ and the constant dimension of $\mathcal{X}|_{U_i}$ on M^m, this basis exists. Thus, on M^m there exists a basis $\{X_1, \ldots, X_m\}$ of vector fields in $\mathcal{T}(M)$ such that $\{X_1, \ldots, X_p\}$ generate $\mathcal{X}$ at every $y \in M^m$.

Let $\{\alpha_i\}_1^m \subset \Lambda^1(M)$ be a dual basis of the vector fields $\{X_1, \ldots, X_m\}$. Then, the Pfaffian forms $\{\alpha_{p+1}, \ldots, \alpha_m\}$ generate $\mathcal{X}^\perp$, and $\mathcal{X} = \{X \in \mathcal{T}(M) : \quad \alpha(x) = 0 \ \ \forall\, \alpha \in X^\perp\}$. It remains only to note that $\mathcal{X}^\perp$ is a Pfaffian system of constant dimension $(m-p)$.

Finally for the converse, if $\dim \mathcal{L}_y^*(I) = m - p$, then $\dim \mathcal{L}_y(I) = p$ for all $y \in M^m$. Thus, $\mathcal{L}(I)$ is a Lie algebra of vector fields on M^m. By the Frobenius theorem, the differential system $\mathcal{L}(I)$ of constant dimension p is integrable on M^m, so there exists an integral manifold (V^p, f), such that $f_* T_y(V^p) = \mathcal{L}_{f(y)}(I)\ \forall\, y \in V^p$. Whence, from the assumptions $i_{_X} I \subset I$, $\mathcal{L}_{_X} I \subset I\ \forall\, X \in \mathcal{L}(I)$ we find that $f^* I = 0$ on V^p. Thus, (V^p, f), $f\colon V^p \to M^m$ is an integrable manifold for the ideal I, so I is integrable on M. $\square$

Then as a straightforward corollary we also obtain the following classical result.

Theorem 14.27. (Cartan) *Let the ideal I be homogeneous, regular and closed ($dI \subset I$) and of constant class $(m-p)$. Then I is integrable.*

Definition 14.52. Let $I(\alpha)$ be an ideal generated by a form $\alpha \in \Lambda^1(M)$. The factor space $def(\alpha(y)) = \{X(y) \in A(I(\alpha)(y))\colon i(X(y))d\alpha(y) = f(y)\alpha(y)\}/A(I(d\alpha(y)), f \in \mathcal{D}(M)$ is called the *defect* of the form α at y.

The *index* of the defect of a form $\alpha \in \Lambda^p(M)$ at $y \in M^m$ is the number $\dim def(\alpha(y))$, and we note that this must equal 0 or 1.

Theorem 14.28. (Darboux) *Let $\alpha \in \Lambda^1(M)$ be a Pfaffian form on M^m without singularities (i.e. is non-vanishing), and suppose the class of the ideal $I(\alpha)$ is constant on M. If the class of the ideal is equal to $2s+1$, then in an open neighborhood $U \ni y$ there exist differentiable functions $(y_1, \ldots, y_{2s+1})$ $((y_1, \ldots, y_{2s}))$ such that $y_1(y) = y_2(y) = \cdots = y_{2s+1}(y) = 0$ $(y_1(y) = y_2(y) = \cdots = y_{2s}(y) = 0)$ and $\alpha|_U = dy_1 + y_2 dy_3 + \cdots + y_{2s} dy_{2s+1}$, if $\dim def\ (\alpha(y)) = 0$ $(\alpha|_U = (1 + y_1)dy_2 + y_3 dy_4 + \cdots + y_{2s-1}dy_{2s}$, if $\dim def(\alpha(y)) = 1)$.*

Proof. We shall use the following pair of lemmas:

Lemma 14.12. *Let $\alpha \in \Lambda^1(M)$ generate an ideal $I(\alpha)$ of constant class $2s+1$ on M^m and suppose that the index of the defect of the form α is zero $\forall\ y \in M^m$. Then $\forall\ y \in M^m$ $\exists f \in \mathcal{D}(V)$, $f(y) = 0$, defined in a neighborhood $V \ni y$ such that the form $\alpha_1 = \alpha|_V - df$ has no singularities, $\dim def(\alpha_1(y)) = 1$ and the ideal $I(\alpha_1)|_V$ is of constant class $2s-1$ on V.*

Lemma 14.13. *Let $\alpha \in \Lambda^1(M^m)$ be a form without singularities and with ideal $I(\alpha)$ of constant class $2s-1$ and with index of defect unity. Then for all $y \in M^m$ there exists a differentiable function $g \in \mathcal{D}(W)$, where W is an open neighborhood of $y \in M^m$, such that $g(y) = 0$ and there exists a non-degenerate form $\alpha_2 = (1+g)\alpha|_W$ having class $2s-1$ on W and $\dim def(\alpha_2(y)) = 0$.*

Proof of Lemma 14.12. Since the index of the defect of the form α is equal to zero, $\mathcal{L}^*(I(\alpha \wedge (d\alpha)^s)) = \mathcal{L}^*(I(\alpha))$ (and $\mathcal{L}^*(I(d\alpha)^s) = \mathcal{L}^*(I(d\alpha)))$, and the system $\mathcal{L}^*(I(d\alpha))$ has dimension $2s+1$. It follows that in a neighborhood $V \ni y$ $\exists$ coordinates $y_1, \ldots, y_{2s+1}$ $(y_1(y) = \cdots = y_{2s+1}(y) = 0)$, such that:

1) $(d\alpha)^s|_V = dy_2 \wedge \cdots \wedge dy_{2s+1}$;

2) $\alpha \wedge (d\alpha)^s|_V = dy_1 \wedge dy_2 \wedge \cdots \wedge dy_{2s+1}$ (because the index of the defect of α $\forall\ y \in M^m$ is zero);

3) $\alpha|_V = dy_1 + \sum_{i=2}^{2s+1} a_i \partial y_1$, where $\sum_{i=2}^{2s+1} a_i^2(z) \neq 0$ (then $d\alpha|_V = 0$ and the class of the ideal $I(\alpha)$ is not $2s+1$). The form $\alpha_1 = \alpha|_V - dy_1$ has no singularities on V and satisfies:

a) $(d\alpha_1)^s = (d\alpha)^s \neq 0$;

b) $(d\alpha_1)^s \wedge \alpha_1 = 0$, thus α_1 generates the ideal $I(\alpha_1)$ of class $2s-1$ and $\dim def(\alpha(y)) = 0$. $\square$

Proof of Lemma 14.13. The submodule of characteristic forms

$$\mathcal{L}^*(I(d\alpha)^s) = \mathcal{L}^*(I(d\alpha))$$

has constant dimension $2s$, and the characteristic system

$$\mathcal{L}(I(d\alpha)^s) = \mathcal{L}I(d\alpha) = \mathcal{L}^*(I(d\alpha))^\perp$$

is an $(m-2s)$-dimensional integrable differential system. Denote by $\mathcal{R}^*$ a family of Pfaffian forms on M^m such that $\forall\ \omega \in \mathcal{R}^*\colon \omega(y) \in A^\perp(I(\alpha \wedge (d\alpha)^{s-1}(y)))\ \forall\ y \in M^m$. The set $\mathcal{R}^*$ is a submodule of $\Lambda^1(M)$ that is stable with respect to locally finite sums. Since the form $\alpha \wedge (d\alpha)^{s-1}(y)$ generates the ideal $I(d \wedge (d\alpha)^{s-1}(y))$ of constant rank $2s-1$, it is easy to verify that the weak regularity condition $I(\alpha \wedge (d\alpha)^{s-1})$ in $y \in M^m$: $A^\perp(I(\alpha \wedge (d\alpha)^{s-1}))(y) = \mathcal{R}^*_y$ holds. Moreover, $\mathcal{R}^* \subset \mathcal{L}^*(I(d\alpha)^s)$. Now we construct a differential system of constant dimension $m-2s+1$: $\mathcal{R} = \{X \in \mathcal{T}(M)\colon i_X(\alpha \wedge (d\alpha)^{s-1}) = 0\}$ that is orthogonal to $\mathcal{R}^*$: $\mathcal{R}^* = \mathcal{R}^\perp$. Since this system is integrable and $\mathcal{L}((d\alpha)^s) \subset \mathcal{R}$ (because the index of the defect of the form α is equal to one), $\mathcal{R}$ is a Lie algebra, and $\forall\ X, Y \in \mathcal{R}$

$$i_{[X,Y]}(\alpha \wedge (d\alpha)^{s-1}) = -i_Y d i_X(\alpha \wedge (d\alpha)^{s-1}) - i_Y i_X (d\alpha)^s = -i_Y i_X (d\alpha)^s = 0.$$

Therefore, we can find a system of local coordinates $(z_1, \dots, z_m)$ in a neighborhood $W \ni y$, such that $z_1(y) = \dots = z_m(y) = 0$, and

1) $(d\alpha)^s|_W = dz_1 \wedge \dots \wedge dz_{2s}$;

2) $\alpha \wedge (d\alpha)^{s-1}|_W = b dz_2 \wedge \dots \wedge dz_{2s}$, where $b(z) \neq 0\ \ \forall\ z \in W$. (since $\mathcal{R}^*_y$ has a basis $dz_1, \dots, dz_{2s}$ of $2s-1$ elements and the form $\alpha \wedge (d\alpha)^{s-1}$ is of degree $2s-1$. The form $\alpha \wedge (d\alpha)^{s-1} \in \Lambda^{2s-1}(M)$ is uniquely determined by the basis $dz_1, \dots, dz_{2s}$ and a function $b(z)$ of the form $\alpha \wedge (d\alpha)^{s-1} = b(z) dz_2 \wedge \dots \wedge dz_{2s},\ b \neq 0$). If h is a differentiable function on W and $\alpha_2 = h\alpha|_W$, then

$$\alpha_2 \wedge (d\alpha_2)^{s-1} = h^s(\alpha \wedge (d\alpha)^{s-1})|_W,$$

$$(d\alpha_2)^s = h^{s-1}[s\ dh \wedge \alpha \wedge (d\alpha)^{s-1}|_W + h(d\alpha)^s|_W].$$

If $h = 1+g$, $g = (exp(-B/s)) - 1$, and $B = \int_0^{z_1}(dz_1/b)$, then $\alpha_2 = (1+g)\alpha|_W$ has class equal to $2s-1$ if $g(y) = 0$ (since $\dim def(\alpha_2(y)) = 0$, the class is equal to $2s-2+1$). $\square$

Proof of Darboux's theorem. We use induction on the class of the ideal $I(\alpha)$ of a Pfaffian form $\alpha \in \Lambda^1(M)$. It is obvious that if a form is of constant class equal to zero, then it is equal to zero. Let us assume that $I(\alpha)$ has $\dim def(\alpha(y)) = 0$ and is of constant class $2s+1$. Then in a neighborhood $V \ni y$ there exists a function f, $f(y) = 0$, such that the form $\alpha_1 = \alpha|_V - df$ has no singularities if $s > 0$ and has a constant class $2s - 1$ on V. Then in a neighborhood $U \subset V$ of a point y there exist $2s$ functions $g_1, \ldots, g_{2s}$, $g_1(y) = \cdots = g_{2s}(y) = 0$ such that $\alpha_1|_U = (1 + g_1)dg_2 + g_3 dg_4 + \cdots + g_{2s-1}dg_{2s}$. Now set $y_1 = f + g_2$ $(y_1 = f_1 s = 0)$, $y_i = g_{i-1}$, $i = 2s + 1$. These functions are equal to zero at y and $\alpha|_U = dy_1 + y_2 dy_3 + \cdots + y_{2s}dy_{2s+1}$. Next, we assume that $I(\alpha)$ is of constant class $2s+1$, $\dim def(\alpha(y)) = 1$, and α has no singularities. Then, in a neighborhood $W \ni y$ there exists a differentiable function g with $g(y) = 0$ such that $\alpha_2 = (1+g)(\alpha|_W)$ is of constant class $2s+1$ on W and $\dim def(\alpha(y)) = 0 \, \forall \, y \in W$. Therefore, in a neighborhood $U \in W$ of y there exist differentiable functions $f_1, \ldots, f_{2s+1}$ $(f_1(y) = \cdots = f_{2s+1}(y) = 0)$ such that $\alpha_2|_U = df_1 + f_2 df_3 + \cdots + f_{2s}df_{2s+1}$. Now, assume that $y_1 = -(g/(1+g))$, $y_i = f_{i-1}/(1+g)$ where $i = 3, 5, \ldots, 2s+1$, $y_i = f_{i-1}$, $i = 2, 4, \ldots, 2s+2$. These functions are equal to zero at y and

$$\alpha|_U = (1 + y_1)dy_2 + y_3 dy_4 + \cdots + y_{2s+1}dy_{2s+2}. \qquad \square$$

Remark 14.4. 1) The functions y_i, $i = 1, 2, \ldots, 2s+2$ introduced in the above proof are independent at y; 2) If a Pfaffian form α is of constant class $2s+1$ and has index of defect equal to zero, then α has no singularities on M^m. If α has index of defect equal to one, then α can have singularities and it is not possible to formulate a general local model.

Theorem 14.29. *Let $\omega^{(2)}$ be a closed differential form of degree 2 and of constant class $2s$ on M^m. Then for any point $y \in M^m$, there exist $2s$ differentiable functions $y_1, \ldots, y_{2s}$ in a neighborhood $U \ni y$ such that $y_1(y) = \cdots = y_{2s}(y) = 0$ and*

$$\omega^{(2)}|_U = dy_1 \wedge dy_2 + \cdots + dy_{2s-1} \wedge dy_{2s}.$$

Proof. It follows from the Poincaré lemma that there exists a Pfaffian form α in a neighborhood $V \ni y$ such that $\partial\alpha = \omega^{(2)}|_V$. The class of the ideal $I(\alpha)$ at y is equal to $2s-1$ or $2s+1$. Let $2s < m$ and $f \in \mathcal{D}(M)$ be such that the class of the form $\alpha + df$ is also equal to $2s+1$ and the index of the defect of the form is zero in a neighborhood $W \subset V \ni y$. Then, in a neighborhood $U \subset W$ there exist smooth functions $y_1, \ldots, y_{2s+1}$ such that $y_j(y) = 0$, $j = 1, \ldots, 2s+1$ and $\alpha|_U = y_1 dy_2 + y_3 dy_4 + \cdots + y_{2s+1}dy_{2s} + y_{2s+1}$. Whence, $d\alpha|_U = \omega^{(2)}|_U = dy_1 \wedge dy_2 + \cdots + dy_{2s+1} \wedge dy_{2s}$. If $2s = m$, we can assume that the index of the defect is one, so the form α has no singularities in $W' \subset V$ and there exist $2s = m$ functions $z_1, \ldots, z_{2s}$ in $U' \subset W' \ni y$, $z_1(y) = \cdots = z_{2m}(y) = 0$, such that $\alpha|_{U'} = (1 + z_1)dz_2 + \cdots + z_{2s-1}dz_{2s}$. Thus, $d\alpha = \omega^{(2)}|_{U'} = dz_1 \wedge dz_2 + \cdots + dz_{2s-1} \wedge dz_{2s}$. $\square$

14.2.10 Dynamical systems on symplectic manifolds. Complete integrability and ergodicity

Definition 14.53. A *dynamical system on a symplectic manifold* $(M^{2n}, \omega^{(2)})$ is a vector field $X \in \mathcal{T}(M)$ such that the Pfaffian form $i_X \omega^{(2)}$ is closed (a form $\omega^{(2)}$ on M of maximal class $2n$ and index of defect unity).

Definition 14.54. If $i_X \omega^{(2)} = -\alpha$ and $d\alpha = 0$, the form $\alpha \in \Lambda^1(M)$ is said to be *quasi-hamiltonian.* If, in addition, $\alpha = dH$, then the function $H \in \mathcal{D}(M)$ is called *Hamiltonian.*

Theorem 14.30. *Let (N^{2n-1}, h) be an integrable manifold of an ideal $I(\alpha)$ on M^{2n}. Then: 1) a vector field X is tangent to $h(N^{2n-1})$ in M^{2n}; 2) $h^*\omega^{(2)}$ is a closed form of degree 2 and of a constant class $2s-2$ on N^{2n-1}; 3) a differential system $\mathcal{L}(I(h^*\omega^{(2)}))$ is generated by a vector field Y induced by a field X on N^{2n-1}.*

Proof. The theorem is obvious if $\alpha(X)$ and $h^*T_y(N^{2n-1}) \ni X(h(y))$. So let $(e^1, \dots, e^{2n})$ be a basis in $T_{h(y)}(M)$ such that for a dual basis $(\varepsilon_1, \dots, \varepsilon_{2n})$ one has $\alpha(h(y)) = \varepsilon_{2n}$; $\omega^{(2)}(h(y)) = \varepsilon_1 \wedge \varepsilon_2 + \dots + \varepsilon_{2n-1} \wedge \varepsilon_{2n}$; $X(h(y)) = e^{2n-1}$.

The linear forms $\eta_i = (h_{y^*})^* \varepsilon_i$, $1 \le i \le 2n-1$, form a basis in $T_y^*(N) = \Lambda_y^1(N)$ and $(h^*\omega^{(2)})(y) = \eta_1 \wedge \eta_2 + \dots + \eta_{2n-3} \wedge \eta_{2n-2}$. Hence, $dh^*\omega^{(2)} \equiv h^* d\omega^{(2)} = 0$; the class of the form $h^*\omega^{(2)}$ is constant and equal to $2n-2$ on N^{2n-1}; and a characteristic subspace $C_y(I(h^*\omega^{(2)}))$ is generated by a vector field Y induced by a field X on N^{2n-1}. In fact, $i(Y(y))(h^*\omega^{(2)})(y) = i(X(h(y)))\omega^{(2)}(y) \equiv 0$ and $\dim\ C_y(I(h^*\omega^{(2)})) = 2n-1 = \dim\ N^{2n-1}$; and the latter also follows from the condition $\dim\ C_y^*(I(h^*\omega^{(2)})) = 2n-2 = \dim\ N^{2n-1} - \dim\ C_y(I(h^*\omega^{(2)}))$. Hence, $\dim\ C_y(I(h^*\omega^{(2)})) = 2n-1-(2n-2) = 1$. $\square$

Theorem 14.31. *There exists a differential form $\pi \in \Lambda^{2n-1}(U)$ such that $(\omega^{(2)}|_U)^n = \alpha \wedge \pi$. For this form $\mathcal{L}_X \pi = \alpha \wedge \rho$, $\rho \in \Lambda^{2n-2}(U)$. If π' is another form on U such that $(\omega^{(2)}|_U)^n = \alpha \wedge \pi'$, then $\pi' = \pi + \alpha \wedge \sigma$, where $\sigma \in \Lambda^{2n-2}(U)$.*

Proof. Since $\alpha \wedge (\omega^{(2)})^n = 0$, there exists a differential form $\pi \in \Lambda^{2n-1}(U)$ such that $(\omega^{(2)}|_U)^n = \alpha \wedge \pi$. Thus, $0 = L_X(\omega^{(2)}|_U)^n = L_X(\alpha \wedge \pi) = \alpha \wedge (L_X \pi)$, and further $L_X \pi = \alpha \wedge \rho$, $\rho \in \Lambda^{2n-2}(U)$. If π' is another form on U, then $\alpha \wedge (\pi - \pi') = 0$, and thus $(\pi - \pi') = \alpha \wedge \sigma$, where $\sigma \in \Lambda^{2n-2}(U)$, $U \in M^{2n}$. $\square$

Let (N^{2n-1}, h) be an integral manifold of an ideal $I(\alpha)$, and $\pi \in \Lambda^{2n-2}(U)$ a differential form such that $(\omega|_U)^n = \alpha \wedge \pi$. Then the form $\Pi = h^*\pi$ has the following properties:

1) Π is independent of a differential form $\pi \in \Lambda^{2n-1}(U)$ satisfying $(\omega|_U)^n = \alpha \wedge \pi$;

2) Π is a volume form on N^{2n-1}, $\pi \neq 0$;

3) for a vector field Y induced by a field X on N^{2n-1}, $L_Y \Pi = 0$.

A proof of the above reduces to the following identity:

$$\mathcal{L}_Y \Pi = \mathcal{L}_Y \cdot h^*\pi = h^*\mathcal{L}_X \pi = h^*(\alpha \wedge \rho) = (h^*\alpha \wedge h^*\rho) = 0.$$

Let X be a dynamical system on a symplectic manifold $(M^{2n}, \omega^{(2)})$ and $\alpha = -i_X \omega^{(2)}$. Let $\beta \in \Lambda^1(M)$ and $\gamma \in \Lambda^1(M)$, $i_{X_\beta}\omega^{(2)} = \beta$, $i_{X_\gamma}\omega^{(2)} = \gamma$, where $X_\beta, X_\gamma \in \mathcal{T}(M)$. Then a Poisson bracket is defined by $(\gamma, \beta) = i_{[X_\gamma; X_\beta]}\omega^{(2)}$. If α and $\beta \in \Lambda^1(M)$ are closed, then $(\alpha, \beta) = -d(\omega^{(2)}(X_\alpha, X_\beta))$. If $\alpha = df, \beta = dg$, and $f, g \in \mathcal{D}(M)$, then $\{f, g\} = -\omega^{(2)}(X_{df}, X_{dg}) = X_{df}g = -X_{dg}f$.

Definition 14.55. Two forms $\alpha, \beta \in \Lambda^1(M)$ are in *involution* if $\omega^{(2)}(X_\alpha, X_\beta) \equiv 0$. Two functions $f, g \in \mathcal{D}(M)$ are in *involution* if $\omega^{(2)}(X_{df}, X_{dg}) = 0$.

Thus, if α and $\beta \in \Lambda^1(M)$ are closed, then $(\alpha, \beta) = -d(\omega^{(2)}(X_\alpha, X_\beta)) = 0$, because of $\omega^{(2)}(X_\alpha, X_\beta) = 0$. A closed form $\beta \in \Lambda^1(M)$ is a first integral of a dynamical system, $X(\beta(X)) = 0$, if and only if $\omega(X_\alpha, X_\beta) = 0$, then α and β are in involution, where $\alpha = -i_X\omega$.

Theorem 14.32 (Galisot–Reeb). *Let $\beta_1, \ldots, \beta_{n-1} \in \Lambda^1(M)$ be first integrals of the dynamical system X on $(M^{2n}, \omega^{(2)})$ that have the following properties: 1) $\beta, \beta_1, \beta_2, \ldots, \beta_{n-1}$ are independent on an open set $U \subset M^{2n}$, where $\beta = -i_X\omega^{(2)}$; 2) forms $\beta, \beta_i, 1 \le i \le n-1$, are in involutions in pairs. Then there exist n Pfaffian forms $\gamma, \gamma_1, \gamma_2, \ldots, \gamma_{n-1}$ on $U \subset M^{2n}$ satisfying: 1) $\omega^{(2)}|_U = \beta \wedge \gamma + \sum_i \beta_i \wedge \gamma_i$; 2) the differentials $d\gamma$ and $d\gamma_i$ belong to the ideal in $\Lambda(U)$ generated by the forms β, β_1, …, β_{n-1}.*

Proof. Let $X_1, \ldots, X_{n-1}$ be vector fields on M^{2n} satisfying $i_{X_i}\omega^{(2)} = \beta_i$, $1 \le i \le n-1$. Then $\beta_i(X) = \beta_i(X_j) = 0$, $\beta(X) = \beta(X_j) = 0$, $1 \le i, j \le n-1$. Hence, $i_X i_{X_{n-1}} \ldots i_{X_1}(\omega^{(2)})^q = \pm q(q-1)\cdots(q-n+1)\beta_1 \wedge \cdots \wedge \beta_{n-1} \wedge \beta \wedge (\omega^{(2)})^{-n+q}$.

Assuming that $q = n+1$, we find: $\beta_1 \wedge \cdots \wedge \beta_{n-1} \wedge \beta \wedge \omega^{(2)} = 0$. Thus, $\omega^{(2)}$ belongs to an ideal in $\Lambda(U)$ generated by forms β, β_1, …,

β_{n-1}. Hence, there exist n Pfaffian forms $\gamma, \gamma_1, \ldots, \gamma_{n-1}$ on U such that $\omega^{(2)}|_U = \beta \wedge \gamma + \sum_i^{n-1} (-\gamma_i) \wedge \beta_i$.

Consequently, the forms α, γ, β_1, $\ldots$, β_{n-1}, γ_1, $\ldots$, γ_{n-1} are independent on U (because dim $C_U^*(I(\omega^{(2)})) = 2n$). Multiplying the equality $(d\omega^{(2)})|_U = -\alpha \wedge d\gamma - \sum_i \beta_i \wedge d\gamma_i \equiv 0$ by $\beta \wedge \beta_1 \wedge \cdots \wedge \beta_{i-1} \wedge \beta_{i+1} \wedge \cdots \wedge \beta_{n-1}$ (respectively on $\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_{n-1}$), we obtain $\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_{n-1} \wedge \beta \wedge d\gamma_i = 0$ (respectively $\beta_1 \wedge \beta_2 \cdots \beta_{n-1} \wedge d\gamma_i = 0$), Thus the proof is complete. $\square$

Let the Pfaffian forms $\beta, \beta_1, \ldots, \beta_{n-1}$ generate an integrable ideal $I(\beta)$ of class n on U. If (M_β^n, h) is an integrable manifold of the ideal $I(\beta)$, then a field X is tangent to $h(M_\beta^n)$ and it induces on M_β^n a vector field Z. Let us assume $\pi = h^*\gamma$, $\pi_i = h^*\gamma_i$, $1 \le i \le n-1$. This leads to the following result on integrating by quadratures and complete integrability.

Theorem 14.33 (Liouville–Arnold–Cartan–Jost). *With the notation and assumptions in the Galisot–Reeb theorem, the following obtain: 1) The forms $\pi, \pi_1, \ldots, \pi_{n-1}$ are independent on M_β^n; 2) The forms $\pi_1, \ldots, \pi_{n-1}$ are first integrals of the field defined by Z: $\partial \pi_i = 0 = i_Z \pi_i$; 3) $\partial \pi = 0$ and $\pi(Z) = 1$.*

Proof. We have $\pi, \pi_1, \ldots, \pi_{n-1}$, which are *independent* on M_β^n and $d\pi = d\pi_1 = \cdots = d\pi_{n-1} = 0$. Since $\pi(Z) = \gamma(X)$, $\pi_i(Z) = \gamma_i(X)$, it follows from $i_X \omega^{(2)} = -\gamma(X)\alpha - \sum_i \gamma_i(X)\beta_i \equiv -\alpha$ that $\gamma(X) = 1$, $\gamma_i(X) = 0$, $i = 1 \ldots, n-1$. $\square$

Remark 14.5. It is easy to see that $\mathcal{L}(I[\beta]) = \{X \in \mathcal{T}(M) \colon i_X I(\beta) \subset I(\beta),\ \mathcal{L}_X I(\beta) \subset I(\beta)\} = \{\sum_{i=1}^{n-1} f_i X_i + fX\}$, where $i_{X_i}\omega^{(2)} = \beta_i$, $\{X_i, X\}$ are linearly independent because $\beta_1, \ldots, \beta_{n-1}, \beta$ are independent on M^{2n}. Now, the ideal is closed, so we have:

$$I(\beta): \quad dI(\beta) \subset I(\beta) \ \ (\text{as } d\beta = d\beta_i = 0).$$

A characteristic system of vector fields $\mathcal{L}(I(\beta))$ orthogonal to $\mathcal{L}^*(I(\beta))$ is generated by the Pfaffian forms $\beta_1, \ldots, \beta_{n-1}, \beta$. Moreover, the class of the ideal $I(\beta)$ is constant on M^{2n} and is equal to n. Now, we assume that all $\beta_i = dF_i$, $F_i \in D(M)$, are exact first integrals. We prove that if the invariant integral manifold (M_β^n, h) is compact, then a dynamical system X on M^{2n} becomes linear on the torus $\mathbb{T}^n$, so the latter is diffeomorphic to the integral submanifold M_β^n. In addition, the motion on the torus $\mathbb{T}^n$ is ergodic if the frequencies $\omega_i \in \mathbb{R}$ are rationally independent, and a trajectory of the flow X on $\mathbb{T}^n$ is dense on $\mathbb{T}^n$ [3, 14, 205, 262]. $\square$

We next prove the following lemma.

Lemma 14.14. *Suppose the vector fields $\{X, X_i\}_{i=1}^{n-1}$ are linearly independent and commutative on the integral manifold (M_β^n, h) of an ideal $I(\beta)$. Then $[X_i, X_j] = [X_i, X] = 0 \ \forall\ i, j = 1, \ldots, n-1$.*

Proof. It follows from that $i_{X_j}\omega^{(2)} = \beta_j$, and the β_j are conservation laws for X: $\beta_j(X) = 0 = i_{X_j}\omega^{(2)}(X) = \omega^{(2)}(X_j, X) = -\beta(X_j) = 0$. Hence, the system of forms $\{\beta_j, \beta\}$ is linearly independent on M^{2n}. $\square$

Lemma 14.15. *Let N^n be a compact and connected differential manifold, on which there are n invariant vector fields, linearly independent at every point on N^n. Then N^n is diffeomorphic to the n-dimensional torus $\mathbb{T}^n$.*

Proof. Denote by $g_i^{t_i}$, $i = 1, \ldots, n$, 1-parameter groups of diffeomorphisms on N^n. Since these fields commute, the groups $g_i^{t_i}, g_j^{t_j}$ commute too. Hence, we can find an action g of a commutative (additive) group $\mathbb{R}^n = \{t\}$ on a manifold N^n, namely $g^t \colon N \to N$, where $g^t = g_1^{t_1} \circ g_2^{t_2} \circ \cdots \circ g_n^{t_n}$ $(t = (t_1, t_2, \ldots, t_n)^\intercal \in \mathbb{R}^n)$.

It is obvious that $g^t \circ g^s = g^s \circ g^t \ \forall\ t, s \in \mathbb{R}^n$. For a given point $x_0 \in N$, we can define a mapping $g \colon \mathbb{R}^n \to N^n$, where $g(t) = g^t x_{(0)}$.

This mapping g naturally yields a chart on N in some neighborhood $U \ni x_0$. Thus, $\exists U \ni x_0$, that $g \colon V \rightleftarrows U$, where $V = g^{-1}(U)$ is diffeomorphism, as follows from the implicit function theorem and the linear independence of vector fields $\{X, X_j : 1 \le j \le n\}$. As a result, we also see that $g \colon \mathbb{R}^n \to N^n$ is a mapping on the whole manifold N^n, where the neighborhoods $U_{i+1} \cap U_i \ne \varnothing \ \forall\ i = 1, \ldots, n$.

Thus $\forall\ x \colon g^t(x_0) = \prod_{i=1}^n g^{t_i}(x_0) = x$, where the g^{t_i} uniquely determine a covering of the curve $\ell(x_0, x)$ by neighborhoods U_i, $1 \le i \le n$. Notice, that the map $g \colon \mathbb{R}^n \to N^n$ cannot be injective, since N^n is compact but $\mathbb{R}^n$ is not. $\square$

Definition 14.56. A *stationary group* of a point x_0 is a set $\Gamma \subset \mathbb{R}^n$ such that $\forall\ t \in \Gamma \quad g^t(x_0) = x_0$ for the group action defined above. Obviously Γ is a subgroup of $\mathbb{R}^n$.

The group Γ is discrete. Indeed, in a sufficiently small neighborhood $V \ni t$ there exists only one element of the group Γ, the element $t = 0$, as follows from the diffeomorphism $g \colon V \rightleftarrows U$, $U \ni x_0$.

It is easy to prove, that there exist $k \in \mathbb{Z}_+$ linearly independent vectors e_i in $\mathbb{R}^n$ such that $\Gamma = \{\sum_{i=1}^k \ell_i e_i\}$, where $\ell_i \in \mathbb{Z}$. To verify this, we first consider the following construction: the product

$$\mathbb{T}^k \times \mathbb{R}^{n-k} = \{(\varphi_1, \ldots, \varphi_k : (\mathrm{mod}\ 2\pi);\ y_1, \ldots, y_{n-k})\}$$

together with natural map $p : \mathbb{R}^n \to \mathbb{T}^k \times \mathbb{R}^{n-k}$, where $p(\varphi, y) = (\varphi \mod 2\pi, y)$. Points $f_j \in \mathbb{R}^n$, where $f_j = (0, \ldots, 2\pi, \ldots, 0;\ 0)$, $1 \le j \le k$, are sent to zero by this mapping. If the stationary subgroup Γ is generated by elements $e_j \in \Gamma$, $j = 1, \ldots, k$, , then the linear space $\mathbb{R}^n = \{(\varphi, y)\}$ can be mapped on the space $\mathbb{R}^n = \{t\}$ so that the vectors f_j are transformed into e_j, $1 \le j \le k$. Denote this isomorphism by $A\colon \mathbb{R}^n \to \mathbb{R}^n$. It is also evident that $\mathbb{R}^n = \{(\varphi, y)\}$ define charts on $\pi^k \times \mathbb{R}^{n-k}$, and $\mathbb{R}^n = \{t\}-$ charts on M^n_β. Owing to the commutativity of the diagram below (Fig. 14.3), we find that the induced map $\tilde{A}\colon \mathbb{T}^k \times \mathbb{R}^{n-k} \to M^n_\beta$ is a diffeomorphism. Since the manifold M^n_β is compact, $k = n$ and M^n_β is diffeomorphic with the n-dimensional torus $\mathbb{T}^n$. □

$$\begin{array}{ccc} \mathbb{R}^n = \{(\varphi, y)\} & \xrightarrow{A} & \mathbb{R}^n = \{t\} \\ p\downarrow & & \downarrow g \\ \mathbb{T}^k \times \mathbb{R}^{n-k} & \xrightarrow{\tilde{A}} & M^n_\beta \end{array}$$

Figure 14.3

From the above lemma it follows that the evolution of the Hamiltonian system on the invariant torus M^n_β is quasiperiodic, since $\varphi = A^{-1}t$ and $d\varphi_i/dt = \omega_i(f)$, $1 \le i \le n$ represent its frequencies. If they are rationally independent, the trajectory on M^n_β is dense, and the evolution is ergodic. If they are rationally dependent, the orbit will be periodic.

To introduce the standard action-angle variables we denote by $\sigma_j \in H^1(M^n_\beta;\ \mathbb{Z})$, $j = 1, \ldots, n$, the set of cycles comprising a basis of the one-dimensional integral cohomology group of the manifold M^n_β. Consider the integrals $I_j = \frac{1}{2\pi} \oint_{\sigma_j} \alpha$, $1 \le j \le n$, where $d\alpha = \omega^{(2)} \in \Lambda^2(M)$ is the resulting symplectic structure on M^{2n}. Since the integrals I_j, $1 \le j \le n$, are functionally independent, solving the equations $I_j = I_j(f)$, with respect to $f \in \mathbb{R}^n$, one finds that the torus M^n_β is to be mapped onto M^n_I. Next, we introduce the multi-valued maps $\Phi(I, Q) = \int_{\sigma(Q,Q_0)} \alpha$, where $I = (I_1, I_2, \ldots, I_n)$, $Q = (Q_1, Q_2, \ldots, Q_n)$, $\sigma(Q, Q_0)$ is some smooth curve lying on the torus M^n_β and starting (ending) at Q_0 (Q). Mapping $\Phi(I, Q)$ we construct a canonical mapping $\tilde{\Phi}(I, \varphi) \to (P, Q)$, where $\varphi = (\varphi_1, \ldots, \varphi_n) \in \mathbb{T}^n$, $P = (P_1, P_2, \ldots, P_n)$, and for all $j = 1, \ldots, n$, $P_j = \frac{\partial \Phi}{\partial Q}$, $\varphi_j = \frac{\partial \Phi}{\partial I_j}$, which includes the invariance of the symplectic structure $\omega^{(2)} \in \Lambda^{(2)}(M)$, namely $\tilde{\Phi}^* \omega^{(2)} = \omega^{(2)}$.

Calculating the variation of functions $\varphi_j \in D(M_\beta^n)$, $1 \leq j \leq n$, by varying $Q \in M_\beta^n$ along cycles $\sigma_k \in H^1(M_\beta^n;\ \mathbb{Z})$, we find that

$$\oint_{\sigma_k} d\varphi_j = \frac{\partial}{\partial I_j}\oint_{\sigma_k} d\Phi = 2\pi\frac{\partial}{\partial I_j}\oint_{\sigma_k}\alpha = 2\pi\frac{\partial I_k}{\partial I_j} = 2\pi\delta_{jk}.$$

Whence, we see that the values $\{\varphi_j \in \mathbb{T}^1 \colon j = 1, \ldots, n\}$ are the *angle* variables on the torus M_I.

As the maps $\tilde{\Phi}$ are canonical, the transformed system has the following form $\{I_j, \varphi_j \colon\ j = 1, \ldots, n\}$

$$\frac{dI_j}{dt} = 0, \qquad \frac{d\varphi_j}{dt} = \frac{\partial H_1(I)}{\partial I_j}$$

for all $j = 1, \ldots, n$. Consequently, $I_j = I_j(f)$ and $\varphi_j = \omega_j(t - t_0) + \varphi_j^{(0)}$, where $\omega_j = \partial H_1(I)/\partial I_j$, $1 \leq j \leq n$, are the frequencies of the motion on the torus M_β^n. Accordingly the motion of the Hamiltonian system on the invariant torus M_β^n is ergodic if frequencies are linearly independent over the ring $\mathbb{Z}$. Moreover, in variables $(I, \varphi) \in M_I^n$, the Hamiltonian system on M^{2n}, with Hamiltonian function $H = f_1 \in \mathcal{D}(M)$, is clearly integrable by quadratures. This approach to integrating Hamiltonian systems is called the *Hamiltonian–Jacobi method*.

14.3 Integrability of Lie-invariant geometric objects generated by ideals in Grassmann algebras

14.3.1 *General setting*

It is well known [40, 41, 314] that motion planning, numerically controlled machining, and robotics are just a few of many areas of manufacturing automation in which the analysis and representation of swept volumes plays a crucial role. Swept volume modeling is also an important part of task - oriented robot motion planning. A typical motion planning problem consists in maneuvering a collection of solid objects around obstacles from an initial to a final configuration. This may include, in particular, solving the collision detection problem.

When a solid object undergoes a rigid motion, the totality of points through which it passes constitutes a region in space called the *swept volume*. To describe the geometrical structure of the swept volume we pose this problem as one of a geometric study of a manifold swept via its surface points using powerful tools from both modern differential geometry and nonlinear dynamical systems theory [3, 14, 40, 173, 326] on manifolds. For some special cases of the Euclidean motion in the space $\mathbb{R}^3$ one can construct a very rich hydrodynamical system modeling a sweep flow, which appears to be a completely integrable Hamiltonian system having a special [227] Lax representation. To describe in detail these and other properties of nonlinear dynamical systems we develop the differential geometric *Cartan theory* of *Lie-invariant geometric objects* generated by closed ideals in the Grassmann algebra [157, 382] as well as investigating some special examples of Euclidean motions in $\mathbb{R}^3$ leading to Lax integrable dynamical systems on functional manifolds.

This theory also serves as a differential-geometric background of the gradient holonomy algorithm developed in Chap. 5 to study the integrability properties of infinite-dimensional dynamical systems on functional manifolds.

Let a Lie group G act transitively on an analytic manifold Y, that is, the action $G \times Y \xrightarrow{\rho} Y$ generates some nonlinear exact representation of the Lie group G on the manifold Y. In the framework of Cartan's differential geometric theory, the representation $G \times Y \xrightarrow{\rho} Y$ can be described by means of a system of differential 1-forms

$$\bar{\beta}^j := dy^j + \sum_{i=1}^{r} \xi_i^j \bar{\omega}^i(a; da) \in \Lambda^1(Y \times G) \tag{14.5}$$

in the *Grassmann algebra* $\Lambda(Y \times G)$ on the product space $Y \times G$, where $\bar{\omega}^i(a; da) \in T_a^*(G)$, $1 \leq i \leq r = \dim G$ is a basis of left invariant Cartan forms of the Lie group G at a point $a \in G$, $y := \{y^j : 1 \leq j \leq n = \dim Y\} \in Y$ and $\xi_i^j : Y \times G \to \mathbb{R}$ are smooth real-valued functions. The following theorem of Cartan [3] is basic in describing a geometric object invariant with respect to the above group action $G \times Y \xrightarrow{\rho} Y$:

Theorem 14.34. (Cartan) *The system of differential forms (14.5) is a system of an invariant geometric objects iff the following conditions are fulfilled: 1) the coefficients $\xi_i^j \in C^\infty(Y; \mathbb{R})$ for all $i = 1, \ldots, r$, $j = 1, \ldots, n$, are analytic functions on Y; 2) the differential system (14.5) is completely Frobenius-Cartan integrable.*

This theorem says that the *differential system* (14.5) can be written as

$$\bar{\beta}^j := dy^j + \sum_{i=1}^{r} \xi_i^j(y) \bar{\omega}^i(a; da), \tag{14.6}$$

where the one-forms $\{\bar{\omega}^i(a; da) : i = 1, \dots, r\}$ satisfy the standard *Maurer–Cartan equations*

$$\bar{\Omega}^j := d\bar{\omega}^j + \frac{1}{2}\sum_{i,k=1}^{r} c^j_{ik}\bar{\omega}^i \wedge \bar{\omega}^k := 0 \tag{14.7}$$

on G, with coefficients $c^j_{ik} \in \mathbb{R}$, $1 \le i, j, k \le r$ that are the structural constants of the Lie algebra $\mathfrak{g}$ of the Lie group G.

Let us consider a case where the set of canonical Maurer–Cartan one-forms $\{\bar{\omega}^i(a; da) \in T^*_a(G) : i = 1, \dots, r\}$ is defined via the scheme:

$$\begin{array}{ccccc} T^*(M \times Y) & \xrightarrow{s^*} & T^*(\bar{M}) & \xleftarrow{\mu^*} & T^*(G \times Y) \\ \downarrow & & \downarrow & & \downarrow \\ M \times Y & \xleftarrow{s} & \bar{M} & \xrightarrow{\mu} & G \times Y \end{array} \tag{14.8}$$

where M is a given smooth finite-dimensional manifold with an embedded submanifold $\bar{M} \subset M$ via $s : \bar{M} \to M \times Y$, and $\mu : \bar{M} \to G \times Y$ is a smooth mapping into $G \times Y$. Under the scheme (14.8), 2-forms can be defined by

$$s^*\Omega^j\big|_{\bar{M}} := \mu^*\bar{\Omega}^j\big|_{\bar{M}} \tag{14.9}$$

for all $1 \le j \le r$ on the integral submanifold $\bar{M} \subset M$, where $\Omega^j \in \Lambda^2(M)$ is a given system of 2-forms on M.

Assume further that $\{\alpha_j \in \Lambda^2(M) : 1 \le j \le m_\alpha\}$ is a basis of two-forms $\{\Omega^j \in \Lambda^2(M) : 1 \le j \le r\}$, generating the ideal $\mathcal{I}(\alpha) \subset \Lambda(M)$. The ideal is $\mathcal{I}(\alpha)$ should be completely integrable within the Cartan integrable, owing to the equations $d\Omega^j \in \mathcal{I}(\Omega)$, $1 \le j \le r$ obtained from (14.7), implying $\mathcal{I}(\Omega) = 0$ on $\bar{M}$. Hence, it follows from the scheme (14.8) that $d\mathcal{I}(\alpha) \subset \mathcal{I}(\alpha)$, since $s^*\mathcal{I}(\alpha) = \mu^*\mathcal{I}(\bar{\Omega})$.

To define a criterion for a Lie group action $G \times Y \xrightarrow{\rho} Y$ to generate a representation of the Lie group G, we need to construct the ideal $\mathcal{I}(\alpha, \beta) \subset \Lambda(M \times Y)$, corresponding to (14.6) and (14.9), for some set of forms $\beta^j \in \Lambda^1(M \times Y)$, $1 \le j \le n$, where $s^*\beta_j := \mu^*\bar{\beta}_j \in \Lambda(\bar{M} \times Y)$, and to insist that it be closed in $\Lambda(M \times Y)$, that is $d\mathcal{I}(\alpha, \beta) \subset \mathcal{I}(\alpha, \beta)$ or

$$d\beta^j = \sum_{k=1}^{m_\alpha} f^j_k \alpha^k + \sum_{i=1}^{n} g^j_i \wedge \beta^i \tag{14.10}$$

for all $1 \le j \le n$ and some $f^j_k \in \Lambda^0(M \times Y)$, $1 \le k \le m_\alpha$, $g^j_i \in \Lambda^1(M \times Y)$, $i, j = 1, \dots, n$. The condition (14.10) guarnatees that there exist smooth submanifold $\bar{M}(Y) \subset M \times Y$ on which a nonlinear Lie group G representation acts exactly. Thus, we have proved the following result [44, 173]:

Theorem 14.35. *The system $\{\beta\}$ of Cartan one-forms $\beta^j \in \Lambda^1(M \times Y)$, $1 \leq j \leq n$, generated by the scheme (14.8), describes an exact nonlinear Lie group G representation on a manifold Y if and only if the adjoint ideal $\mathcal{I}(\alpha, \beta)$ generated by the system $\{\beta\}$ and a basic system $\{\alpha\}$ of the curvature 2-forms $\Omega^j \in \Lambda^2(M)$, $j = 1, \ldots, r$, of (14.9), is closed together with the corresponding ideal $\mathcal{I}(\alpha) = \mathcal{I}(\alpha, 0)$ in the Grassmann algebras $\Lambda(M \times Y)$ and $\Lambda(M)$, respectively.*

In view of the above, it is natural to try to specialize Cartan's geometric construction by means of the theory of principal fiber bundles [197, 198, 384]. To proceed, let us interpret the Cartan differential system $\{\beta\}$ on $M \times Y$ as generating a linear $(r \times r)$ matrix adjoint representation [179, 273, 274] of the Lie algebra $\mathfrak{g}$, setting $\xi_i^j(y) := \sum_{k=1}^r c_{ik}^j y^k$, $1 \leq i, j \leq r$, when $\dim Y = n := r$:

$$\beta^j := dy^j + \sum_{i,k=1}^r c_{ik}^j y^k b^i(z) \in \Lambda^1(M \times Y), \tag{14.11}$$

where $z \in M$, and the 1-forms $b^i(z)$ on M satisfy the necessary embedding conditions $s^* b^i = \mu^* \bar{\omega}^i$ upon $\bar{M} \subset M$ for all $i, j = 1, \ldots, r$ in accordance with (14.9).

The Lie group G acts on the linear r-dimensional space Y by the usual left translations as follows: $Y \times G \ni y \times a \xrightarrow{\rho} ay \in Y$ for all $a \in G$. Whence, we can easily calculate the following infinitesimal translations in the Lie group G:

$$da_k^j + \sum_{s=1}^r c_{si}^j b^s(z) a_k^i \in \Lambda^1(M \times G). \tag{14.12}$$

These expressions ultimately induce the following $\mathfrak{g}$-valued Ad-invariant connection 1-form ω on $M \times G$ via the isomorphic mapping $\rho^* : \Lambda^1(M \times Y) \to \Lambda^1(M \times G) \otimes \mathfrak{g}$:

$$\{\beta\} :\xrightarrow{\rho^*} \omega := a^{-1} da + Ad_{a^{-1}} \Gamma(z), \tag{14.13}$$

where the one-forms matrix $\Gamma(z) := (\Gamma_k^j(z)), 1 \leq j, k \leq r$, belongs to the $(r \times r)$-matrix representation of the Lie algebra $\mathfrak{g}$ owing to construction: $(\Gamma_k^j(z)) := (\sum_{i=1}^r c_{ik}^j b^i(z)) \in T^*(M) \otimes \mathfrak{g}$. The results above can naturally be interpreted as defining [44, 173, 197, 198] a $\mathfrak{g}$-valued *connection* Γ on a *principal fiber space* $P(M; G)$, carrying the $\mathfrak{g}$-valued connection 1-form (14.13). The corresponding Cartan 1-form determines the *horizontal* *subspace* for parallel transport of vectors of the fiber bundle $P(M;G,Y)$ associated with $P(M;G)$ according to the general theory [173, 197, 198] of fiber spaces with connections.

Thus, we have constructed the $\mathfrak{g}$-valued connection 1-form (14.13) at a point $(z,a)\in P(M;G)$ as $\omega := \bar{\omega}(a) + Ad_{a^{-1}}\Gamma(z)$, where $\bar{\omega}(a)\in T^*(G)\otimes\mathfrak{g}$ is the standard Maurer–Cartan left-invariant $\mathfrak{g}$-valued 1-form on the Lie group G. The connection 1-form (14.13) vanishes on the above-mentioned horizontal subspace, consisting of vector fields on $P(M;G)$ that generate a G representation of the Lie group on the space Y. Thus, this horizontal subspace necessarily defines a completely integrable differential system on $P(M;G)$, or equivalently, the corresponding *curvature* $\Omega\in\Lambda^2(M)\otimes\mathfrak{g}$ of the connection Γ vanishes on the integral submanifold $\bar{M}\subset M$:

$$\Omega := d\omega + \omega\wedge\omega = Ad_{a^{-1}}(d\Gamma(z) + \Gamma(z)\wedge\Gamma(z))$$

$$= \frac{1}{2}Ad_{a^{-1}}\sum_{j=1}^{m}\Omega_{jk}dz^j\wedge dz^k\bigg|_{\bar{M}} = 0, \tag{14.14}$$

whence we obtain

$$\Omega_{ij}(z) := \frac{\partial\Gamma_j(z)}{\partial z_i} - \frac{\partial\Gamma_i(z)}{\partial z_j} + [\Gamma_i(z),\Gamma_j(z)], \tag{14.15}$$

$$\Gamma(z) := \sum_{j=1}^{m}\Gamma_j(z)dz^j := \sum_{j=1}^{m}\sum_{k=1}^{r}\Gamma_j^k(z)dz^jA_k.$$

The vanishing curvature Ω (14.14) on the submanifold $\bar{M}\subset M$ is easily explained by means of the following commuting diagram:

$$\begin{array}{ccccccc} T^*(G) & \xrightarrow{\mu^*} & T^*(P(\bar{M};G)) & \xleftarrow{s^*} & T^*(P(M;G)) & \xleftarrow{\rho^*} & T^*(P(M;G,Y)) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ G & \xleftarrow{\mu} & P(\bar{M};G) & \xrightarrow{s} & P(M;G) & \xrightarrow{\rho} & P(M;G,Y) \end{array} \tag{14.16}$$

We can now derive from (14.16) that, owing to (14.15),

$$\rho^*\{\beta\} = \omega, \qquad s^*\Gamma_k^j = \sum_{i=1}^{r}c_{ik}^j\mu^*\bar{\omega}^i, \qquad s^*\Omega = \mu^*\bar{\Omega} = 0, \tag{14.17}$$

which implies (14.9) on $\bar{M}$.

Thus, if an integrable ideal $\mathcal{I}(\alpha)\subset\Lambda(M)$ is given on the manifold M, we can use the equation corresponding to (14.14) in $\Lambda(M)$:

$$\sum_{j,k=1}^{m}\Omega_{jk}dz^j\wedge dz^k\subset\mathcal{I}(\alpha)\otimes\mathfrak{g} \tag{14.18}$$

both for determining the $\mathfrak{g}$-valued 1-forms $\Gamma_j(z) \in T^*(M) \otimes \mathfrak{g}$, $1 \leq j \leq m$, and a Lie algebra structure of $\mathfrak{g}$, taking into account the *QTR holonomy Lie group reduction theorem* of Ambrose, Singer and Loos [12, 238]. Namely, the holonomy Lie algebra $\mathfrak{g}(h) \subset \mathfrak{g}$ generated by the covariant derivative compositions of the $\mathfrak{g}$-valued curvature form $\Omega \in T^*(M) \otimes \mathfrak{g}$ represented as

$$\mathfrak{g}(h) := \mathrm{span}_{\mathbb{R}}\{\nabla_1^{j_1}\nabla_2^{j_2}\ldots\nabla_n^{j_n}\Omega_{si} \in \mathfrak{g}: \ j_k \in \mathbb{Z}_+,\ s,i,k = 1,\ldots,n\} \tag{14.19}$$

where the covariant derivative $\nabla_j : \Lambda(M) \to \Lambda(M)$, $1 \leq j \leq n$, is defined as

$$\nabla_j := \partial/\partial z^j + \Gamma_j(z). \tag{14.20}$$

If the identity $\mathfrak{g}(h) = \mathrm{span}_{\mathbb{C}}\{\Omega_{sl} \in \mathfrak{g}: \ 1 \leq s,l \leq n\}$ holds, that is, the embedding $[\mathfrak{g}(h), \mathfrak{g}(h)] \subset \mathfrak{g}(h)$ is satisfied, the *holonomy Lie algebra* $\mathfrak{g}(h)$ is said to be *perfect.* Thus we have proved the following equivalence theorem.

Theorem 14.36. *Given a closed ideal $\mathcal{I}(\alpha)$ on a manifold M, $d\mathcal{I}(\alpha) \subset \mathcal{I}(\alpha)$, its 1-form augmentation $\mathcal{I}(\alpha,\beta)$ on $M \times Y$ by means of a special set $\{\beta\}$ of 1-forms*

$$\{\beta\} := \{\beta^j = dy^j + \sum_{k=1}^{n}\xi_k^j(y)b^k(z) : \ b^j(z) \in T^*(M),\ 1 \leq j \leq n\}, \tag{14.21}$$

compatible with the scheme (14.16), is Frobenius–Cartan integrable if and only if there exists a Lie group G action on Y such that the adjoint connection (14.13) on a fiber space $P(M;G)$ with structure group G vanishes on the integral submanifold $\bar{M} \subset M$ of the ideal $\mathcal{I}(\alpha) \subset \Lambda(M)$. The latter can serve as an algorithm for determining the structure of the Lie group G based on the holonomy Lie algebra reduction theorem of Ambrose–Singer–Loos.

If the conditions of Theorem 14.35 are satisfied, the set of 1-forms $\{\beta\}$ (14.21) generates a representation of the Lie group G on the analytic manifold Y according to Cartan's theorem 14.35. The Lie algebra $\mathfrak{g}$ of the Lie group G can be reduced to the holonomy Lie algebra $\mathfrak{g}(h)$, generated via (14.21) by the curvature 2-form Ω of the connection Γ on the principal fiber bundle $P(M;G,Y)$ constructed above.

14.3.2 *The Maurer–Cartan one-form construction*

To continue the study of the integrability of Lie invariant geometric objects generated by the scheme (14.16) with a mapping $s : \bar{M} \to G$, one often needs to have an effective way of constructing the *Maurer–Cartan forms*

$\bar{\omega}^j(a;da) \in T_a^*(G) \otimes \mathfrak{g}$, $1 \leq j \leq r$, corresponding to the Lie algebra $\mathfrak{g} \simeq T_e^*(G)$. Next, we shall describe a direct procedure for constructing these forms on G.

Let G be a Lie group with the Lie algebra $\mathfrak{g} \simeq T_e(G)$, having the basis $\{A_i \in \mathfrak{g} : 1 \leq i \leq r\}$, where $r = \dim G = \dim \mathfrak{g}$. Also let $U_0 \subset \{a^i \in \mathbb{R} : 1 \leq i \leq r\}$ be an open neighborhood of zero in $\mathbb{R}^r$. The exponential map $\exp : U_0 \to G_0$ defined as

$$\mathbb{R}^r \supset U_0 \ni (a^1, \ldots, a^r) :\xrightarrow{\exp} \exp(\sum_{i=1}^{r} a^i A_i) := a \in G_0 \subset G, \tag{14.22}$$

is an analytic mapping of U_0 onto an open neighborhood G_0 of the unit element $e \in G$. It is easy to show that $T_e(G) = T_e(G_0) \simeq \mathfrak{g}$, where $e := \exp(0) \in G$. Now define the following left-invariant $\mathfrak{g}$-valued differential one-form on $G_0 \subset G$:

$$\bar{\omega}(a;da) := a^{-1}da = \sum_{j=1}^{r} \bar{\omega}^j(a,da)A_j \in \mathfrak{g}, \tag{14.23}$$

where $\bar{\omega}^j(a;da) \in T_a^*(G)$, $a \in G_0$, $j = 1, \ldots, r$.

To construct the unknown forms $\{\bar{\omega}^j(a;da) : 1 \leq j \leq r\}$, consider the analytic one-parameter one-form $\bar{\omega}_t(a;da) := \bar{\omega}(a_t;da_t)$ on G_0, where $a_t := \exp(t\sum_{i=1}^{r} a^i A_i)$, $t \in [0,1]$, and differentiate this form with respect to the parameter $t \in [0,1]$, thereby obtaining

$$d\bar{\omega}_t/dt = -\sum_{j=1}^{r} a^j A_j a_t^{-1} da_t + \sum_{j=1}^{r} a_t^{-1} a_t da^j A_j \tag{14.24}$$

$$+\sum_{j=1}^{r} a_t^{-1} da_t a^j A_j = -\sum_{j=1}^{r} a^j [A_j, \bar{\omega}_t] + \sum_{j=1}^{r} A_j da_j.$$

Using the Lie identity $[A_j, A_k] = \sum_{i=1}^{r} c_{jk}^i A_i$, $j,k = 1, \ldots, r$, and the right-hand side of (14.23) in the form

$$\bar{\omega}^j(a;da) := \sum_{k=1}^{r} \bar{\omega}_k^j(a) da^k, \tag{14.25}$$

we ultimately obtain

$$\frac{d}{dt}(t\bar{\omega}_i^j(ta)) = \sum_{k=1}^{r} \mathcal{A}_k^j t\bar{\omega}_i^k(ta) + \delta_i^j, \tag{14.26}$$

where the matrix $\mathcal{A}_i^k$, $1 \leq i, k \leq r$, is defined as follows:

$$\mathcal{A}_i^k := \sum_{j=1}^{r} c_{ij}^k a^j. \tag{14.27}$$

Thus, the matrix $W_i^j(t) := t\bar{\omega}_i^j(ta)$, $1 \leq i, j \leq r$, satisfies the differential equation that follows from (14.26) [87]

$$dW/dt = \mathcal{A}W + E, \quad W|_{t=0} = 0, \tag{14.28}$$

where $E = (\delta_i^j)$ is the identity matrix. The solution of (14.28) is representable as

$$W(t) = \sum_{n=1}^{\infty} \frac{t^n}{n!} \mathcal{A}^{n-1} \tag{14.29}$$

for all $t \in [0, 1]$. Whence, recalling the definition of $W(t)$, we readily compute that

$$\bar{\omega}_k^j(a) = W_k^j(t)\Big|_{t=1} = \sum_{n=1}^{\infty} (n!)^{-1} \mathcal{A}^{n-1}. \tag{14.30}$$

Thus, the task of finding the Maurer–Cartan one-form for a given Lie algebra $\mathfrak{g}$ is solved in an effective and constructive way that is completely algebraic. This is expressed in the following theorem for the left invariant one-form $\bar{\omega}(a; da) \in T_a^*(G) \otimes \mathfrak{g}$ corresponding to a Lie algebra $\mathfrak{g}$ at any $a \in G$:

Theorem 14.37. *Let $\mathfrak{g}$ be a Lie algebra with the structural constants $c_{ij}^k \in \mathbb{R}$, $i, j, k = 1, \ldots, r = \dim \mathfrak{g}$, related to a basis $\{A_j \in \mathfrak{g} : 1 \leq j \leq r\}$. Then the adjoint to $\mathfrak{g}$ of left-invariant Maurer–Cartan one-form $\bar{\omega}(a; da)$ may be constructed as follows:*

$$\bar{\omega}(a; da) = \sum_{k,j=1}^{r} A_j \bar{\omega}_k^j(a) da^k, \tag{14.31}$$

where the matrix $W := (\bar{w}_k^j(a))$, $1 \leq j, k \leq r$, is

$$W = \sum_{n=1}^{\infty} (n!)^{-1} \mathcal{A}^{n-1}, \quad \mathcal{A}_k^j := \sum_{i=1}^{r} c_{ki}^j a^i. \tag{14.32}$$

In the sequel, we shall use the above approach to solve an analogous problem in the theory of connections over a principal fiber space $P(M; G)$ as well as the associated fiber bundle $P(M; Y, G)$.

14.3.3 *Cartan–Frobenius integrability of ideals in a Grassmann algebra*

Given two-forms generating a closed ideal $\mathcal{I}(\alpha)$ in the Grassmann algebra $\Lambda(M)$, we denote as above by $\mathcal{I}(\alpha,\beta)$ an augmented ideal in $\Lambda(M;Y)$, where the manifold Y will be called the *representation space* of an adjoint Lie group G action: $G \times Y \xrightarrow{\rho} Y$. Whence, we can determine relationships for the set of one-forms $\{\beta\}$ and 2-forms $\{\alpha\}$

$$\{\alpha\} : = \{\alpha^j \in \Lambda^2(M) : 1 \le j \le m_\alpha\}, \tag{14.33}$$

$$\{\beta\} : = \{\beta^j \in \Lambda^1(M \times Y) : j = 1, \dots, n = \dim Y\},$$

satisfying the equations

$$d\alpha^i = \sum_{k=1}^{m_\alpha} a_k^i(\alpha) \wedge \alpha^k, \tag{14.34}$$

$$d\beta^j = \sum_{k=1}^{m_\alpha} f_k^j \alpha^k + \sum_{s=1}^{n} \omega_s^j \wedge \beta^s,$$

where $a_k^i(\alpha) \in \Lambda^1(M)$, $f_k^j \in \Lambda^0(M \times Y)$ and $\omega_s^j \in \Lambda^1(M \times Y)$ for all $1 \le i,k \le m_\alpha$, $1 \le j,s \le n$.

Since the identity $d^2\beta^j = 0$ holds for all $1 \le j \le n$, from (14.34) we deduce that

$$\sum_{k=1}^{n} \left(d\omega_k^j + \sum_{s=1}^{n} \omega_s^j \wedge \omega_k^s \right) \wedge \beta^k \tag{14.35}$$

$$+ \sum_{s=1}^{m_\alpha} \left(df_s^j + \sum_{k=1}^{n} \omega_k^j f_s^k + \sum_{l=1}^{m_\alpha} f_l^j a_s^l(\alpha) \right) \wedge \alpha^s = 0.$$

Hence, from (14.35) we obtain

$$d\omega_k^j + \sum_{s=1}^{n} \omega_s^j \wedge \omega_k^s \in \mathcal{I}(\alpha,\beta), \tag{14.36}$$

$$df_s^j + \sum_{k=1}^{n} \omega_k^j f_s^k + \sum_{l=1}^{m_\alpha} f_l^j a_s^l(\alpha) \in \mathcal{I}(\alpha,\beta)$$

for all $1 \le j,k \le n$, $1 \le s \le m_\alpha$.

Owing to the second embedding in (14.36), we can define the 1-forms $\theta_s^j := \sum_{l=1}^{m_\alpha} f_l^j a_s^l(\alpha)$ satisfying

$$d\theta_s^j + \sum_{k=1}^{n} \omega_k^j \wedge \theta_s^k \in \mathcal{I}(\alpha,\beta) \oplus \sum_{l=1}^{m_\alpha} f_l^j c_s^l(\alpha), \tag{14.37}$$

which have been obtained from the identities $d^2\alpha^j = 0$, $1 \le j \le m_\alpha$, in the form $\sum_{s=1}^{m} c_s^j(\alpha) \wedge \alpha^s = 0$, where

$$c_s^j(\alpha) = da_s^j(\alpha) + \sum_{k=1}^{m_\alpha} a_l^j(\alpha) \wedge a_s^l(\alpha), \tag{14.38}$$

which follows from (14.34). We further assume that at $s = s_0$ the 2-forms $c_{s_0}^j(\alpha) = 0$ for all $1 \le j \le m_\alpha$. Then at $s = s_0$, we can define a set of 1-forms $\theta^j := \theta_{s_0}^j \in \Lambda^1(M \times Y)$, $1 \le j \le n$, satisfying the exact embeddings

$$d\theta^j + \sum_{k=1}^{n} \omega_k^j \wedge \theta^k := \Theta^j \in \mathcal{I}(\alpha,\beta) \tag{14.39}$$

together with a set of embeddings for 1-forms $\omega_k^j \in \Lambda^1(M \times Y)$

$$d\omega_k^j + \sum_{s=1}^{n} \omega_s^j \wedge \omega_k^s := \Omega_k^j \in \mathcal{I}(\alpha,\beta). \tag{14.40}$$

It follows from the general theory [173, 197, 198, 396] of connections on the fibered frame space $P(M; GL(n))$ over a manifold M that we can interpret (14.40) as the equations defining the curvature 2-forms $\Omega_k^j \in \Lambda^2(P)$ and the equations (14.39) as defining the torsion 2-forms $\Theta^j \in \Lambda^2(P)$. Since $\mathcal{I}(\alpha) = 0 = \mathcal{I}(\alpha,\beta)$ on the integral submanifold $\bar{M} \subset M$, the reduced fibered frame space $P(\bar{M}; GL(n))$ will be flat and be torsion free, so it is trivial over $\bar{M} \subset M$. Consequently, we have proved the following result.

Theorem 14.38. *Let the condition above on the ideals $\mathcal{I}(\alpha)$ and $\mathcal{I}(\alpha,\beta)$ be satisfied. Then the set of 1-forms $\{\beta\}$ generates the integrable augmented ideal $\mathcal{I}(\alpha,\beta) \subset \Lambda(M \times Y)$ if and only if there exists a connection 1-form $\omega \in \Lambda^1(P) \otimes gl(n)$ and torsion 1-form $\theta \in \Lambda^1(P) \otimes \mathbb{R}^n$ on the adjoint fiber bundle frame space $P(M; GL(n))$, satisfying the embeddings*

$$d\omega + \omega \wedge \omega \in \mathcal{I}(\alpha,\beta) \otimes \ gl(n), \tag{14.41}$$

$$d\theta + \omega \wedge \theta \in \mathcal{I}(\alpha,\beta) \otimes \mathbb{R}^n.$$

On the reduced fiber bundle frame space $P(\bar{M}, GL(n))$ the corresponding curvature and torsion vanish, where $\bar{M} \subset M$ is the integral submanifold of the ideal $\mathcal{I}(\alpha) \subset \Lambda(M)$.

We can see from Theorem 14.38 that some of its conditions coincide with those of Theorem 14.35 concerning the properties of adjoint curvature forms $\omega \in \Lambda^1(P) \otimes \mathfrak{g}$. Accordingly it follows that the existence of a connection 1-form $\omega \in \Lambda^1(P) \otimes \mathfrak{g}$, whose curvature form $\Omega \in \Lambda^2(P) \otimes \mathfrak{g}$ necessarily vanishes on the integral submanifold of the ideal $\mathcal{I}(\alpha) \subset \Lambda(M)$. The nature of the second embedding of (14.41) is at present not completely understood; namely, concerning the existence of the integrable augmented ideal $\mathcal{I}(\alpha, \beta) \subset \Lambda(M \times Y)$. In what follows, we shall present a detailed analysis of some special examples of the construction suggested above, concerning integrable dynamical systems on an invariant jet submanifold.

14.3.4 *The differential-geometric structure of a class of integrable ideals in a Grassmann algebra*

Consider a set $\{\beta\}$ defining Cartan's Lie group G invariant object on a manifold $M \times Y$:

$$\beta^j := dy^j + \sum_{k=1}^{r} \xi_k^j(y) b^k(z), \tag{14.42}$$

where $1 \leq i \leq n = \dim Y$, $r = \dim G$, satisfying the mapping scheme (14.8) for a chosen integral submanifold $\bar{M} \subset M$. Thus, the set (14.42) defines on the manifold Y a set $\{\xi\}$ of vector fields inducing a representation $\rho : \mathfrak{g} \to \{\xi\}$ of a given Lie algebra $\mathfrak{G}$; that is, vector fields $\xi_s := \sum\limits_{j=1}^{n} \xi_s^j(y) \frac{\partial}{\partial y^j} \in \{\xi\}$, $1 \leq s \leq r$, satisfying the Lie algebra $\mathfrak{g}$ relationships

$$[\xi_s, \xi_l] = \sum_{k=1}^{r} c_{sl}^k \xi_k \tag{14.43}$$

for all $s, l, k = 1, \ldots, r$.

We can now compute the differentials $d\beta^j \in \Lambda^2(M \times Y)$, $1 \leq j \leq n$, using (14.42) and (14.43) as follows:

$$d\beta^j = \sum_{l=1}^{n} \sum_{k=1}^{r} \frac{\partial \xi_k^j(y)}{\partial y^l} \left(\beta^l - \sum_{s=1}^{r} \xi_s^l(y) b^s(z) \right) \wedge b^k(z) + \sum_{l=1}^{n} \sum_{k=1}^{r} \xi_k^j(y) db^k(z) \tag{14.44}$$

$$= \sum_{l=1}^{n} \sum_{k=1}^{r} \frac{\partial \xi_k^j(y)}{\partial y^l} \beta^l \wedge b^k(z) - \sum_{l=1}^{n} \sum_{k,s=1}^{r} \frac{\partial \xi_k^j(y)}{\partial y^l} \xi_s^l(y) b^s(z) \wedge b^k(z)$$

$$+ \sum_{k=1}^{r} \xi_k^j(y) db^k(z) = \sum_{l=1}^{n} \sum_{k=1}^{r} \frac{\partial \xi_k^j(y)}{\partial y^l} \beta^l \wedge b_k(z)$$

$$+\frac{1}{2}\sum_{l=1}^{n}\sum_{k,s=1}^{r}\left[\frac{\partial\xi_k^j(y)}{\partial y^l}\xi_s^l(y)-\frac{\partial\xi_s^j(y)}{\partial y^l}\xi_k^l(y)\right]\times db^k(z)\wedge db^s(z)+\sum_{k=1}^{r}\xi_k^j(y)db^k(z)$$

$$\Rightarrow\sum_{l=1}^{n}\sum_{k=1}^{r}\frac{\partial\xi_k^j(y)}{\partial y^l}\beta^l\wedge b_k(z)+\frac{1}{2}\sum_{k,s=1}^{r}[\xi_s,\xi_k]^j db^k(z)\wedge db^s(z)+\sum_{k=1}^{r}\xi_k^j(y)db^k(z)$$

$$\Rightarrow\sum_{l=1}^{n}\sum_{k=1}^{r}\frac{\partial\xi_k^j(y)}{\partial y^l}\beta^l\wedge b_k(z)+\frac{1}{2}\sum_{l=1}^{n}\sum_{k,s=1}^{r}c_{ks}^l\xi_l^j db^k(z)\wedge db^s(z)$$

$$+\sum_{k=1}^{r}\xi_k^j(y)db^k(z)\Longrightarrow\sum_{l=1}^{n}\sum_{k=1}^{r}\frac{\partial\xi_k^j(y)}{\partial y^l}\beta^l\wedge b_k(z)$$

$$+\sum_{l=1}^{r}\xi_l^j\left(db^l(z)+\frac{1}{2}\sum_{k,s=1}^{r}c_{ks}^l db^k(z)\wedge db^s(z)\right)\in\mathcal{I}(\alpha,\beta)\subset\Lambda(M\times Y),$$

where $\{\alpha\}\subset\Lambda^2(M)$ is a given integrable system of 2-forms on M, vanishing on the integral submanifold $\bar{M}\subset M$. It is obvious that the embeddings (14.44) hold if and only if the following conditions hold for all $1\leq j\leq r$

$$db^j(z)+\frac{1}{2}\sum_{k,s=1}^{r}c_{ks}^j db^k(z)\wedge db^s(z)\in\mathcal{I}(\alpha). \tag{14.45}$$

The embeddings (14.45) mean, in particular, that on the integral submanifold $\bar{M}\subset M$ of the ideal $\mathcal{I}(\alpha)\subset\Lambda(M)$ the equalities

$$\mu^*\bar{\omega}^j=s^*b^j \tag{14.46}$$

hold, where $\bar{\omega}^j\in T_e^*(G)$, $1\leq j\leq r$, are the left-invariant Maurer–Cartan forms on the Lie group G. Thus, by the embeddings (14.45) all the conditions of Cartan's are satisfied. Consequently, we can obtain the set of forms $b^j(z)\in\Lambda^1(M)$.

To do this, we define a $\mathfrak{g}$-valued connection 1-form $\omega\in\Lambda^1(P(M;G))\otimes\mathfrak{g}$ as

$$\omega:=Ad_{a^{-1}}\left(\sum_{j=1}^{r}A_jb^j\right)+\bar{\omega}, \tag{14.47}$$

where $\bar{\omega}\in\mathfrak{g}$ is the standard Maurer–Cartan 1-form on G, as above. It follows from (14.45) that this 1-form satisfies the canonical structure embedding (14.14) for $\Gamma:=\sum_{j=1}^{r}A_jb^j\in\Lambda^1(M)\otimes\ \mathcal{G}$:

$$d\Gamma+\Gamma\wedge\Gamma\in\mathcal{I}(\alpha)\otimes\ \mathcal{G}, \tag{14.48}$$

serving as a main relationships for determining the form (14.47) in accordance with results of subsection 14.3.3. To proceed further, we would need to express the set of 2-forms $\{\alpha\}\subset\Lambda^2(M)$ in explicit form.

14.3.5 *Example: Burgers' dynamical system and its integrability*

Consider Burgers dynamical system on a functional manifold $M \subset C^{\infty}(\mathbb{R};\mathbb{R})$ given as

$$u_t = uu_x + u_{xx}, \tag{14.49}$$

where $u \in M$ and $t \in \mathbb{R}$ is an evolution parameter. The flow (14.49) on M can be recast into a set of 2-forms $\{\alpha\} \subset \Lambda^2(J^{(\infty)}(\mathbb{R}^2;\mathbb{R}))$ on the adjoint jet manifold $J^{(\infty)}(\mathbb{R}^2;\mathbb{R})$ as follows:

$$\{\alpha\} = \Big\{du^{(0)} \wedge dt - u^{(1)}dx \wedge dt = \alpha^1,\ du^{(0)} \wedge dx + u^{(0)}du^{(0)} \wedge dt \tag{14.50}$$
$$+du^{(1)} \wedge dt = \alpha^2 :\ (x,t;u^{(0)},u^{(1)})^{\top} \in M^4 \subset J^1(\mathbb{R}^2;\mathbb{R})\Big\},$$

where M^4 is a finite-dimensional submanifold in $J^1(\mathbb{R}^2;\mathbb{R})$ with coordinates $(x, t,\ u^{(0)} = u,\ u^{(1)} = u_x)$.

The set of 2-forms (14.50) generates the closed ideal $\mathcal{I}(\alpha)$ since

$$d\alpha^1 = dx \wedge \alpha^2 - u^{(0)}dx \wedge \alpha^1, \quad d\alpha^2 = 0, \tag{14.51}$$

whose integral submanifold $\bar{M} = \{x, t \in \mathbb{R}\} \subset M^4$ is defined by the condition $\mathcal{I}(\alpha) = 0$. We now look for a reduced connection 1-form $\Gamma \in \Lambda^1(M^4)\otimes\mathfrak{g}$ belonging to a not yet determined Lie algebra $\mathfrak{g}$. This 1-form can be represented using (14.50) as

$$\Gamma := b^{(x)}(u^{(0)}, u^{(1)})dx + b^{(t)}(u^{(0)}, u^{(1)})dt, \tag{14.52}$$

where the elements $b^{(x)}, b^{(t)} \in \mathfrak{g}$ satisfy determining equations (14.48) taking the form

$$\Omega = \frac{\partial b^{(x)}}{\partial u^{(0)}}du^{(0)} \wedge dx + \frac{\partial b^{(x)}}{\partial u^{(1)}}du^{(1)} \wedge dx + \frac{\partial b^{(t)}}{\partial u^{(0)}}du^{(0)} \wedge dt \tag{14.53}$$
$$+\frac{\partial b^{(t)}}{\partial u^{(1)}}du^{(1)} \wedge dt + [b^{(x)}, b^{(t)}]dx \wedge dt$$
$$= g_1(du^{(0)} \wedge dt - u^{(1)}dx \wedge dt) + g_2(du^{(0)} \wedge dx$$
$$+u^{(0)}du^{(0)} \wedge dt + du^{(1)} \wedge dt) \in \mathcal{I}(\alpha) \otimes \mathfrak{g}$$

for $\mathfrak{g}$-valued functions g_1, g_2 on M.

From (14.53) one finds that

$$\frac{\partial b^{(x)}}{\partial u^{(0)}} = g_2, \quad \frac{\partial b^{(x)}}{\partial u^{(1)}} = 0, \quad \frac{\partial b^{(t)}}{\partial u^{(0)}} = g_1 + g_2 u^{(0)}, \tag{14.54}$$

$$\frac{\partial b^{(t)}}{\partial u^{(1)}} = g_2, \quad [b^{(x)}, b^{(t)}] = -u^{(1)} g_1,$$

which has the following unique solution

$$b^{(x)} = A_0 + A_1 u^{(0)}, \quad b^{(t)} = u^{(1)} A_1 + \frac{{u^{(0)}}^2}{2} A_1 + [A_1, A_0] u^{(0)} + A_2, \tag{14.55}$$

where $A_j \in \mathfrak{g}$, $j = 0, 1, 2$, are constant elements on M of the Lie algebra $\mathfrak{g}$ under investigation that satisfy the structural equations

$$[A_0, A_2] = 0, \quad [A_0, [A_1, A_0]] + [A_1, A_2] = 0, \tag{14.56}$$

$$[A_1, [A_1, A_0]] + \frac{1}{2}[A_0, A_1] = 0.$$

From (14.54) one sees that the curvature 2-form $\Omega \in \mathrm{span}_{\mathbb{R}}\{A_1, [A_0, A_1] : A_j \in \mathfrak{G}, j = 0, 1\}$. Therefore, using reducing via the Ambrose–Singer reduction of the associated principal fiber bundle frame space $P(M, GL(n))$ to the principal fiber bundle $P(M; G(h))$, where $G(h) \subset G$ is the corresponding holonomy Lie group of the connection Γ on P, we need to verify the following conditions for the set $\mathfrak{g}(h) \subset \mathfrak{g}$ to be a Lie subalgebra in $\mathfrak{g}$: $\nabla_x^m \nabla_t^n \Omega \in \mathfrak{g}(h)$ for all $m, n \in \mathbb{Z}_+$.

Let us complete the transfinite procedure requiring that

$$\mathfrak{g}(h) = \mathfrak{g}(h)_0 := \mathrm{span}_{\mathbb{R}}\{\nabla_x^m \nabla_x^n \Omega \in \mathfrak{g} : m + n = 0\}. \tag{14.57}$$

This means that

$$\mathfrak{g}(h)_0 = \mathrm{span}_{\mathbb{R}}\{A_1, A_3 = [A_0, A_1]\}. \tag{14.58}$$

To satisfy the set of relations (14.56), we need to use expansions over the basis (14.58) of the external elements $A_0, A_2 \in \mathfrak{g}(h)$:

$$A_0 = q_{01} A_1 + q_{13} A_3, \tag{14.59}$$

$$A_2 = q_{21} A_1 + q_{23} A_3.$$

Substituting the expansions (14.59) into (14.56), we obtain that $q_{01} = q_{23} = \lambda$, $q_{21} = -\lambda^2/2$ and $q_{03} = -2$ for some arbitrary real parameter $\lambda \in \mathbb{R}$; that is, $\mathfrak{g}(h) = \mathrm{span}_{\mathbb{R}}\{A_1, A_3\}$, where

$$[A_1, A_3] = A_3/2; \quad A_0 = \lambda A_1 - 2A_3, \tag{14.60}$$

$$A_2 = -\lambda^2 A_1/2 + \lambda A_3.$$

It follows from (14.60) that the holonomy Lie algebra $\mathfrak{g}(h)$ is real and two-dimensional, with the (2×2) matrix representation

$$A_1 = \begin{pmatrix} 1/4 & 0 \\ 0 & -1/4 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{14.61}$$

$$A_0 = \begin{pmatrix} \lambda/4 & -2 \\ 0 & -\lambda/4 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -\lambda^2/8 & \lambda \\ 0 & \lambda^2/8 \end{pmatrix}.$$

Therefore, from (14.52), (14.55) and (14.61) we obtain the reduced connection 1-form $\Gamma \in \Lambda^1(M) \otimes \mathfrak{g}$

$$\Gamma = (A_0 + uA_1)dx + ((u_x + u^2/2)A_1 - uA_3 + A_2)dt, \tag{14.62}$$

characterizing parallel transport of vectors from the representation space Y of the holonomy Lie algebra $\mathfrak{g}(h)$

$$dy + \Gamma y = 0 \tag{14.63}$$

on the integral submanifold $\bar{M} \subset M^4$ of the ideal $\mathcal{I}(\alpha)$ generated by the set of 2-forms (14.50). This also means that the dynamical system (14.49) is endowed with the standard Lax representation [12, 227, 262, 278, 326, 406], having the spectral parameter $\lambda \in \mathbb{R}$ necessary for its integrability in quadratures.

If

$$\mathfrak{g}(h) = \mathfrak{g}(h)_1 := \mathrm{span}_{\mathbb{R}}\{\nabla_x^m \nabla_t^n \Omega \in \mathfrak{g} : \, m + n = 0, 1\}$$

is satisfied, one can compute that

$$\mathfrak{g}(h)_1 := \mathrm{span}_{\mathbb{R}}\{\nabla_x^m \nabla_t^n g_j \in \mathfrak{g} : \, j = 1, 2, \, m + n = 0, 1\} \tag{14.64}$$

$$= \mathrm{span}_{\mathbb{R}}\{g_j \in \mathfrak{g}; \, \partial g_j/\partial x + [g_j, A_0 + A_1 u^{(0)}] \, ,$$

$$\partial g_j/\partial t + [g_j, u^{(1)} A_1 + u^{(0)} A_1/2 + [A_1, A_0] u^{(0)} + A_2] \in \mathfrak{g} : \, j = 1, 2\}$$

$$= \mathrm{span}_{\mathbb{R}}\{A_1, [A_1, A_0], [[A_1, A_0], A_0], [[A_1, A_0], A_1], [A_1, A_2], [[A_1, A_0], A_2] \in \mathfrak{g}\}$$

$$= \mathrm{span}_{\mathbb{R}}\{A_{j \neq 2} \in \mathfrak{g} : \, 1 \le j \le 7\},$$

where, by definition,

$$[A_1, A_0] = A_3, \quad [A_3, A_0] = A_4, \quad [A_3, A_2] = A_7, \tag{14.65}$$

$$[A_3, A_1] = A_5, \quad [A_1, A_2] = A_6.$$

Consequently, we have the following expansions for the undetermined hidden elements $A_0, A_2 \in \mathfrak{g}$

$$A_0 := \sum_{j=1, j\neq 2}^{7} q_{0j} A_j, \quad A_2 := \sum_{j=1, j\neq 2}^{7} q_{2j} A_j, \tag{14.66}$$

where $q_{0j}, q_{2j} \in \mathbb{R}$ are real numbers to be found from conditions (14.64) and (14.65) as well as from the standard Jacobi identities. Having found a finite-dimensional representation of the Lie algebra $\mathcal{G}(h) = \mathfrak{g}(h)_1$ (14.64) and substituted it into (14.62), we will be able to write down the parallel transport equation (14.63) in a new Lax form useful for the study of exact solutions to the Burgers dynamical system (14.49). Analogous calculations can be carried out for other Lax integrable nonlinear dynamical systems.

Bibliography

[1] Abe E., Hopf Algebras, Cambridge University Press, Cambridge, UK, 1977

[2] Ablowitz M. and Segur H., Solitons and the Inverse Scattering Transform, SIAM, Philadelphia, 1981

[3] Abraham R. and Marsden J., Foundations of Mechanics, Second Edition, Benjamin Cummings, NY, 1978

[4] Adams M.R., Harnard J. and Hurtubise J., Dual moment maps into loop algebras, Lett. Math. Phys. 20, (N2), (1980) 299-308

[5] Adams M. R., Harnard J. and Hartubise J. Darboux coordinates and Liouville-Arnold integration in loop algebras, Comm. Math. Phys. 155, (N3) (1993) 385-413

[6] Adams M.R., Harnard J. and Previato E., Isospectral Hamiltonian flows in finite and infinite dimensions, Comm. Math. Phys. 117, (N4) (1988) 451-500

[7] Adler M., On a trace functional for formal pseudo-differential operators and the sympletic structure of the Korteweg-de Vries equations, Invent. Math. 50 (N3) (1979) 219-248

[8] Adler M. and van Moerbeke P., Linearization of Hamiltonian systems, Adv. Math. 38, (N3)(1980) 267-379

[9] Adler M. and van Moerbeke P., Linearization of Hamiltonian systems. Adv. Math. 38, 267-379, 1980

[10] Adler M. and van Moerbeke P., Algebraic completely integrable systems:a systematic approach, Semin. Math. (N1) (1987) 165-310

[11] Albeverio S., Kondratiev Y.G. and Streit L., How to generalize white noise analysis to non-gaussian measures. Preprint Bi-Bo-S, Bielefeld, 1992

[12] Ambrose N. and Singer J.M., A theorem on holonomy, Trans. AMS 75 (1953) 428-443

[13] Ankiewicz A. and Pask C. J., Phys. A 16 (1983) 4203-4208

[14] Arnold V.I., Mathematical Methods of Classical Mechanics, Springer-Verlag, NY, 1978

[15] Arnold V.I. and Khesin B.A., Topological Methods in Hydrodynamics. Springer-Verlag, NY, 1998

[16] Avan J., Babelon O. and Talon M. Construction of the classical R-matrices

for the Toda and Calogero models. Preprint LPTHE University Paris VI, CNRR UA 280, par IPTHE 93-31, 22p., 1993

[17] Avan J. and Talon M., Alternative Lax structures for the classical and quantum Neumann model, Phys. Lett. B 268 (N2) (1991) 209-216

[18] Babelon O. and Vialett C-M., The generalized Lie-Poisson structure for integrable dynamical systems, Phys. Lett. B 237 (N3-4) (1990) 411-415

[19] Balakrishnan A.V., Applied Functional Analysis, Nauka, Moscow, 1980 (in Russian)

[20] Balk A.M. and Ferapontov E.V., Wave systems with an infinite number of invariants, Physica D 70 (1994) 100-114

[21] Ballesteros A. and Ragnisco O., A systematic construction of completely integrable Hamiltonian flows from co-algebras.solv-int/9802008-6 Feb 1998

[22] Barbashov B.M., On the canonical treatment of Lagrangian constraints. arXiv:hepth/0111164, 2001

[23] Barbashov B.M. and Chernikov N.A., Solution and quantization of a nonlinear Born-Infeld type model, Zhurnal of Theoret. Mathem. Phys. 60 (N5) (1966) 1296-1308 (in Russian)

[24] Barbashov B.M. and Nesterenko V.V., Introduction to Relativistic String Theory, World Scientific, Singapore, 1990

[25] Barbashov B.M., Pervushin V.N., Zakharov A.F. and Zinchuk V.A., The Hamiltonian approach to general relativity and CNB-primordal spectrum. arXiv:hep-th/0606054, 2006

[26] Basarab-Horwath P., Integrability by quadratures for systems of involutive vector fields. Ukrainian Math. Journal, (Plenum Pres Publ., USA) 43(10) (1992) 1236-1242

[27] Benney D.J., Some properties of long nonlinear waves, Stud. Appl. Math. 52 (1973) 45–50

[28] Berezansky Yu. M., Eigenfunctions Expansions Related with Selfadjoint Operators, Nauk. Dumka Publ., Kiev, 1965 (in Russian)

[29] Berezansky Yu.M., A generalization of white noise analysis by means of theory of hypergroups, Rep. Math. Phys. 38 (N3) (1966) 289-300

[30] Berezansky Yu.M. and Kondratiev Y.G., Spectral Methods in Infinite Dimensional Analysis, v.1 and 2, Kluwer, Dordrecht, 1995

[31] Berezin F.A., The Second Quantization Method. Nauka Publ., Moscow, 1986 (in Russian)

[32] Berezin F.A. and Shubin M.A., Schrodinger Equation, Moscow University Press, Moscow 1983 (in Russian)

[33] Bertrand J. and Irac-Astaud M., Invariance quantum groups of the deformed oscillator algebra, J. Phys.A: Math.&Gen. 30 (1997) 2021-2026

[34] Bialynicky-Birula I., Phys. Rev. 155 (1967) 1414; 166,(1968) 1505

[35] Bjorken J.D. and Drell S.D., Relativistic Quantum Fields, McGraw-Hill, NY, 1965

[36] Blackmore D., Hamiltonian analysis of vortex filaments, Proc. Fourth Int. Conf. on Nonlinear Mechanics, Shanghai, China, Aug. 2002, pp. 807-813

[37] Blackmore D. and Dave R., Chaos in one-dimensional granular flows with oscillating boundaries, Powders & Grains 97, R. Behringer and J. Jenkins

(eds.), Balkema, Rotterdam, 1997, pp. 409-412

[38] Blackmore D. and Knio O., KAM theory analysis of the dynamics of three coaxial vortex rings, Physica D 140 (2000) 321-348

[39] Blackmore D. and Knio O., Hamiltonian structure for vortex filament flows, ZAMM 81S (2001) 145 - 148

[40] Blackmore D. and Leu M.C., Analysis of swept volume via Lie groups and differential equations, Int. J. Rob. Res. 11 (1992) 516-537

[41] Blackmore D., Leu M.C. and Shih F., Analysis and modelling of deformed swept volumes. Computer-Aided Des. 26 (1994) 315-326

[42] Blackmore D. and Prykarpatsky A.K., Versal deformations of a Dirac type differential operator, J. Nonlin. Math. Phys. 6 (1999) 246-254

[43] Blackmore D. and Prykarpatsky A. K., On a class of factorized operator dynamical systems and their integrability, Math. Methods and Physics-Mechanical Fields 46 (N2) (2003)

[44] Blackmore D., Prykarpatsky Ya. and Samulyak R., On the Lie invariant geometric objects generated by integrable ideals in Grassmann algebra, J. Nonlin. Math. Phys. 5(1) (1998) 54-67

[45] Blackmore D., Samulyak R. and Rosato A., New mathematical models for particle flow dynamics, J. Nonlin. Math. Phys. 6 (1999) 198-221

[46] Blackmore D., Ting L. and Knio O., Studies of perturbed three vortex dynamics, J. Math. Phys. 48 (2007) 065402

[47] Blackmore D., Rosato A., Tricoche X. and Urban K.,Tapping dynamics for a column of particles and beyond (2010), preprint

[48] Błaszak M., Miura map and bi-Hamiltonian formulation for restricted flows of the KdV hierarchy, J. Phys. A: Math. Gen. 26 (1993) 5985

[49] Błaszak M., Lagrangian-Hamiltonian formulation for stationary flows of some class of nonlinear dynamical systems, J. Phys. A: Math. Gen. 26 (1993) L263

[50] Błaszak M., Bi-Hamiltonian field Gardner system, Phys. Lett. A. 174 (1993) 85

[51] Błaszak M., On a non-standard algebraic description of integrable nonlinear systems, Physica A 198 (1993) 637

[52] Błaszak M., Newton representation for stationary flows of some class of nonlinear dynamical systems, Physica A 215 (1995) 201

[53] Błaszak M., Bi-Hamiltonian formulation for the KdV hierarchy with sources, J. Math. Phys. 36 (1995) 4826

[54] Błaszak M., Multi-Hamiltonian Theory of Dynamical Systems, Springer-Verlag, Berlin, 1998

[55] Bogolubov N.N. Collected Works, v.2, Naukova Dumka, Kiev, 1960 (in Russian)

[56] Bogolubov N.N. and Bogolubov N.N. (Jr.), Introduction to Quantum Statistical Mechanics, World Scientific, NJ, 1986

[57] Bogolubov N.N. and Shirkov D.V., Introduction to Quantized Field Theory, Nauka, Moscow, 1984 (in Russian)

[58] Bogolubov N.N. and Shirkov D.V., Quantum Fields, Nauka, Moscow, 1984

[59] Bogolubov N.N., Logunov A.A., Oksak A.I. and Todorov I.T., Introduc-

tion to Axiomatic Field Theory, W.A.Benjamin Advanced Book Program, Massachusetts, 1975

[60] Bogolubov N.N. (Jr.) and Prykarpatsky A.K., Quantum method of generating Bogolubov functionals in statistical physics: current Lie algebras, their representations and functional equations, Physics of Elementary Particles and Atomique Nucleus 17 (N4)(1986) 791-827 (in Russian)

[61] Bogolubov N.N. (Jr.) and Prykarpatsky A.K., Complete integrability of nonlinear Ito and Benney-Kaup sytems: gradient algorithm and Lax representation, Teor. i Matem. Phys. 67 (N3) (1986) 410-425 (in Russian)

[62] Bogolubov N.N. (Jr.) and Prykarpatsky A.K., Quantum current Lie algebra - universal algebraic structure for symmetries of completely integrable nonlinear dynamical systems in theoretical and mathematical physics, Teor. i Matem. Phys. 75 (N1) (1988) 3-17, (in Russian)

[63] Bogolubov N.N. (Jr.) and Prykarpatsky A.K., The Lagrangian and Hamiltonian formalisms for the classical relativistic electrodynamical models revisited. arXiv:0810.4254v1 [gr-qc], 2008

[64] Bogolubov N.N. (Jr.) and Prykarpatsky A.K., The analysis of Lagrangian and Hamiltonian properties of the classical relativistic electrodynamics models and their quantization. Foundations of Physics, DOI: 10.1007/s10701-009-9399-1, 2010

[65] Bogolubov N.N. (Jr.), Prykarpatsky A.K. and Blackmore D., On Benney type hydrodynamical systems and their Boltzmann-Vlasov equations kinetic models. Available at: http://www.ict it/ pub-off Preprint ICTP-IC/2006/006, 2006

[66] Bogolubov N.N. (Jr.), Prykarpatsky A.K. and Samoylenko V.H., Hamiltonian structure of hydrodynamic equation of Benney type and associated with them Boltzmann-Vlasov equations on an axis, Kiev: NAS Inst. Math., Preprint N 91-25, 1991

[67] Bogolubov N.N. (Jr.), Samoylenko V.H. and Prykarpatsky A.K., The Hamiltonian structure of hydrodynamical Benney type systems and associated with them Boltzmann-Vlasov equations, Ukrainian Phys. J. 37, (N1) (1992) 147-156 (in Ukrainian)

[68] Bogolubov N.N. (Jr.), Prykarpatsky A., Gucwa I. and Golenia J., Analytical properties of an Ostrovsky-Whitham type dynamical system for a relaxing medium with spatial memory and its integrable regularization. Preprint ICTP-IC/2007/109, Trieste, Italy. (available at: http://publications.ictp.it)

[69] Bogolubov N.N. (Jr.), Prykarpatsky A.K., Kurbatov A.M. and Samoylenko V.H., A nonlinear Schrödinger type model: conservation laws, Hamiltonian structure and complete integrability, Theoret. and Math. Phys. 65 (N2) (1985) 271-284

[70] Bogolubov N.N. (Jr.), Prykarpatsky Y.A., Samoilenko A.M. and Pryakrpatsky A.K., A generalized de Rham-Hodge theory of multi-dimensional Delsarte transformations of differential operators and its applications for nonlinear dynamic systems, Physics of Particles and Nuclei. 36 (N1) (2005) 1110-121

[71] Bogolubov N.N. (Jr.), Golenia J., Popowicz Z., Pavlov M. and Prykrpatsky, A.K., A new Riemann type hydrodynamical hierarchy and its integrbility analysis. Preprint ICTP, IC/2009/095, 2010

[72] Bogoyavlensky O.I. and Novikov S.P., On connection of Hamiltonian formalisms of stationary and non-stationary problems, Func. Anal. Appl. 10 (N1) (1976) 9-13

[73] Bonora L., Liu Q.P. and Xiong C.S., The integrable hierarchy constructed from a pair of KdF-type hierarchies and associated W-algebra, ArXiv: hep-th/9408035, Preprint SISSA-ISAS-118/94/EP and AS-ITP-94-43, 1994

[74] Brans C.H. and Dicke R.H., Mach's principle and a relativistic theory of gravitation, Phys. Rev. 124 (1961) 925

[75] Brillouin L., Relativity Reexamined, Academic Press, New York and London, 1970

[76] Brunelli L. J. and Das A. J., Math. Phys. 45 (2004) 2633

[77] Bugajski S., Klamka J. and Wegryzn S., Foundations of quantum computing, Archiwum Informatyki Teoretycznej i Stosowanej, Part 1, vol 13, No. 2, 2001; Part 2, vol. 14, No. 4, 2002

[78] Bukheim A.L., Volterra Equations and Inverse Problems, Nauka, Moscow, 1983 (in Russian)

[79] Bulyzhenkov-Widicker I.E., Einstein's gravitation for Machian relativism of nonlocal energy charges, Int. J. Theor. Phys. 47 (2008) 1261-1269

[80] Bulyzhenkov I.E., Einstein's curvature for nonlocal gravitation of Gesamt energy carriers. arXiv:math-ph/0603039, 2008

[81] Calini A., Recent developments in integral curve dynamics, Geometric Approaches to Differential Equations, Lect. Notes Australian Math. Soc., Vol. 15, Cambridge University Press, 2000, pp. 56-99

[82] Calogero F., Classical many-body problems amenable to exact treatments, Lecture Notes in Physics Monograph, Vol. 66, Springer-Verlag, New York, 2001

[83] Calogero F. and Degasperis A., Spectral Transform and Solitons, v.1, North-Holland, Amsterdam,1982

[84] Calogero F. and Degasperis A., Spectral Transform and Solitons, North-Holland, Amsterdam, 1982

[85] Cartan E., Lecons sur invariants integraux. Hermann, Paris, 1971

[86] Chern-S.S., Complex Manifolds, Foreign Literature Publisher, Moscow, 1960 (in Russian)

[87] Chevalley K., Lie Group Theory. v.2, Foreign Literature Publisher, Moscow, 1958 (in Russian)

[88] Chorin A. and Marsden J., A Mathematical Introduction to Fluid Mechanics, Third Edition, Springer-Verlag, New York, 1993

[89] Chruściński D. and Jamiołkowski A., Faza geometryczna: teoria i zastosowania, Nicolai Copernicus University, Publisher, Torun, Poland, 1996 (in Polish)

[90] Coddington E. and Levinson N., Theory of Ordinary Differential Equations, McGraw-Hill, New York, 1955

[91] Crespo T. and Hajto Z., Introduction to Differential Galois Theory.

Preprint IMUJ, Cracow, 2007/11

[92] DeFilippo S. and Salerno M., A geometric approach to discretization of nonlinear integrable evolution equations: Burgers hierarchy, Phys. Lett. A 101 (N2) (1984) 75-80

[93] Deligne P., Equations differentielles a points singulairs, Springer Lect. Notes in Math. 163, 1970

[94] Delsarte J., Sur certaines transformations fonctionelles relative aux equations lineaires aux derives partielles du second ordre, C.R. Acad. Sci. Paris 206 (1938) 178-182

[95] Delsarte J. and Lions J., Transmutations d'operateurs differentielles dans le domain complex, Comment. Math. Helv. 52 (1957) 113-128

[96] De Rham G., Sur la theorie des formes differentielles harmoniques, Ann. Univ. Grenoble 22 (1946) 135-152

[97] De Rham G., Varietes Differentielles, Hermann, Paris, 1955.

[98] Deser S. and Jackiw R., Time travel? arXiv:hep-th/9206094, 1992

[99] Deutsch D., Quantum theory, the Church-Turing principle and the universal quantum computer, Proc. Roy. Soc. (London) A400 (1985) 97-117

[100] Dickey L.A., Soliton equations and Hamiltonian systems. World Scientific, NJ, 1991, 310 p.

[101] Dickey L.A., Soliton Equations and Hamiltonian Systems, World Scientific, Singapore, 1991

[102] Dickey L.A., On the constrained KP hierarchy, ArXiv: hep-th/9407038/9411008, 1994

[103] Dirac P.A.M., Principles of Quantum Mechanics, Clarendon Press, Oxford, 1935

[104] Dirac P.A.M., Generalized Hamiltonian dynamics, Canad. J. Math. 2 (N2) (1950) 129-148

[105] Dirac P.A.M, Fock W.A. and Podolsky B., Phys. Zs. Soviet. 2 (1932) 468

[106] Dittman J. and Rudolph G., On a connection governing transport along (2 + 2)-density Matrices, J. Geom. Phys. 10 (1992) 93-106

[107] Dixmier J., Algebres Enveloppantes, Gauthier-Villars Editeur, Paris, 1974

[108] Donaldson S.K., An application of gauge theory to four dimansional topology, J. Diff. Geom. 17 (1982) 279-315

[109] Drinfeld V.G., Quantum groups, Proc. Int. Congress of Mathematicians, MRSI Berkeley, p. 798, 1986

[110] Drobotska I.S., Analysis of complete integrability of a system inverse to the nonlinear Benney-Kaup system, Ukr. Math. J. 45 (N3) (1993) 373-382 (English transl.)

[111] Dubrovin B.A., Matveev V.B and. Novikov S.P., Nonlinear equations of the Korteweg-de Vries type, finite zonal linear operators and Abel manifolds, Russian Math. Surveys 31 (1976) 55-136

[112] Dubrovin B.A., Novikov S.P. and Fomenko A.T., Modern Geometry, Nauka, Moscow, 1986 (in Russian)

[113] Dubrovin B.A., Fomenko A.T., Novikov S.P. and Burns R.G., Modern Geometry: Methods and Applications: The Geometry of Surfaces, Transformations Groups and Fields Pt. 1 (Graduate Texts in Mathematics),

Springer-Verlag, Berlin, 1991

[114] Duistermaat J.J., On global action-angle coordinates, Comm. Pure and Appl. Math. 33 (1980) 687-706

[115] Dunford N. and Schwartz J.T. Linear Operators, InterSci. Publ., NY, 1963

[116] Dunner G. and Jackiw R. "Peierles substitution"and Chern-Simons quantum mechanics. arXiv:hep-th/92004057, 1992

[117] L. D. Faddeev, Quantum inverse scattering problem II, in Modern Problems of Mathematics, M: VINITI Publ.3, 93-180, 1974 (in Russian)

[118] Faddeev L. D., Quantum completely integrable models in field theory, Sov. Sci. Rev. C.: Math. Phys. 1, (N10) (1980) 107-155

[119] Faddeev L.D., Energy problem in the Einstein gravity theory, Russian Physical Surveys 136 (N 3) (1982) 435-457 (in Russian)

[120] Faddeev L.D. and Sklyanin E.K., Quantum mechanical approach to completely integrable field theories. Proc. USSR Academy of Sciences (DAN) 243, 1430-1433, 1978 (in Russian)

[121] Faddeev L.D. and Takhtajan L.A., Hamiltonian Approach to Soliton Theory, Springer-Verlag, Berlin, 1987

[122] Feynman R., Lectures on gravitation. Notes at California Inst. of Technology, 1971

[123] Feynman R., Simulating physics with computers, Int. J. Theor. Phys. 21 (1982) 467-488

[124] Feynman R., Quantum mechanical computers, Found. Phys. 16 (1986) 507-531

[125] Feynman R., Leighton R. and Sands M., The Feynman Lectures on Physics. The Modern Science on Nature, Mechanics, Space, Time, Motion, v. 1, Addison-Wesley, Massachusetts, 1963

[126] Feynman R., Leighton R. and Sands M., The Feynman Lectures on Physics. Electrodynamics, v. 2, Addison-Wesley, Massachusetts, 1964

[127] Fock V.A., Konfigurationsraum und zweite Quantelung, Zeischrift Phys. Bd. 75, 622-647, 1932

[128] Fokas A.S. and Anderson R.L., The use of isospectral eigenvalue problems for obtaining hereditary symmetries for Hamiltonian systems, J. Math. Phys. 23 (N6) (1982) 1066-1073

[129] Fokas A. and Gelfand I., Bi-Hamiltonian structures and integrability, Important Developments in Solition Theory, Springer-Verlag, New York, 1992

[130] Fokas A. S. and Santini P. M., The recursion operator of the Kadomtsev-Petviashvili equation and the squared eigenfunctions of the Schrödinger operator, Stud. Appl. Math. 75 (N2) (1986) 179-186

[131] Fokas A.S. and Santini P.M., Recursion operators and bi-Hamiltonian structure in multidimensions II, Comm. Math. Phys. 116 (N3) (1988) 449-474

[132] Fomenko A.T., Differential Geometry and Topology. Supplementary chapters. Moscow University Publ., Moscow, 1983

[133] Fomenko A.T., Symplectic Geometry, Moscow University Publ., Moscow, 1988 (in Russian)

[134] Fomenko A.T., Integrability and Nonintegrability in Classical Mechanics,

Reidel, Dordrecht, 1986

[135] Francoise J.P., Monodromy and the Kowalewskaya top, Asterisque 150/151 (1987) 87-108

[136] Francoise J.P., Arnold's formula for algebraically completely integrable systems. Bull. AMS 17 (1987) 301-303

[137] Fuchssteiner B. and Fokas A.S., Symplectic structures, their Backlund transformations and nereditary symmetries, Physica D 4 (N1) (1981) 47-66

[138] Fujii K., More on optical holonomic quantum computers, arXiv:quant-ph/0005129, 31 May 2000; From Geometry to Quantum Computation, 26 May 2001 (quant-ph/0107128)

[139] Fushchych W.I. and Nikitin A.G., Symmetry of Equations of Quantum Mechanics, Nauka, Moscow, 1990 (in Russian)

[140] Gelfand I.M. and Dikij L., Integrable nonlinear equations and Liouville theorem, Funct. Anal. Appl. 13 (N1) (1979) 8-20

[141] Gelfand I.M. and Shilov G.E., Generalized Functions and Actions upon Them, Second Edition, Nauka, Moscow, 1959 (in Russian)

[142] Gelfand I. M. and Vilenkin N., Generalized Functions, Academic Press, New York, 1964

[143] Glauber R.J. Quantum Theory of Optical Coherence. Selected Papers and Lectures, Wiley-VCH, Weinheim, 2007

[144] Godbillon C., Geometrie differentielle et mechanique analytique, Paris, Hermann, 1969

[145] Gokhberg I. C. and Krein M. G., Theory of Volterra Operators in Hilbert Spaces and Its Applications, Nauka, Moscow, 1967 (in Russian)

[146] Goldin G.A., Nonrelativistic current algebras as unitary representations of groups, J. Math. Phys. 12(3) (1971) 462-487

[147] Goldin G.A., Grodnik J., Powers R.T. and Sharp D. Nonrelativistic current algebra in the N/V limit, J. Math. Phys. 15 (1974) 88-100

[148] Goldin G.A., Menikoff R. and Sharp F.H., Diffeomorphism groups, gauge groups, and quantum theory, Phys. Rev. Lett. 51 (1983) 2246-2249

[149] Golenia J., Bogolubov N. (Jr.), Popowicz Z., Pavlov M. and Prykarpatsky A.K., A new Riemann type hydrodynamical hierarchy and its integrability analysis, Preprint ICTP, IC/2009/095, 2010

[150] Golenia J., Pavlov M., Popowicz Z. and Prykarpatsky A.K., On a nonlocal Ostrovsky-Whitham type dynamical system, its Riemann type inhomogenious regularizations and their integrability, SIGMA 6 (2010) (in press)

[151] Golenia J., Popowicz Z., Pavlov M. and Prykarapatsky A.K., On a nonlocal Ostrovsky-Whitham type dynamical system, its Riemann type inhomogeneous regularizations and their integrability, SIGMA, (2010) (in press)

[152] Golenia J., Prykarpatsky Y.A., Samoilenko A.M. and Prykarpatsky A.K., The general differential-geometric structure of multi-dimensional Delsarte transmutation operators in parametric functional spaces and their applications in soliton theory, Part 2, Opuscula Mathematica, 24 (2004) 71-83

[153] Gotay M.J., On symplectic submanifolds of cotangent bundles, Letters in Math Phys. 29 (N4) (1993) 271-279

[154] Goto T., Relativistic quantum mechanics of one-dimensional mechanical

continuum and subsidiary condition of dual resonance model, Prog. Theor. Phys. 46 (N5) (1971) 1560-1569
[155] Goto M. and Grosshans F., Semisimple Lie Algebras, Marcel Dekker, New York, 1978
[156] Green B., The Fabric of the Cosmos, Vintage Books Inc., New York, 2004
[157] Griffiths P., Exterior Differential Systems and the Calculus of Variations, Birkhäuser, 1982
[158] Griffiths P. and Harris J., Princeples of Algebraic Geometry, Wiley-Interscience, New York, 1978
[159] Gromov M., Partial Differential Relations, Springer-Verlag, New York, 1986
[160] Grover L.K., Quantum mechanics helps in searching for a needle in a haystack, Phys. Rev. Lett. 79 (1997) 325-328
[161] Gu C.H., Generalized self-dual Yang-Mills flows, explicit solutions and reductions, Acta Applicandae Math. 39 (1995) 349-360
[162] Guillemin V. and Sternberg S., Geometric Asymptotics, AMS, Providence, Rhode Island, 1977
[163] Guillemin V. and Sternberg S., On the equations of motion of a classical particle in a Yang-Mills field and the principle of general covariance, Hadronic J. 1 (1978) 1 - 32
[164] Guillemin V. and Sternberg S., The moment map and collective motion, Ann. Phys. 127 (1980) 220-253
[165] Guillemin V. and Sternberg S., Convexity properties of the moment mapping, Invent. Math. 67 (1982) 491-513
[166] Guillemin V. and Sternberg S., On collective complete integrability according to the method of Thimm, Ergod. Theory and Dynam. Syst. 3 (N2) (1983) 219-230
[167] Guillemin V. and Sternberg S., Symplectic Techniques in Physics, Cambridge Univ. Press, Cambridge, 1984
[168] Gurevich A.V. and Zybin K.P., Nondissipative gravitational turbulence, Sov. Phys. - JETP 67 (1988) 1-12
[169] Gurevich A.V. and Zybin K.P., Large-scale structure of the Universe, Analytic Theory Sov. Phys. Usp. 38 (1995) 687-722
[170] Hasimoto H., A soliton on a vortex filament, J. Fluid Mech. 51 (1972) 477-485
[171] Hazewinkel M., Riccati and soliton equations, Report AMPR-9103, CWI, Amsterdam, The Netherlands, March 1995
[172] Helgason S., Differential Geometry and Symmetric Spaces, Academic Press, New York, 1962
[173] Hentosh O.Ye., Prytula M.M. and Prykarpatsky A.K., Differential-geometric integrability fundamentals of nonlinear dynamical systems on functional menifolds, Second Edition, Lviv University Publ., Lviv, Ukraine, 2006
[174] Hentosh O., Prytula M. and Prykarpatsky A.K., Differential-geometric and Lie-algebraic foundations of investigating nonlinear dynamical systems on functional manifolds, Second Edition, Lviv University Publ., Lviv, Ukraine, 2006 (in Ukrainian)

[175] Hogg T., Highly structured searches with quantum computers, Phys. Rev. Lett. 80 (1998) 2473-2476
[176] Holm D. and Kupershmidt B., Poisson structures of superfluids, Phys. Lett. 91 (1982) 425-430
[177] Holm D. and Kupershmidt B., Superfluid plasmas: multivelocity nonlinear hydrodynamics of superfluid solutions with charged condensates coupled electromagnetically, Phys. Rev. 36A(8) (1987) 3947-3956
[178] Holm D., Marsden J., Ratiu T. and Weinstein A., Nonlinear stability of fluid and plasma equilibria, Phys. Rep. 123/(1 and 2) (1985) 1-116
[179] Holod P.I. and Klimyk A.U., Mathematical Foundations of Symmetry Theory, Naukova Dumka, Kyiv, 1992 (in Ukrainian)
[180] Hopf H., Noncommutative associative algebraic structures, Ann. Math. 42 (N1) (1941) 22
[181] Hörmander L., Course at Lund University, 1974
[182] Hörmander L., An Introduction to Complex Analysis in Several Variables, Van Nostrand Reinhold, New York, 1986
[183] Hunter J. and Saxton R., Dynamics of director fields, SIAM J. Appl. Math. 51 (1991) 1498-1521.
[184] Hurst C.A., Dirac's theory of constraints, in Recent Developments in Mathematical Physics, Springer-Verlag, New York, 18-51, 1987
[185] Jackiw R., Lorentz violation in a diffeomorphism-invariant theory, arXiv:hep-th/0709.2348, 2007
[186] Jackiw R. and Polychronakos A.P., Dynamical Poincaré symmetry realized by field-dependent diffeomorhisms. arXiv:hep-th/9809123, 1998
[187] Jacob A. and Sternberg S., Coadjoint structures, solitons and integrability, Lect. Notes Phys. No. 120, 52-84, 1989
[188] Jaffe A. and Quinn F., Theoretical mathematics: toward a cultural synthesis of mathematics and theoretical physics, Bull. Amer. Math. Soc. 29 (1993) 1-13
[189] Kaplansky I., An Introduction to Differential Algebra, Actualites Sci. Indust., no. 1251, Hermann, Paris, 1957
[190] Kato T., Perturbation Theory for Linear Operators, Springer-Verlag, Berlin, 1966
[191] Kazhdan D., Kostant B. and Sternberg S., Hamiltonian group actions and dynamical systems of Calogero type, Comm. Pure Appl. Math. 31 (N3) (1978) 481-507
[192] Kirillov A.A., Unitary Representations of Nilpotent Lie Groups, Uspekhi Matem. Nauk 17 (N4) (1962) 52-96 (in Russian)
[193] Kirillov A. A., Infinite dimensional Lie groups, their orbits, invariants and representations. The geometry of moments, Lecture Notes in Math. 970, 101-123, 1982
[194] Kitaev A.Yu et al., Classical and Quantum Computation (Graduate Studies in Mathematics) AMS, Providence, 2002
[195] Kleinert H., Path Integrals, Second Edition, World Scientific, Singapore, 1995
[196] Klymyshyn I.A., Relativistic Astronomy, Naukova Dumka, Kyiv, 1980 (in

Ukrainian)
[197] Kobayashi Sh. and Nomizu K., Foundations of Differential Geometry, v.1, Interscience, New York, 1963
[198] Kobayashi Sh. and Nomizu K., Foundations of Differential Geometry, v.2, Interscience, New York, 1969
[199] Kolchin E.R., Differential Algebra and Algebraic Groups, Academic Press, New York, 1973
[200] Kondratiev Y.G., Streit L., Westerkamp W. and Yan J.-A. Generalized functions in infinite dimensional analysis. II AS preprint, 1995
[201] Konopelchenko B.G., On the integrable equations and degenerate dispersion laws in multidimensional sources, J. Phys. A: Math. and Gen. 16 (1983) L311-L316
[202] Konopelchenko B. and Oevel W., An R-matrix approach to nonstandard classes of integrable equations, Publ. RIMS 29 (1993) 581-666
[203] Kopych M., Prykarpatsky Y. and Samulyak R., Proc. NAS Ukraina (Mathematics) 2 (1997) 32-38
[204] Korepin V., Bogolubov N. and Izergin A., Quantum Inverse Scattering Method and Correlation Functions. Cambridge University Press, Cambridge, 1993
[205] Kornfeld I.P., Sinai Ya.G. and Fomin S.V., Ergodic Theory, Nauka, Moscow, 1980 (in Russian) and Naukova Dumka, Kyiv, 1992 (in Ukrainian)
[206] Kostant B., The solution to generalized Toda lattice and representation theory, Adv. Math. 34, (N2) (1979) 195-338
[207] Kostant B., Quantization and Representation, London Math. Soc. Lect. Notes, Ser. A, No. 34, 287-316, 1979
[208] Kourensky M., Procedings London Math. Soc. 24 (2) (1926) 202-210
[209] Kowalski K., Methods of Hilbert Spaces in the Theory of Nonlinear Dynamical Systems, World Scientific, Singapore, 1994
[210] Kowalski K. and Steeb W.-H., Symmetries and first integrals for nonlinear dynamical systems: Hilbert space approach. I and II. Prog. Theoretical Phys. 85 (N4) (1991) 713-722 and 975-983
[211] Kowalski K. and Steeb W.-H., Nonlinear Dynamical Systems and Carleman Linearization, World Scientific, Singapore, 1991
[212] Kozlov V.V. and Kolesnikov N.N., On dynamics theorems, Prykl. Mathem. and Mechanics, 42(1) (1978), 28-33
[213] Krichever I. M., Algebro-geometric methods in theory of nonlinear equations, Russian Math. Surveys 32(6) (1977) 183-208 (in Russian)
[214] Kryloff N.M. and Bogoliubov N.N., La theorie generale de la mesure et son application a letude des systemes dynamiques de la mechanique nonlineaire, Ann. Math., II, 38 (1937) 65113
[215] Kulish, Sklyanin E.K., Lecture Notes in Physics 151, Springer-Verlag, Berlin, 1982, 61-119
[216] Kummer J., On the construction of the reduced phase space of a Hamiltonian system with symmetry, Indiana University Math. J. 30(2) (1981) 281
[217] Kummer M., On the regularization of the Kepler problem, Commun. Math.

Phys. 84 (1982) 133-152

[218] Kupershmidt B., Geometry of jet-bundles and the structure of Lagrangian and Hamiltonian formalisms, Lect. Notes in Math. 775, 162-218, 1980

[219] Kupershmidt B., Discrete Lax equations and differential-difference calculus, Asterisque 123 (1985) 5-212

[220] Kuperschmidt B., Hydrodynamic Poisson brackets and local Lie algebras, Phys. Lett. 21A (4) (1987) 167-174

[221] Kupershmidt B., Lie algebras of Korteweg-de Vries equations, Physica D 27 (2) (1987) 294-310

[222] Kupershmidt B., Infinite-dimensional analogs of the minimal coupling principle of W.J. Le and of the Poincare lemma for differential two-forms, Diff. Geom. and Appl. 2 (1992) 275-293

[223] Kupershmidt B. and Manin Ju.I., Long wave equations with a free surface. II. The Hamiltonian structure and the higher equations, Funktsional. Anal. i Prilozhen. 12(1) (1978) 25–37

[224] Kupershmidt B. and Ratiu T., Canonical maps between semiderect products with applications to elasticity and superfluids, Commun. Math. Phys. 90 (N2) (1983) 235-250

[225] Landau L.D. and Lifshitz E.M., Field Theory, v. 2, Nauka, Moscow, 1973

[226] Landau L.D. and Lifshitz E.M., Quantum Mechanics, v. 6, Nauka, Moscow, 1974

[227] Lax P.D., Periodic solutions of the KdV equation, Commun. Pure and Appl. Math. 28 (1975) 141-188

[228] Lebedev D.R. and Manin Yu.I., Benney's long wave equations: Lax representation and conservation laws. Zapiski nauchnykh seminarov LOMI.- 1980-96; Boundary Value Problems of Mathematical Physics and Adjacent Function Theory Problems. 169-178, 1980 (in Russian)

[229] Lenells J., The Hunter-Saxton equation: a geometric approach, SIAM J. Math. Anal. 40 (2008) 266-277

[230] Levi D., Pilloni L. and Santini P.M., Backlund transformations for nonlinear evolution equations in (2+1)-dimensions, Phys. Lett. 81A (N8) (1981) 419-423

[231] Levi D. and Winternitz P., Continuous symmetries of discerete equations, Phys. Lett. A 152 (1991) 335-338

[232] Levitan B. M., Sturm-Liouville Inverse Problems, Nauka Publ., Moscow, 1984 (in Russian)

[233] Levitan B. M. and Sargsian I. S., Sturm-Liouville and Dirac Operators, Nauka Publ., Moscow, 1988 (in Russian)

[234] Logunov A.A., Lectures on Relativity Theory and Gravitation, Nauka Publ., Moscow, 1987

[235] Logunov A.A., The Theory of Gravity, Nauka Publ., Moscow, 2000

[236] Logunov A.A., Relativistic Theory of Gravitation, Nauka, Moscow, 2006 (In Russian)

[237] Logunov A.A. and Mestvirishvili M.A., Relativistic Theory of Gravitation, Nauka, Moscow, 1989 (In Russian)

[238] Loos H.G., Internal holonomic groups of Yang-Mills fields, J. Math. Phys.

8 (N10) (1967) 2114-2124

[239] Lopatynski Y. B., On harmonic fields on Riemannian manifolds, Ukr. Math. J. 2 (1950) 6-60 (in Russian)

[240] Lund F., Constrained Hamiltonian dynamics and the Poisson structure of some integrable systems, Physica D 18 (N3) (1986) 420-422

[241] Lytvynov E.W., Rebenko A.L. and Shchepaniuk G.V., Wick calculus on spaces of generalized functions compound Poisson white noise, Rep. Math. Phys. 39 (N2) (1997) 219-247

[242] Magri F., A simple model of the integrable Hamiltonian equation, J. Math. Phys. 19 (1978) 1156-1162

[243] Magri F., On the geometry of soliton equations, Acta Applicandae Mathematicae 41 (1995) 247-270

[244] Manakov S.V., The method of inverse scattering problem and two-dimensional evolution equations, Adv. Math. Sciences 31 (1976) 245-246

[245] Manin Yu.I., Computable and uncomputable. Moscow, Sov. Radio, 1980 (in Russian).

[246] Manin Yu.I., Classical computation, quantum computation and P. Shor's factorizing algorithm, Seminaire N. Bourbaki, exp. n 862, 375-404, 1998-1999

[247] Marchenko V. A., Spectral Theory of Sturm-Liouville Operators, Nauk. Dumka Publ., Kiev, 1972 (in Russian)

[248] Marsden J., Ratiu T. and Weinstein A., Semidirect products and reduction in mechanics, Trans. Amer. Math. Soc. 281 (N1) (1984) 147-177

[249] Marsden J., Ratiu T. and Weinstein A., Reduction and Hamiltonian structures on duals of semidirect product Lie algebras, Contem Math. 25 (1984) 55-100

[250] Marsden J. and Weinstein A., Reduction of symplectic manifolds with symmetry, Rep. Math. Phys. 5(1) (1974) 121-130

[251] Marsden J. and Weinstein A., The Hamiltonian structure of the Maxwell-Vlasov equations, Physica D 4 (1982) 394-406

[252] Marsden J. and Weinstein A., Coadjoint orbits, vortices and Clebsch variables for incompressible fluids, Physica D 7 (1983) 305-323

[253] Matveev V.B. and Salle M.I., Darboux-Backlund Transformations and Applications, Springer-Verlag, New York, 1993

[254] Melnikov V. K., Some new nonlinear evolution equations integrable by the inverse problem method, Math. USSR Sbornik 49 (N2) (1984) 461-489

[255] Mermin N.D., Relativity without light, Amer. J. Phys. 52 (1984) 119-124

[256] Mermin N.D., It's About Time: Understanding Einstein's Relativity, Princeton University Press, Princton, NJ, 2005

[257] Mikhailov A.V., Shabat A.B. and Yamilov R.I., A symmetry approach to the classification of nonlinear equations, Uspekhi Mat. Nauk 42 (N4) (1987) 3-53

[258] Mikhailov A.V., Shabat A.B. and Yamilov R.I., Extension of the module of invertible tranformations. Classification of integrable systems, Commun. Math. Phys. 115 (N1) (1988) 1-19

[259] Mishchenko A.S. and Fomenko A.T., Generalized Liouville method of in-

tegrating Hamiltonian systems, Funct. Anal. and Appl. 12(2) (1978) 46-56 (in Russian)

[260] Mishchenko A.S. and Fomenko A.T., Introduction to Differential Geometry and Topology, Moscow University Publ., Moscow, 1983 (in Russian)

[261] Mitropolsku Yu., Bogolubov N.N. (Jr.), Prykarpatsky A.K. and Samoylenko V.H., Integrable Dynamical System: Spectral and Differential-Geometric Aspects, Naukova Dumka, Kiev, 1987 (in Russian)

[262] Mitropolsky Yu.A., Bogolubov N.N. (Jr.), Prykarpatsky A.K. and Samoylenko V.H., Integrable Dynamical Systems, Naukova Dumka, Kyiv, 1987 (in Russian)

[263] Mitropolsky Yu.O., Prykarpatsky A.K. and Fil' B.M., Some aspects of a gradient-holonomic algorithm in the theory of integrability of nonlinear dynamic systems and computer algebra problems, Ukrainian Math. J. 43 (N1) (1991) 63-74

[264] Mitropolsky Yu.A., Prykarpatsky A.K. and Samoylenko V.H., Ideal integrability in Grassmann algebra and its applications, Ukr. Matem. Z. 36 (N4) (1984) 451-456 (in Russian)

[265] Mitropolsky Yu.O., Prykarpatsky A.K. and Samoylenko V.H., An asymptotic method of constructing implectic and recursion operators of completely integrable dynamical systems, Soviet Math. Dokl. 33 (N2) (1986) 542-546 (English transl.)

[266] Moffat H.K., The degree of knottedness of tangled vortex lines, J. Fluid Mech. 35/1 (1969) 117-129

[267] Moore J.D., Lectures on Seiberg-Witten Invariants, Second Edition, Springer-Verlag, New York, 2001

[268] Morosi C. and Tondo G., Yang-Baxter equations and intermediate long wave hierarchies, Commun. Math. Phys. 122 (1989) 91-103

[269] Morrison A.J., Parkes E.J. and Vakhnenko V.O., The N-loop soliton of the Vakhnenko equation, Nonlinearity 12 (1999) 1427-1437,

[270] Moser J., Geometry of quadrics and spectral theory, Chern Symposium, Berkeley; Springer-Verlag, Berlin, 147-188, 1979-1980

[271] Moser Yu., Some aspects of integrable Hamiltonian systems, Uspekhi Matem. Nauk 36 (N5) (1981) 109-151 (in Russian)

[272] Mykytiuk Ya. V., Factorization of Fredholmian operators, Mathematical Study, Proc. Lviv Mathematical Society, 20(2) (2003) 185-199 (in Ukrainian)

[273] Naimark M.A., Linear Differential Operators, Nauka, Moscow, 1969 (in Russian)

[274] Naimark M.A., Group Representation Theory, Nauka, Moscow, 1976 (in Russian)

[275] Nambu Y., Strings, monopoles, and gauge fields, Phys. Rev. D. 10 (N12) (1974) 4262-4268

[276] Nesterenko V., Propagation of nonlinear compression pulses in granular media, J. Appl. Mech. Tech. Phys. 24 (1984), 733-743

[277] Neumann J. von, Mathematische Grundlagen der Quanten Mechanik, Springer-Verlag, Berlin, 1932

[278] Newell A.C., Solitons in Mathematics and Physics, SIAM Lectures, 1985

[279] Newlander A. and Nirenberg L., Complex analytical coordinates in almost-complex manifolds, Annals of Math. 65 (1957) 391-404

[280] Newman R.P., The global structure of simple space-times, Commun. Math. Phys. 123 (1989) 17-52

[281] Niekhoroshev N.N., Action-angle variables and their generalization, Trudy mosk. Matem. Obshchestva 26 (1972) 181-198 (in Russian)

[282] Nimmo J.C.C., Darboux transformations from reductions of the KP-hierarchy. Preprint of the Dept. of Maths., University of Glasgow, November 8, 1-11, 2002

[283] Nyzhnyk L.P., Integration of many-dimensional nonlinear equations by means of the inverse scattering problem, Proc. USSR Academy of Sciences 254 (N2) (1980) 332-335 (in Russian)

[284] Nyzhnyk L.P., Inverse Scattering Problems for Hyperbolic Equations, Nauk. Dumka Publ., Kiev, 1991 (in Russian).

[285] Nyzhnyk L.P. and Pochynaiko M.D., The integration of a spatially two-dimensional Schrodinger equation by the inverse problem method, Func. Anal. and Appl. 16 (N1) (1982) 80-82 (in Russian)

[286] Oevel W., Dirac constraints in field theory: lifts of Hamiltonian systems to the cotangent bundle, J. Math. Phys. 29 (N1) (1988) 210-219

[287] Oevel W., R-structures, Yang-Baxter equations and related evolution theorems. J. Math. Phys. 30 (1989) 1140-1149

[288] Oevel W. and Strampp W., Constraint KP-hierarchy and Bi-Hamiltonian structures, Comm. Math. Phys. 157 (N1) (1993) 51-81

[289] del Olmo M., Rodrigues M. and Winternitz P., Integrable systems based on SU(p,q) homogeneous manifold, Report CRM-1834, Université de Montreal, QC, Canada, Sept. 1992

[290] Olver P., On the Hamiltonian structure of evolution equations, Math. Proc. Cambridge Phil. Soc. 88 (N1) (1980) 71-88

[291] Olver P., Applications of Lie Groups to Differential Equations, Springer-Verlag, New York, 1986

[292] Ostrovsky L.A., Nonlinear ininternal waves in a rotating ocean, Oceanology 18 (1978) 119-125

[293] Owczarek R., Topological defects in superfluid Helium, Int. J. Theor. Phys. 30/12 (1991) 1605-1612

[294] Pachos J. and Chountasis S., Optical holonomic quantum computers, quant-ph/9912093

[295] Parasiuk I.O., Non-Poisson commutative symmetries and multidimensional invariant toris of Hamiltonian systems, Dokl. Akad. Nauk UkrSSR, Ser. A (N10) (1984) 10-16 (in Russian)

[296] Parasiuk I.O., Coisitropic quasiperiodic motions for a constrained set of rigid bodies, J. Nonl. Math. Phys. 1 (N2) (1994) 189-201

[297] Parasiuk I.O., Coisitropic quasiperiodic motions near relative equilibrium of Hamiltonian systems, J. Nonl. Math. Phys. 1 (N4) (1994) 340-357

[298] Pavlov M., The Gurevich-Zybin system. J. Phys. A: Math. Gen. 38 (2005) 3823-3840

[299] Pavlov M. et al., A new Riemann type hydrodynamical hierarchy and its integrability analysis Preprint ICTP, IC/2009/095, 2010 (in press)

[300] Parkes E.J., The stability of solution of Vakhnenko's equation, J. Phys. A: Math. Gen. 26 (1993) 6469-6475

[301] Parkes E.J., Explicit solutions of the reduced Ostrovsky equation, Chaos, Solitons and Fractals 26(5) (2005) 1309-16

[302] Pauli W., Theory of Relativity, Oxford Publ., Oxford, 1958

[303] Peradzyński Z., Helicity theorem and vortex lines in superfluid ^{4}He, Int. J. Theor. Phys. 29/11 (1990) 1277-1284

[304] Perelomov A.M., Integrable Systems of Classical Mechanics and Lie Algebras, Birkhäuser, 1990

[305] Pochynaiko M.D. and Sydorenko Yu.M., Integrating some (2+1)-dimensional integrable systems by methods of inverse scattering problem and binary Darboux transformations, Matematychni Studii, N20, 119-132, 2003

[306] Popowicz Z. and Prykarpatsky A.A., The non-polynomial conservation laws and integrability analysis of generalized Riemann type hydrodynamical equations. arXiv:submit/0044844 [nlin.SI] 21 May 2010

[307] Postnikov M., Lie Groups and Lie Algebras. M.: Mir Publishers, 1982

[308] Prykarpatsky A.K., The gradient holonomic algorithm of construction of integrability criteria for nonlinear dynamical systems, Proc. USSR Academy of Sciences 287, N4) (1986) 827-832 (in Russian)

[309] Prykarpatsky A.K., Elements of integrability theory for discrete dynamical systems, Ukrainian Math. J. 39 (N1) (1987) 87-92 (in Russian)

[310] Prykarpatsky A.K., The non-abelian Liouville-Arnold integrability problem: a symplectic approach. J. Nonl. Math. Physics 6(4) (1999) 384-410

[311] Prykarpatsky A.K., Quantum Mathematics and Its Applications, Part 1, Automatyka, vol. 6, AGH Publ., Krakow, 2002, No. 1, pp. 234-2412; Holonomic Quantum Computing Algorithms and Their Applications, Part 2, Automatyka, vol. 7, No. 1, 2004

[312] Prykarpatsky A.K., Artemovich O.D., Popowicz Z. and Pavlov M., The Differential-algebraic Lax type integrability analysis of a generalized Riemann type differential system. Abstracts of the All-Ukrainian Scientific Seminar "Modern Problems of Probability Theory and Mathematical Analysis" held 25-28 March 2010 in Vorokhta, Ivano-Frankivsk region, Ukraine, pp. 3-6

[313] Prykarpatsky A.K., Artemovich O.D., Popowicz Z. and Pavlov M., Differential-algebraic integrability analysis of the generalized Riemann type and Korteweg–de Vries hydrodynamical equations, J. Phys. A: Math. Theor. 43 (2010) 295205

[314] Prykarpatsky A.K., Blackmore D. and Bogolubov N.N. (Jr.), Swept volume dynamical systems and their kinetic models, Ukrainian Math. J. 48 (1996) 1620-1627

[315] Prykarpatsky A.K., Blackmore D. and Bogolubov N.N. (Jr.), Hamiltonian structure of Benney type hydrodynamic systems and Boltzmann-Vlasov kinetic equations on an axis and some applications to manufacturing science,

Open Systems and Information Dynamics 6 (1999) 335-373

[316] Prykarpatsky A.K., Blackmore D. and Hentosh O., The finite-dimensional Moser type reductions of modified Boussinesq and super Korteweg-de Vries Hamiltonian systems via thegradient holonomic algorithm and dual momentum map, in T. Gill (ed.), Proc. Int'l. Conf. on New Frontiers in Physics, vol. 11, Institute for Basic Research, Monteroduni, Italy, Hadronic Press, 1996, 271-292

[317] Prykarpatsky A.K., Bogolubov N.N. (Jr.) and Golenia J., A symplectic generalization of the Peradzynski Helicity Theorem and some applications, Int. J. Theor. Phys. 47 (2008) 1919-1928

[318] Prykarpatsky A.K., Bogolubov N.N. (Jr.) and Golenia J., A Symplectic Generalization of the Peradzynski Helicity Theorem and Some Applications. Available at: http://publications.ictp.it, Preprint IVTP-IC/2007/118

[319] Prykarpatsky A.K., Bogolubov N.N. (Jr.), Golenia J. and Taneri U., Introductive backgrounds of modern quantum mathematics with application to nonlinear dynamical systems. Available at: http://publications.ict it Preprint ICTP-IC/2007/108

[320] Prykarpatsky A.K., Bogolubov N.N. (Jr.), Golenia J. and Taneri U., Introductive backgrounds to modern quantum mathematics with application to nonlinear dynamical systems, Int. J. Theor. Phys. 47 (2008) 2882-2897

[321] Prykarpatsky A.K., Bogolubov N.N. (Jr.) and Taneri U., The vacuum structure, special relativity and quantum mechanics revisited: a field theory nogeometry approach, Theoretical and Mathematical Physics, RAS, Moscow, 2008 (in print) (arXiv lanl: 0807.3691v.8 [gr-gc] 24.08.2008)

[322] Prykarpatsky A.K., Bogolubov N.N. (Jr.) and Taneri U., The field structure of vacuum Maxwell equations and relativity theory aspects. Preprint ICTP, Trieste, IC/2008/051 (http://publications.ictp.it)

[323] Prykarpatsky A.K., Bogolubov N.N. (Jr.) and Taneri U., The relativistic electrodynamics least action principles revisited: New charged point particle and hadronic string models analysis, Int. J. Theor. Phys. 49 (2010) 798-820

[324] Prykarpatsky A.K. and Fil B.M., Category of topological jet-manifolds and certain applications in the theory of nonlinear infinite-dimensional dynamical systems, Ukrainian Math. J. 44 (N9) (1993) 1136-1147 (English transl.)

[325] Prykarpatsky A.K. and Hentosh O.Ye, The Lie-algebraic structure of (2+1)-dimensional Lax type integrable nonlinear dynamical systems, Ukrainian Math. J. 56 (2004) 939-946

[326] Prykarpatsky A.K. and Mykytyuk I., Algebraic Integrability of Nonlinear Dynamical Systems on Manifolds: Classical and Quantum Aspects. Kluwer, Dordrecht, 1998

[327] Prykarpatsky A.K. and Prytula M.M., The gradient-holonomic integrability analysis of a Whitham type nonlinear dynamical model for a relaxing medium with spacial memory. Proceeding of the National Academy of Sciences of Ukraine, Math. Series, N.5, (2006), pp. 13-18 (in Ukrainian)

[328] Prykarpatsky A.K. and Prytula M.M., The gradient-holonomic integrabil-

ity analysis of a Whitham-type nonlinear dynamical model for a relaxing medium with spatial memory, Nonlinearity 19 (2006) 2115-2122

[329] Prykarpatsky A.K. , Samoilenko A.M. and Blackmore D., Embedding of integral submanifolds and associated adiabatic invariants of slowly perturbed integrable Hamiltonian systems, Rep. Math. Phys. 37(5) (1999) 171-182

[330] Prykarpatsky A.K., Samoilenko A.M. and Blackmore D., Hopf algebras and integrable flows related to the Heisenberg–Weil coalgebra, Ukrainian Math. J. 56 (N1) (2004) 88-96

[331] Prykarpatsky A.K., Samoilenko A.M. and Prykarpatsky Y.A., The multidimensional Delsarte transmutation operators, their differential-geometric structure and applications, Part1, Opuscula Mathematica 23 (2003) 71-80

[332] Prykarpatsky A.K., Samoylenko V.H. and Andrushkiw R., Gradient-holonomic algorithm for the Lax type integrable dynamical systems, J. Math. Phys. 35 (N8) (1994) 6115-6126

[333] Prykarpatsky A.K., Samoylenko V.Hr., Andrushkiw R.I., Mitropolsky Yu.O. and Prytula M.M., Algebraic structure of the gradient-holonomic algorithm for Lax integrable nonlinear systems, I, J. Math. Phys. 35 (1994) 1763-1777

[334] Prykarpatsky A.K., Taneri U. and Bogolubov N.N. (Jr.), Quantum Field Theory and Application to Quantum Nonlinear Optics, World Scientific, New York, 2002

[335] Prykarpatsky A.K. and Zagrodzinski J.A., Lagrangian and Hamiltonian aspects of Josephson type media, Annales de Inst. H. Poincare 70A (N5) (1999) 497-524

[336] Prykarpatsky A.K., Zagrodzinski J.A. and Blackmore D., Lax type flows on Grassmann manifolds and dual momentum mappings, Rep. Math. Phys. 40 (1997) 539-549

[337] Prykarpatsky Y.A., The structure of integrable Lax type flows on nonlocal manifolds: dynamical systems with sources, Math. Methods and Phys.-Mech. Fields 40 (N 4) (1997) 106-115 (in Ukrainian)

[338] Prykarpatsky Y.A. and Samoilenko A.M., Algebraic-analytic aspects of integrable nonlinear dynamical systems and their perturbations, Kyiv, Inst. Mathematics Publ. 41, 2002 (in Ukrainian)

[339] Prykarpatsky Y.A., Samoilenko A.M. and Prykarpatsky A.K., The geometric properties of canonically reduced symplectic spaces with symmetry, their relationship with structures on associated principal fiber bundles and some applications, Opuscula Mathematica 25 (N2) (2005) 287-298

[340] Prykarpatsky Y.A., Samoilenko A.M. and Prykarpatsky A.K., The de Rham-Hodge-Skrypnik theory of Delsarte transmutation operators in multi-dimension and its applications, Rep. Math. Phys. 55 (N3) (2005) 351-363

[341] Prykarpatsky Ya., Samoilenko A., Prykarpatsky A.K., Bogolubov N.N. (Jr.) and Blackmore D., The differential-geometric aspects of integrable dynamical sytems. Available at: http://www.ictp.it/ pub-off Preprint ICTP-IC/2007/030, 2007

[342] Prykarpatsky Y.A., Samoilenko A.M., Prykarpatsky A.K. and Samoylenko

V.H., The Delsarte–Darboux type binary transformations and their differential-geometric and operator staructure. lanl-arXiv: math-ph/0403055 1, 29 Mar 2004

[343] Prykarpatsky Y.A., Samoilenko A.M. and Samoylenko V.H., The structure of binary Darboux type transformations and their applications in soliton theory, Ukrainian Math. J. 55 (2003) 1704-1719 (in Ukrainian)

[344] Prytula M., Prykarpatsky A.K. and Mykytyuk I., Fundamentals of the Theory of Differential-Geometric Structures and Dynamical Systems, Ministry of Educ. Publ., Kiev, 1988 (in Ukrainian)

[345] Puta M., Hamiltonian Mechanical Systems and Geometric Quantization, Kluwer Acad.Publ., Dordrecht, 1993

[346] Putterman S.J., Superfluid Hydrodynamics, North Holland, Amsterdam, 1974.

[347] Rainville E.D., Necessary and sufficient conditions for polynomial solutions of certain Riccati equations, Amer. Math. Monthly 43 (1936) 473-476

[348] Rajeev S., Kalyana R. and Siddhartha S., Symplectic manifolds, coherent states and semiclassical approximation, J. Math. Phys. 35 (5) (1994) 2259-2269

[349] Ravoson V., Gavrilov L. and Caloz R., Separability and Lax pairs for Henon-Heiles system, J. Math. Phys. 34(6) (1993) 2385-2393

[350] Reed M. and Simon B., Functional Analysis, Academic Press, New York, 1972

[351] Repchenko O., Field Physics, "Galeria" Publ., Moscow, 2005

[352] Reyman A.G. and Semenov-Tian-Shansky M.A., Reduction of Hamiltonian systems, affine Lie algebras and Lax equations, Part 1; and 2, Inv. Math. 54 (1979) 81-100; and 63 (1981) 423-432

[353] Reyman A.G. and Semenov-Tian-Shansky M.A., A new integrable case of the motion of the 4-dimensional rigid body, Comm. Math. Phys. 105 (1986) 461-472

[354] Reyman A.G. and Semenov-Tian-Shansky M.A., The Hamiltonian structure of Kadomtsev-Petviashvili type equations, LOMI Proceedings, Nauka, Leningrad, 164, 212-227, 1987 (in Russian)

[355] Reymann A.G. and Semenov-Tian-Shansky M.A., Hamiltonian structure hierarchy of Hamiltonians and reduction for first degree matrix differential operators, Func. Anal. and Appl. 14 (N2) (1990) 77-78 (in Russian)

[356] Reyman A. and Semenov-Tian-Shansky M.A., A set of Hamiltonian structures, a hierarchy of Hamiltonians and reduction for first order matrix differential operators. Funct. Anal. and Appl., 14 (1990) 77-78 (in Russian)

[357] Reyman A.G., Semenov-Tian-Shansky M.A., The integrable systems, Computer Science Institute Publ., Moscow-Izhevsk, 2003 (in Russian)

[358] Reyman A.G., Semenov-Tian-Shansky M.A., Integrable Systems, The Computer Research Institute Publ., Moscow-Izhvek, 2003 (in Russian)

[359] Richtmyer R.D., Principles of Advanced Mathematical Physics. V.1, Springer-Verlag, New York, 1978

[360] Rieffel E. and Polak W., An introduction to quantum computing for non-physicists, xxxlanl archive: quant-ph/9809016

[361] Ritt J.F., Differential Algebra. AMS-Colloqium Publications, vol. XXXIII, Dover Publ., New York, 1966

[362] Roekaerts R. and Scwarz F., Painleve analysis, Yoshida's theorems and direct methods in the search for integrable Hamiltonians, J. Phys. A: Mat. Gen. 20 (1987) L127-L133

[363] Sakovich S. On a Whitham-Type Equation. SIGMA 5 (2009), 101

[364] Salerno M., Enolski V.Z. and Leykin D.V., Canonical transformation between integrable Henon-Heiles systems. Phys. Rev. E, 49(6) (1994) 5897-5899

[365] Samoilenko A.M. and Prykarpatsky Ya.A., Algebraic-analytic Aspects of Completely Integrable Dynamical Systems and Their Pertubations, Institute of Mathematics Publ., v. 41, Kyiv, 2002 (in Ukrainian)

[366] Samoilenko A.M., Prykarpatsky A.K. and Prykarpatsky Ya.A., The spectral and differential-geometric aspects of a generalized de Rham-Hodge theory related with Delsarte tramsmutation operators in multi-dimension and its applications to spectral and soliton problems, Nonlin. Anal. 65 (2006) 395-432

[367] Samoilenko A., Prykarpatsky Ya., Taneri U., Prykarpatsky A. and Blackmore D., A geometrical approach to quantum holonomic computing algorithms, Math. and Computers in Simulation 66 (2004) 120

[368] Samoilenko A.M., Samoylenko V.H. and Sydorenko Y.M., The Kadomtsev-Petviashvili equation hierarchy with nonlocal constraints: multidimensional generalizations and exact solutions of reduced systems, Ukrainian Math. J. 49 (1999) 78-97 (in Ukrainian)

[369] Sato M., Soliton equations as dynamical systems on infinite-dimensional Grassmann manifolds, RIMS Kokyuroku 439 (1981) 30-40

[370] Satzer J., Canonical reduction of mechanical systems invariant under abelian group actions with an application to celestial mechanics, Indiana Univ. Math. J. 26(5) (1977) 951-976

[371] Schutz B.F., Geometrical Methods of Mathematical Physics, Cambridge University Press, Cambridge, 1982

[372] Schwarz K.W., Phys. Rev. B 38 (1988) 2398-2417

[373] Semenov-Tian-Shansky M. A., Poisson Lie groups, quantum duality principle, and the quantum double, In: Contem Math. 175, Amer. Math. Soc., Providence, 1994, 219-248

[374] Semenov-Tian-Shansky A.M., What is $\mathcal{R}$-matrix?, Func. Anal. and Appl. 17 (N4) (1983) 17-33

[375] Shor P.W., Polynomial time algorithms for prime factorization and discrete logarithms on a quantum computer, SIAM J. Comput. 26 (N5) (1997) 1484-1509

[376] Shubin M. A., Pseudo-Differential Operators and Spectral Theory, Nauka Publ., Moscow, 1978 (in Russian)

[377] Skrypnik I.V., Periods of a-closed forms, Proc. USSR Academy of Sciences 160(4) (1965) 772-773 (in Russian)

[378] Skrypnik I.V., A harmonic fields with peculiarities, Ukr. Math. J. 17(4) (1965) 130-133 (in Russian)

[379] Skrypnik I.V., The generalized De Rham theorem, Proc. UkrSSR Acad. of Sci. 1 (1965) 18-19 (in Ukrainian)

[380] Skrypnik I.V., A-harmonic forms on a compact Riemannian space, Proc. UkrSSR Acad. of Sci. 2 (1965) 174-175 (in Ukrainian)

[381] Souriau J.-M., Structures des systemes dynamiques, Dunod, Paris, 1970

[382] Sternberg S., Lectures on Differential Geometry, Prentice-Hall, Englewood Cliffs, 1964

[383] Sternberg S., Some preliminary remarks on the formal variational calculus of Gelfand and Dikij, Lect. Notes Math. 150, 399-407, 1980

[384] Sulanke R. and Wintgen P., Differential Geometry und Faser-bundel; bound 75, Veb. Deutscher Verlag der Wissenschaften, Berlin, 1972

[385] Symes W.W., Systems of Toda type, inverse spectral problems, representation theory, Invent. Math. 59 (1980) 13-51

[386] Teleman R., Elemente de topologie si varietati diferentiabile, Bucuresti Publ., Romania, 1964

[387] Thirring W., Classical Mathematical Physics, Third Edition, Springer-Verlag, Berlin, 1992

[388] Troshkin O.V., Nontraditional Methods in Mathematical Hydrodynamics, Transl. Math. Monogr., AMS, Providence, 114, 1995

[389] Tu G.Z., Andrushkiw R.I. and Huang X.C., A trace identity and its application to integrable systems in $(1 + 2)$-dimensions, J. Math. Phys. 32 (N7)(1991) 1900-1907

[390] Uhlmann A., Parallel transport and quantum holonomy along density operators, Rep. Math. Phys. 24 (1986) 229-240

[391] Vakhnenko V.A. Solitons in a nonlinear model medium. J.Phys. A: Math. Gen. 25 (1992) 4181-4187

[392] Wahlquist H. and Estabrook F., Prolongation structures of nonlinear evolution equations, J. Math. Phys. 16 (N1) (1975) 1-7; 17, (N1) (1975) 1293-1297

[393] Warner F., Foundations of Differential Manifolds and Lie Groups, Academic Press, NY, 1971.

[394] Weil J.-A. Introduction to Differential Algebra and Differential Galois Theory. CIMPA- UNESCO- VIETNAM Lectures: Hanoi, 2001

[395] Weinstock R., New approach to special relativity, Amer. J. Phys. 33 (1965) 640-645

[396] Wells R.O., Differential Analysis on Complex Manifolds, Prentice-Hall, Englewood Cliffs, NJ, 1973

[397] Weyl H., The Theory of Groups and Quantum Mechanics, Dover, New York, 1931

[398] Whitham G.B., Linear and Nonlinear Waves, Wiley-Interscience, New York, 1974

[399] Wilchek F., QCD and natural philosophy. Ann. Henri Poincaré 4 (2003) 211-228

[400] Wilson G., On the quasi-Hamiltonian formalism of the KdV equation, Phys. Lett. 132(8/9) (1988) 445-450

[401] Witten E., Nonabelian bozonization in two dimensions, Commun. Math.

Phys. 92 (1984) 455-472
[402] Woronowicz S.L., Commun. Math. Phys. 149 (1992) 637
[403] Zakharov V.E., Integrable Systems in Multi-dimensional Spaces, Lect. Notes Phys. 153, 190-216, 1983
[404] Zakharov V.E. and Manakov S.V., On a generalization of the inverse scattering problem, Theoret. Math. Phys. 27 (N3) (1976) 283-287
[405] Zakharov V.E. and Shabat A.B., A scheme of integration of nonlinear equations of mathematical physics via the inverse scattering problem, Func. Anal. Appl., Part1, 8 (N3) (1974) 43-53; Part2, 13 (N3) (1979) 13-32 (in Russian)
[406] Zakharov V., Manakov S., Novikov S. and Pitaevsky L., Theory of solitons: the inverse scattering problem, Moscow, Nauka Publisher, 1980 (in Russian)
[407] Zakrzewski S., Induced representations and induced Hamiltonian actions, J. Geometry and Physics 3(2) (1986) 211-219
[408] Zanetti P. and Rasetti M., Holonomic quantum computation, Phys. Lett. A 264 (1999) 94, quant-ph/9904011, quantph/9907103
[409] Zverovich E., Boundary problems of the theory of analytical functions in Hölder classes on Riemanniann surfaces, Russian Math. Surveys 26(1) (1971) 113-176 (in Russian)

Index